Boston University Theses & Dissertations

2013

The p-adic Shintani cocycle and p-adic L-functions

BOSTON UNIVERSITY

GRADUATE SCHOOL OF ARTS AND SCIENCES

Dissertation

THE P-ADIC SHINTANI COCYCLE AND P-ADIC L-FUNCTIONS

by

GEORGE ALEXANDER STEELE IV

B.Sc., Georgia Institute of Technology, 2007

Submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

2013

Approved by

First Reader ______________________________

Glenn Stevens, PhD
Professor of Mathematics

Second Reader ______________________________

Robert Pollack, PhD
Associate Professor of Mathematics

Third Reader ______________________________

David Rohrlich, PhD
Professor of Mathematics

Acknowledgments

It is a great pleasure to thank the many people who have made writing this thesis possible.

First of all, it is an honor to thank my advisor, Glenn Stevens, who has been guiding me since I was a student in PROMYS. Glenn has been a remarkable advisor, and I thank him for his patient mentorship, extraordinary generosity and friendship. I am enormously grateful for his generous financial support and for giving so much of his time and energy. I can't recall Glenn ever turning me away from his door, and I have left each of our meetings inspired and with renewed enthusiasm. I thank him for sharing his beautiful perspective of mathematics and for teaching me to think deeply of simple things.

It has been a great pleasure to be part of the vibrant number theory community at Boston University. I have learned a great deal of my mathematics from David Rohrlich, Rob Pollack, and Jay Pottharst, and I thank them for enthusiastically sharing their wealth of knowledge. I would also like to thank David Fried and Sam Isaacson for many stimulating conversations.

I am fortunate to have benefitted from many exceptional teachers, but there are two teachers in particular who profoundly influenced my path. I thank my undergraduate advisor, Matt Baker, for preparing me for this journey. My sincerest thanks to Ms. Yoder, my first great teacher, for inspiring me to pursue mathematics.

To the friends I have met over the past years: I warmly thank you for making the good times memorable and the difficult times endurable. There are too many names to list, but I especially want to thank Aly, Andrés, Ike, Jay, Katie, Ross, Sam, and Wes. I am incredibly fortunate to have so many good people in my life.

To my wonderful girlfriend, Kerri: Thank you for your love and patient understanding. The past two years have been an amazing time and I am enormously thankful to have you in my life.

Lastly, I am incredibly grateful for the love and support of my family. To my dad, Alec Steele and my aunt, Memi Turkington: Thank you both for your encouragement, your sage advice, and for always lending an ear. To my mom, Gini Steele: Thank you for everything. I miss you dearly.

THE P-ADIC SHINTANI COCYCLE AND P-ADIC L-FUNCTIONS

(Order No.)

GEORGE ALEXANDER STEELE IV

Boston University, Graduate School of Arts and Sciences, 2013

Major Professor: Glenn Stevens, Professor of Mathematics

ABSTRACT

The Shintani cocycle on $\mathrm{GL}_n(\mathbb{Q})$, as constructed by Hill, gives a cohomological interpretation of special values of Shintani zeta functions. We interpret this cocycle as taking values in a module, introduced by Stevens, of locally polynomial p-adic distributions localized with *rational poles*. Our first theorem shows that the Shintani cocycle takes values in the sub-module of locally analytic distributions with rational poles. Next, we give a simple criterion for the Shintani cocycle to specialize to true p-adic distributions which interpolate the special values of Shintani zeta functions. The theorem shows that such specializations are automatically p-adic measures.

We give two applications of these theorems: First, we recover the p-adic L-functions of totally real fields. Our construction greatly simplifies the original constructions of Deligne-Ribet, Cassou-Noguès, and Barsky, and we anticipate the methods will be useful for studying refinements of the Gross-Stark conjecture. Second, we give a simple construction of the p-adic L-function of critical slope Eisenstein series, proving a conjecture of Pasol and Stevens. The results of this construction were recently and independently proven by Bellaiche and Dasgupta, but our methods side-step many of the technical hurdles present in their construction.

