1993-06

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June, 1993

Technical Report CAS/CNS–93–044

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Abstract

A biomechanical model of the human oculomotor plant kinematics in 3–D as a function of muscle length changes is presented. It can represent a range of alternative interpretations of the data as a function of one parameter. The model is free from such deficits as singularities and the nesting of axes found in alternative formulations such as the spherical wrist (Paul, 1981). The equations of motion are defined on a quaternion based representation of eye rotations and are compact and computationally efficient.

Introduction

The position of the eye and, consequently, the position of images upon the retina are a result of the actions of the oculomotor muscles. This result depends upon the manner in which the muscles are attached to the eye. Miller (1989) describes two muscle models, namely the pulley and the no–pulley models. These two models have very different implications for the control of the oculomotor plant. His data are most closely fit by the pulley model. Because there may be some elasticity in the 'pulleys', the real eye may lie somewhere between the two models.

All of the eye muscles wrap around the globe somewhat. That is, the point of attachment for each eye muscle lies beyond its point of tangency with the globe. According to the "shortest path" hypothesis (Krewson 1950; Boeder 1962), the muscle's point of tangency slips along the globe via the shortest path as the globe is rotated by other antagonistic pairs. For instance, consider the muscles operating in the horizontal plane, the lateral (LR) and medial (MR) recti. These muscles rotate the eye around a vertical axis. If the shortest path hypothesis holds, then as the eye is elevated, by, say, the superior and inferior recti, the points of tangency for the LR and MR would move along a great circle in the direction of motion, or, in this case, 'up'. As a consequence the LR and MR could actually rotate the eye upwards by contracting simultaneously. This has been called the "bridle effect" (Robinson, 1975). According to Miller (1989) the shortest path hypothesis doesn't match the actual

*Supported in part by AFOSR (AFOSR 90-0128 and AFOSR F49620-92-J-0225).
†Supported in part by DARPA (AFOSR 90-0083) and ONR (ONR N00014-92-J-1309).

movement of the eye muscles and no bridle force exists in normal eyes. He notes that the points of tangency can sometimes move in the direction opposite to that predicted by the shortest path hypothesis.

Miller describes two alternative eye muscle models, the no–pulley and pulley models. In the no-pulley model, though the anterior part of the muscle is fixed to the globe, the eye rotates as if the muscle were inserted at the head–fixed point of tangency. This model implies that the eye muscles move the eye around a set of head fixed axes, see Figure 1A. Accurate intentional visual fixation commands depend upon eye position: The change in a target's retinotopic position is a function not only of length changes in a single ag–antag muscle pair, but also of the lengths of the other muscle pairs. A simple mechanical realization of the no-pulley model is not apparent. In the pulley model, the muscle sheaths are fixed to the wall of the orbit. This is interpreted, herein, to mean that the points of tangency are head–fixed. The pulley model implies that each muscle pair rotates the eye about the same eye fixed axis, see Figure 1B. For the pulley model, Miller observes that visual fixation commands can be given in retinotopic coordinates independent of eye position. That is, changes in a muscle pair's lengths will result in the same motion of the target on the retina, regardless of the lengths of the other muscle pairs. Miller points out that the eye may actually act as if it is somewhere in between the two models due to elasticity in the pulleys. Until this issue is resolved, a physical model of the oculomotor plant should be able to capture not only the pulley and no–pulley cases, but intermediate cases as well.

The spherical wrist model from robotics (Paul, 1981) may seem to be the obvious choice as a model of the globe's kinematics but it is not appropriate for the following reasons. In the spherical wrist, three fixed orthogonal joint axes are chosen. All rotations are parameterized by rotations about these axes in a specified order. However, The rotation axes of the oculomotor plant, as determined by the muscle planes, are not orthogonal. Hence, some mapping from the actual muscle axes to the joint axes of the spherical wrist would be necessary. Furthermore, rotations in the spherical wrist are always performed in the same order. Thus, the first axis is fixed, but later axes are not. It is inconsistent with both the no–pulley and the pulley models to have one axis fixed while others are not.

Nested axes present a problem not only to the theoretician, but to the experimentalist and the clinician as well. Eye rotations have typically been described in terms of horizontal and vertical components: The eye is looking so far over and so far up. Different coordinate frames have been defined by specifying different orders of rotation. Fick coordinates specify rotation first about a fixed vertical axis and then about the horizontal axis (Fick, 1854) Helmholtz coordinates specify rotation first about a (head) fixed horizontal axis and then about the vertical axis (Helmholtz, 1962). Unfortunately, both of these coordinate systems introduce an erroneous specification of the amount of rotation about the axis of gaze, this has been called "false torsion" (Boeder, 1962). Such difficulties led Westheimer (1957) to introduce Quaternion Algebra (Hamilton, 1866) to the oculomotor literature. This algebra is now used by some leading experimentalists (e.g. Tweed and Vilis, 1990).