Contents

List of Symbols

V	A finite dimensional $\mathbb{Q}$-vector space
V_p	A finite dimensional $\mathbb{Q}_p$-vector space
$\mathrm{GL}(V)$	The group of automorphisms of V
$\mathcal{LP}(V)$	Locally polynomial functions on V.
$\mathcal{S}(V)$	Test functions on V
$\mathcal{A}(V_p)$	Locally analytic functions on V_p
$\mathcal{D}(V, A)$ ...	Distributions on V valued in A
$\mathcal{D}_{poly}(V, A)$	Locally polynomial distributions on V valued in A.
$\mathcal{D}(V_p)$	Locally analytic distributions on V_p valued in $\mathbb{Q}_p$.
$\mathcal{M}(V_p)$	Measures on V_p valued in $\mathbb{Q}_p$.

Chapter 1

Introduction

The goal of this thesis is to construct a cocycle on $\mathrm{GL}_n(\mathbb{Q})$, valued in a space of p-adic distributions, which parameterizes (among other things) the p-adic L-functions of totally real fields of degree n. This idea begins with the works of Stevens [25] with the *Eisenstein modular symbol* and Sczech [20] with his eponymous $\mathrm{GL}_2(\mathbb{Q})$-cocycle. In these papers, the authors start with period integrals of Eisenstein series and (using very different methods) construct cocycles for $\mathrm{GL}_2(\mathbb{Q})$ valued in a space of distributions. A theorem of Siegel shows that these distributions parameterize special values of zeta functions for real quadratic fields. This constructions gives us interesting results on the special values: for example, it is clear from the construction that the special values at negative integers are rational. Moreover, these cocycles can be used to efficiently *compute* zeta values using, for example, the continued fraction trick of Manin.

It was observed by Stevens that the resulting distributions might be used for constructing p-adic zeta functions for real quadratic fields. Partial progress was made by Campbell [6], who showed that certain values of Stevens modular symbol can be put into p-adic analytic families of modular symbols. The idea bloomed a decade later in a series of papers by Solomon. In [22], Solomon constructed an Eiseinstein $\mathrm{GL}_2(\mathbb{Q})$ cocycle by adopting the methods of Shintani's cone zeta functions. Shintani showed that the Artin L-functions of totally real fields decompose as sums of these so-called Shintani zeta functions, and thus studying the special values of the cone zeta functions one deduces results on totally real fields [21]. For example, Cassou-Noguès [7] and Barsky [3] used Shintani's methods to construct p-adic L-functions for totally real fields. After making a "smoothing" modification similar to Cassou-Noguès's trick (see [7] and [15]), Solomon obtained p-adic zeta functions for real quadratic fields [23]. However, the case of arbitrary degree totally real fields remained off limits for two reasons. First, the $\mathrm{GL}_2(\mathbb{Q})$ cocycles only parameterize the zeta values of real quadratic fields– the zeta values for a degree n totally real field should come from a $\mathrm{GL}_n(\mathbb{Q})$-cocycle. Unfortunately, Solomon's construction only extends to $\mathrm{PGL}_3(\mathbb{Q})$, where we run into the next problem: It is not at all clear how to adopt Solomon's smoothing technique to the higher dimensional setting.

Recently, Hill has constructed a Shintani cocycle on $\mathrm{GL}_n(\mathbb{Q})$ for all n [13]. Hill's $n-1$ cocycle is valued in a module of cone functions (indicator functions of rational cones in $\mathbb{R}^n$) and can be roughly described as follows: An n-tuple of matrices $\alpha_1, \ldots, \alpha_n$ gives us a cone in $\mathbb{R}^n$ by taking the positive span of $\alpha_1 e_1, \ldots, \alpha_n e_n$. To get a (homogeneous) $n-1$ cocycle, we need to be able to "glue" together cones. When $n = 2$, this amounts to $\mathrm{Cone}(\alpha, \beta) + \mathrm{Cone}(\beta, \gamma) = \mathrm{Cone}(\alpha, \gamma)$. In order to glue cones, we must decide which faces to include. Hill addresses this problem by infinitesimally deforming his cones in a systematic way. Finally, Hill uses these cone functions to construct distributions (on the finite adeles of $\mathbb{Q}^n$) which parameterize the special values of Shintani zeta functions. We will sketch the details of his construction in §3.6.

Our starting point is Stevens' module of p-adic distributions *with rational poles*, which we recall in Chapter 2. In Chapter 3, we show how to interpret the Solomon-Hu pairing to get p-adic distributions (with rational poles) from a cone function and an auxiliary test function f' away from p. We use this recipe and Hill's cocycle to construct the p-adic Shintani cocycle. In §3.3, we introduce the *Vanishing Hypothesis*, which describes exactly when a cone function gives rise to a p-adic distribution (without poles). Theorem **??** is the main technical result of this section. In §3.8, we prove one of the main result of this thesis:

Theorem (3.7.1)**.** *Suppose f' satisfies the vanishing hypothesis for w_1. Then $\Phi_{f'}(\alpha_1, \ldots, \alpha_n)$ is a measure on V_p for all non-degenerate $(\alpha_1, \ldots, \alpha_n) \in \Gamma^n_{f'}$.*

In Chapter , we show that the construction of p-adic L-functions of totally real fields is an easy corollary of the above theorem. Recently, Charollois and Dasgupta [8] have obtained similar results constructions for totally real fields with Sczech's GL_n cocycle as part of a program to study the Gross-Stark units. In their work, they define an ℓ-smoothed Sczech cocycle and deduce integrality results from explicit formulas in terms of Dedekind sums. As a consequence of these integrality results, they construct the p-adic measures corresponding to the zeta values of totally real fields. They have announced similar results for a version of the Shintani cocycle, but their techniques are substantially different from ours. Concurrently, Spiess [24] has constructed p-adic measures from the Shintani cocycle, adapting the argument of Cassou-Noguès. Again, our techniques differ substantially and we believe our results are more general. Rather than constructing measures from integrality results, we find the p-adic pseudo-measures as specializations of the Shintani cocycle, then show that these specializations are in fact measures via our elementary arguments.

In Chapter , we give a very different application of our methods to construct the p-adic L-functions of critical slope Eisenstein series. This settles a conjecture of Pasol and Stevens who computed an approximation of (what appeared to be) a critical slope eigensymbol in $\Phi \in \mathrm{Symb}_{\Gamma_0(33)}(\mathcal{D}(\mathbb{Z}_3))$. This modular symbol specialized to the critical 3-stabilization of the classical modular symbol coming from the weight 2 Eisenstein series

$$E_{2,\ell} = \frac{\ell - 1}{24} + \sum_{n \geq 1} \left(\sum_{\substack{d \mid n \\ (\ell, d) = 1}} d \right) q^n, \tag{1.1}$$

with $\ell = 11$. We confirm their numerical experiments and construct a p-adic family of modular eigensymbols specializing to the p-adic L-functions of critical slope Eisenstein series of any weight. This result extends and generalizes the previous work of Kostadinov [16]. Prior and independently, Bellaïche and Dasgupta ([5]) constructed the same p-adic L-functions, using very different techniques and detailed knowledge of the eigencurve.

1.1 Notation and Definitions

We will reserve the letter V to denote a finite dimensional $\mathbb{Q}$-vector space. For each prime p, we will write $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$. If $L \subset V$ is a lattice, we will write $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We write $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$. If $V_{\mathbb{R}}$ has been equipped with a basis $e_1, \dots, e_n$, then we will denote by $(V_{\mathbb{R}})_+$ the positive orthant $\mathbb{R}_+ e_1 + \cdots \mathbb{R}_+ e_n$, where $\mathbb{R}_+ = (0, \infty)$. We will suppress this choice of basis from our notation.

Unlesss we specify otherwise, our convention will be to let $\mathrm{GL}(V)$ act on V on the left. By duality, this endows $\mathcal{S}(V)$ with a *right* $\mathrm{GL}(V)$ action, $(f|\gamma)(v) := f(\gamma v)$.

If $v_1, \dots, v_r$, $r \leq n$, are linearly independent vectors in $V_{\mathbb{R}}$, we write $C^o(v_1, \dots, v_r)$ for the set of all positive linear combinations $C^o(v_1, \dots, v_r) = \{\sum a_i v_i | a_i \in \mathbb{R}_+\}$. $C(v_1, \dots, v_r)$ will denote the closed cone $C(v_1, \dots, v_r) = \{\sum a_i v_i | a_i \in \mathbb{R}, a_i \geq 0\}$. In either case, we will call the rays in the directions $v_1, \dots, v_r$ the *extremal rays*. More generally, a *simplicial* cone C is a finite union of open cones (glued along boundaries). A *pointed* cone is a cone that does not contain any lines.