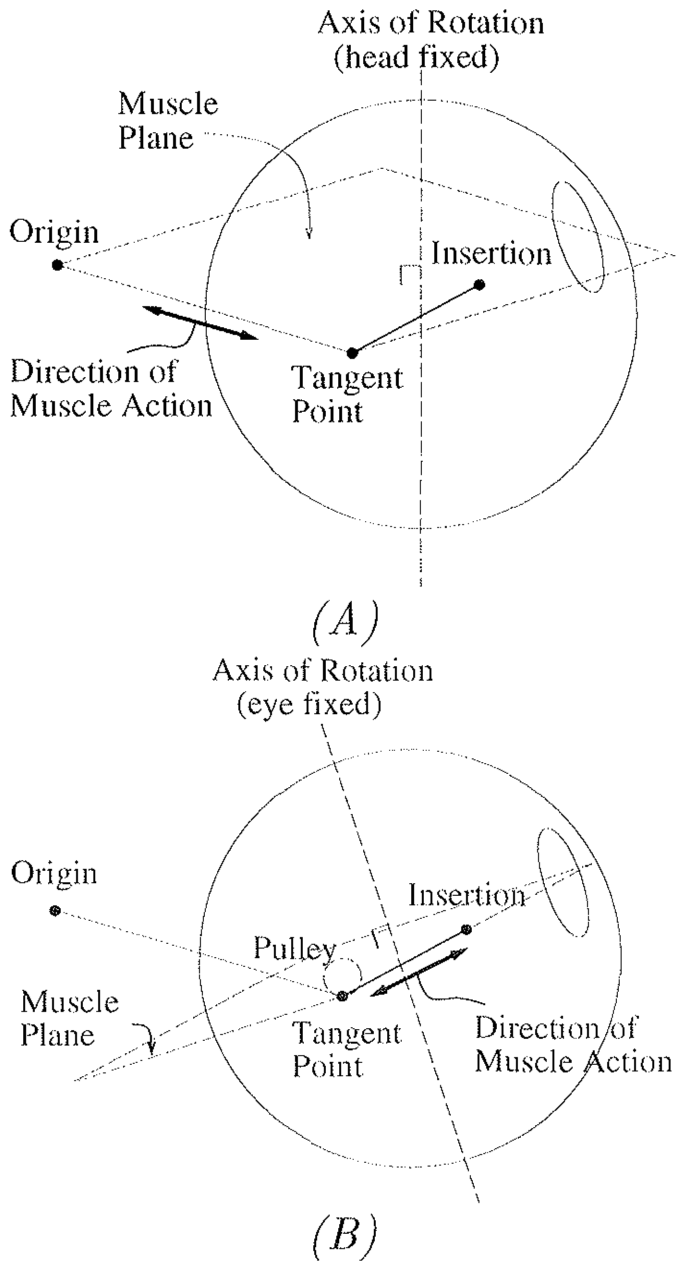

Figure 1: *(A)*: The no–pulley model of extraocular muscle action: The muscle plane is defined by the directions of action of the agonist and antagonist muscles. In this case, the effect of changing muscle length is a change in the distance between a muscle's point of tangency and its origin point. Thus, the eye is rotated about a head fixed axis. *(B)*: The pulley model of extraocular muscle action: In this case, the effect of changing muscle length is change in the distance between a muscle's head–fixed point of tangency and its eye–fixed insertion point. This results in an eye fixed axis.

A Formal Oculomotor Articulator Model

Let the no-pulley and pulley models be bounding cases on the movement of the axes of rotation. The no–pulley model is simpler to describe mathematically and is treated first. In no–pulley case, the axes of rotation axes stay fixed in the head. And, in the pulley model, the axes rotate with the eye. Each of the three ag–antag muscle pairs corresponds to a rotation axis perpedicular to the muscle plane defined by that pair. In all cases, define a saccade of the eye as the application of a spinor[1], S, to the eye or gaze attitude, g:

$$\Theta_s(g) = SgS^{-1} \tag{1}$$

A change in gaze attitude depends upon a change in the saccade spinor:

$$\dot{\Theta}(g) = \dot{S}gS^{-1} - SgS^{-1}\dot{S}S^{-1} \tag{2}$$

This can be reduced to

$$\dot{\Theta} = [\dot{S}S^{-1}, \Theta_s] \tag{3}$$

where $[A, B]$ is the commutator, $AB - BA$, of A and B.