We will say a pointed open cone $C \subset V_{\mathbb{R}}$ is rational if it is generated by vectors $v_1, \dots, v_r \in V$. More generally, a simplicial cone is rational if it us the union of rational open cones.

Chapter 2

Distribution with rational poles

Fix V a finite dimensional vector space with distinguished lattice L. We topologize V by declaring the affine lattices to be open.

Definition 2.0.1. Let $\mathcal{S}(V)$ denote the abelian group of *test functions* $f : V \longrightarrow \mathbb{Z}$ satisfying the following conditions

a) f is locally constant: There exists a lattice $P \subset V$ such that, for all $v \in V$ and $\ell \in P$, $f(v + \ell) = f(v)$.

b) f has bounded support: There exist a lattice $S \subset V$ such that, for all $v \notin S$, $f(v) = 0$.

For example, if $U \subset V$ is an affine lattice, then the characteristic function of U, denoted $[U]$, is a test function on V. It is not hard to see that the test functions on V are exactly finite linear combinations of such characteristic functions.

The group of test functions on V can be identified with the abelian group of Bruhat-Schwartz functions on the finite adeles $\mathcal{S}(\mathbb{A}_V^{(\infty)})$. Let us quickly recall the definition.

Definition 2.0.2. For each prime p, write $\mathcal{S}(V_p)$ for the space of locally constant, compactly supported functions $f_p : V_p \to \mathbb{Z}$, with respect to the usual p-adic topology induced by L_p. The group of Bruhat-Schwartz functions is

$$\mathcal{S}(\mathbb{A}_V^{(\infty)}) = \bigotimes_p{}' \mathcal{S}(V_p) \tag{2.1}$$

where the restricted product means $f_p = [L_p]$ almost everywhere.

For each prime p, we will denote by $\mathcal{S}(V^{(p)})$ the space

$$\bigotimes_{\ell \neq p}{}' \mathcal{S}(V_\ell), \tag{2.2}$$

and refer to elements $f' \in \mathcal{S}(V^{(p)})$ as a test functions *away from p*

The diagonal embedding $V \hookrightarrow \mathbb{A}_V^{(\infty)}$ induces, via restriction, a canonical map $\mathcal{S}(\mathbb{A}_V^{(\infty)}) \longrightarrow \mathcal{S}(V)$. By the Chinese Remainder Theorem, this map is an isomorphism. From now on, we will implicitly identify these spaces and adopt the notation $\mathcal{S}(V)$ for both.

Definition 2.0.3. Let A be an abelian group equipped with a *left* $\mathrm{GL}(V)$ action. We define $\mathcal{D}(V, A) := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V), A)$. This is a *left* $\mathrm{GL}(V)$-module via the action $(\gamma \cdot \mu)(f) = \gamma \cdot \mu(f|\gamma)$. To invoke the analogy with integration, we will often write $\int_V f d\mu$ for $\mu(f)$.

There are two fundamental examples of distributions, both valued in $\mathbb{Q}$ (with the trivial $\mathrm{GL}(V)$-action). For each $v \in V$, there is the Dirac delta distribution $\delta_v(f) := f(v)$. We also have a Haar distribution, which depends on our fixed choice of lattice L. It can be defined in two equivalent ways. First, for each $f \in \mathcal{S}(V)$, there exist a lattice L_f for which f is periodic: $\forall \ell \in L_f$ and $v \in V$, $f(v + \ell) = f(v)$. One can define (see [25]) the global Haar measure h_V by putting

$$h_V(f) := \frac{1}{[L : L_f]} \sum_{v \in V/L_f} f(v). \tag{2.3}$$

Since f has bounded supported, the sum is finite, and it's easy to see that it is independent of choice of L_f.

On the other hand, we have at each local component V_ℓ, a local Haar measure h_ℓ normalized so that $h_\ell([L_\ell]) = 1$. Given $f = \bigotimes_\ell f_\ell \in \mathcal{S}(V)$, we can also define

$$h'_V(f) := \prod_\ell h_\ell(f_\ell) \tag{2.4}$$

and extend to all of $\mathcal{S}(V)$ by linearity.

Lemma 2.0.4. *The distributions h_V and h'_V are equal.*

Proof. See [25], section 3.3. $\square$

We define the Haar distribution of test functions away from p by defining $h^{(p)}(f') := h_V(f' \otimes [L_p])$. If f' is factorizable (i.e. $f' = \bigotimes_\ell f_\ell \in \mathcal{S}(V)$), then $h^{(p)}(f') = \prod_{\ell \neq p} h_\ell(f')$.

2.1 Locally polynomial distributions

It is useful to think of the test functions on $\mathcal{S}(V)$ as the locally constant functions on V, valued in $\mathbb{Z}$. In this section, we recall the notion of a locally polynomial function on V and, dually, a locally polynomial distribution. The locally polynomial distributions on V will play the role of products of locally polynomial distributions at each finite place, and will allow us to construct p-adic analytic distributions.

Write $V^* = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ for the linear dual of V. The symmetric algebra $\mathbb{Q}[V^*] = \bigoplus_{k=0}^{\infty} \mathrm{Symm}^k(V^*)$ will play the role of the ring of polynomial on V. Indeed, there is a natural $\mathbb{Q}$-algebra homomorphism

$$\mathbb{Q}[V^*] \to \mathrm{Funct}(V, \mathbb{Q})$$

which associates to each λ in $V^* \subset \mathbb{Q}[V^*]$ the associated linear function $\lambda : V \to \mathbb{Q}$, viewed as an element of $\mathrm{Funct}(V, \mathbb{Q})$.

Definition 2.1.1. We say a function $f : V \to \mathbb{Q}$ is *polynomial* if it is in the image of the map.

More generally, suppose W is another finite dimensional $\mathbb{Q}$-vector space.

Definition 2.1.2. A function $f : V \to W$ is called a polynomial function if, for every linear functional $\lambda : W \to \mathbb{Q}$, the composition $\lambda \circ f : V \to \mathbb{Q}$ is polynomial in the sense of definition 2.1.1.

Definition 2.1.3. A function $\varphi : V \longrightarrow W$ is said to be a *locally polynomial* function (of compact support) on V if (1) f is pre-compactly supported and (2) there is a lattice $L \subseteq V$such that for all $v \in V$ there is a polynomial function (in the above sense) $P_v : V \longrightarrow W$ such that φ is the restriction of P_v to $v + L$. Let $\mathcal{LP}_c(V, W)$ denote the space of all locally polynomial functions of compact support from V to W. When $W = \mathbb{Q}$, we will simply write $\mathcal{LP}_c(V)$.

As a concrete example, if $f \in \mathcal{S}(V)$ is a Schwartz function, then the map $V \to \mathrm{Symm}^n(V)$ by $v \mapsto f(v)v^n$ is a locally polynomial function (of compact support).

When $W = \mathbb{Q}$, the product of a Schwartz function $f \in \mathcal{S}(V)$ and a polynomial $P \in \mathbb{Q}[V^*]$ is a locally polynomial function from V to $\mathbb{Q}$. Conversely, on can express any locally polynomial function in $\mathcal{LP}_c(V)$ as the sum of finitely many terms $[v + L]P_v$. Therefore, one has

Lemma 2.1.4. *The natural map* $\mathcal{S}(V) \otimes_{\mathbb{Q}} \mathbb{Q}[V^*] \to \mathcal{LP}_c(V)$ *is an isomorphism.*

Definition 2.1.5. A locally polynomial distribution is a linear functional $\mu : \mathcal{LP}_c(V) \to \mathbb{Q}$. The vector space of ($\mathbb{Q}$-valued) locally polynomial distributions on V will be denoted

$$\mathcal{D}_{poly}(V) = \mathrm{Hom}_{\mathbb{Q}}(\mathcal{LP}_c(V), \mathbb{Q}).$$

Write $\mathcal{D}_{poly}(V, W)$ for $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{LP}_c(V), W)$ for a $\mathbb{Q}$-vector space W.