No–pulley: In the no-pulley model, the saccade spinor, S, is constructed from three head–fixed unit bivectors, $\hat{P}$, $\hat{Q}$, and $\hat{R}$ defined by the rotation axes of the three muscle pairs. Since we wish to determine the position of the eye in terms of muscle coordinates, let α, β, and γ be determined by the muscle lengths and express angles of rotation about $\hat{P}$, $\hat{Q}$, and $\hat{R}$, respectively, in a right handed coordinate system. Below are the differential equations specifying a saccade as a function of these rotations.

In terms of α, β, and γ,

$$\dot{S} = (\dot{\alpha}\hat{P} + \dot{\beta}\hat{Q} + \dot{\gamma}\hat{R})S, \tag{4}$$

so

$$\dot{\Theta} = [\dot{\alpha}\hat{P} + \dot{\beta}\hat{Q} + \dot{\gamma}\hat{R}, \Theta]. \tag{5}$$

Pulley: In the pulley, model the axes of rotation remain fixed in the eye. To capture this, define, as vectors, head fixed points of tangency, x_t, y_t, and z_t, and eye fixed insertion points, x_i, y_i, and z_i. These vectors are used to compute the current bivector for each muscle pair.

In this case,

$$\dot{S} = (\dot{\alpha}\Theta(x_i) \wedge x_t + \dot{\beta}\Theta(y_i) \wedge y_t + \dot{\gamma}\Theta(z_i) \wedge z_t)S, \tag{6}$$

so

$$\dot{\Theta} = [\dot{\alpha}\Theta(x_i) \wedge x_t + \dot{\beta}\Theta(y_i) \wedge y_t + \dot{\gamma}\Theta(z_i) \wedge z_t, \Theta]. \tag{7}$$

Note that if S begins with unit length, it will stay so. This is true because the spinor or quaternion on the left hand side is a pure bivector.

[1]Quaternions are spinors. The terminology and notation used are from the Geometric Algebra (Hestenes, 1986). The Geometric Algebra of 3–D Euclidean space is a similar to the Quaternion Algebra (Hamilton, 1866), though the former has a right–handed basis and the latter a left–handed basis.

Unified Model: A unified mathematical model of both the pulley, no-pulley, and all cases lying in between the two can be constructed by defining a scalar function, $\zeta(t) \subseteq [0,1]$, that determines the degree to which each model holds, viz:

$$\dot{\Theta} = [\zeta(\dot{\alpha}\hat{P} + \dot{\beta}\hat{Q} + \dot{\gamma}\hat{R}) + (1-\zeta)(\dot{\mu}\Theta(x_i) \wedge x_t + \dot{\nu}\Theta(y_i) \wedge y_t + \dot{\xi}\Theta(z_i) \wedge z_t), \Theta]. \qquad (8)$$

The pulley model holds exclusively when $\zeta = 0$ and the no-pulley model holds exclusively when $\zeta = 1$. In general, ζ can be a function of $\Theta, t, \dot{\alpha}, \dot{\beta}$, $\dot{\gamma}$, $\dot{\mu}$, $\dot{\nu}$, and/or $\dot{\xi}$. If $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, $\dot{\mu}$, $\dot{\nu}$, and $\dot{\xi}$ represent true rates of change of muscle lengths then, of necessity, $\dot{\alpha} = \dot{\mu}, \dot{\beta} = \dot{\nu}$, and $\dot{\gamma} = \dot{\xi}$.

Conclusion

The above model was derived for use as the physical complement to a 3-D oculomotor control system (Bullock and Pribe, *in preparation*). This model takes rates of muscle length changes as input and gives eye attitude in space as output. This is the forward kinematics for converting muscle length derivatives to eye attitude in space. It is straightforward to compute retinotopic target positions from the spatial eye and target attitudes using spherical trigonometry. This computes a kind of inverse kinematics: Eye attitude in space is converted to attitude dependent retinotopic target position. Conversely, the oculomotor control model takes as input retinotopic target positions and yields muscle length changes as output.

Quaternion or Geometric Algebra simplifies our formulation and computations in several useful ways. Because the sines and cosines in the rotation matrices of linear algebra can be replaced by arithmetic operations, the actual computational cost is lower than matrix multiplication. Through assiduous grouping of terms, a quaternion rotation can be reduced to 16 multiplications and 12 additions (Field, 1987). The equivalent matrix multiplication requires 27 multiplications and 18 additions. Further, a representation such as the one above has no singularities (Geradin and Cardon, 1989). The equations presented here may be the simplest formulation of the equations of motion for the no-pulley and pulley models. This simplicity stems from expressing these equations in Geometric Algebra.

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