Remark 2.1.6. If V and W are both equipped with a left G-action (for any group G), then dually G acts on $\mathcal{LP}_c(V)$ by $f|\gamma(v) = f(\gamma v)$ and G acts on $\mathcal{D}_{poly}(V, W)$ by $(\gamma \star \mu)(f) = \gamma \cdot_W (\mu(f|\gamma))$.

Differential operators and rational poles

A vector $v \in V$ gives us a $\mathbb{Q}$-derivation $D_v : \mathbb{Q}[V^*] \to \mathbb{Q}[V^*]$ by putting

$$D_v(\lambda_1 \cdots \lambda_n) = \sum_{i=1}^{n} \lambda_i(v) \lambda_1 \cdots \widehat{\lambda_i} \cdots \lambda_n$$

and extending linearly. It is easy to see that the map $v \to D_v$ is a linear map $V \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[V^*])$. Moreover, the derivations commute, so the linear map $V \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[V^*])$ naturally extends to a $\mathbb{Q}$-algebra homomorphism $\mathbb{Q}[V] \to \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[V^*])$. We will write $D_{v_1 v_2}$ for $D_{v_1} D_{v_2}$, and if Q is a polynomial in $\mathbb{Q}[V]$, we will write D_Q for the corresponding differential operator $D_Q : \mathbb{Q}[V^*] \to \mathbb{Q}[V^*]$. By duality, we have an action of $\mathbb{Q}[V]$ on $\mathcal{D}_{poly}(V)$ and $\mathcal{D}_{poly}(V, W)$. That is, $(D_Q^* \mu)(f \otimes P) = \mu(f \otimes D_Q \cdot P)$.

Remark 2.1.7. If V has a left-action by a group G, then this extends to an action on $\mathbb{Q}[V]$. Furthermore, V^* has a right-action by G: $\lambda|\gamma(v) := \lambda(\gamma v)$ and this extends to an action on $\mathbb{Q}[V]$. It is easy to check that $D_{\gamma \cdot Q} P|\gamma^{-1} = D_Q \cdot P$.

As a concrete example, take $V = \mathbb{Q}^2$ with the standard basis, and let x, y be the standard coordinate functions. Then $D_{e_i} = \frac{\partial}{\partial x_i}$ and $D_{a_1 e_1 + b e_2} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$. A polynomial $Q(e_1, e_2) \in \mathbb{Q}[e_1, e_2]$ maps to the differential operator $D_Q = Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$

Let $S \subset \mathbb{Q}[V]$ be the multiplicative set generated by $V \backslash \{0\}$, and let $\mathbb{Q}[V]_S$ denote the localization of $\mathbb{Q}[V]$ with respect to S. Note that S is preserved by the natural action of $\mathrm{GL}(V)$ on $\mathbb{Q}[V]$.

Definition 2.1.8. (Distributions with rational poles) We define

$$\widetilde{\mathcal{D}}_{poly}(V, W) := S^{-1}\mathcal{D}_{poly}(V, W) = \mathcal{D}_{poly}(V, W) \otimes_{\mathbb{Q}[V]} \mathbb{Q}[V]_S.$$

We will call the elements of $\widetilde{\mathcal{D}}_{poly}(V, W)$ *polynomial distributions with rational poles.* If G acts on V and W, it acts on $\mathcal{D}_{poly}(V, W)$ and $\mathbb{Q}[V]_S$, and thus acts on $\widetilde{\mathcal{D}}$.

Proposition 2.1.9. *The canonical map $\mathcal{D}_{poly}(V, W) \to \widetilde{\mathcal{D}}_{poly}(V, W)$ is injective.*

Proof. It suffices to show that for an arbitrary non-zero $v \in V$, the differential operator d_v is injective on $\mathcal{D}_{poly}(V, W)$. But it is clear that the map $d_v : \mathcal{LP}_c(V) \to \mathcal{LP}_c(V)$ is surjective, hence the dual map is injective. □

Fourier Transform

For each $f \in \mathcal{S}(V)$, the map $v \to f(v)v^n$ is a locally polynomial function $V \to \mathrm{Symm}^n(V)$ in the sense of Defintion 2.1.3. If μ is a polynomial distribution $\mu \in \mathcal{D}_{poly}(V)$, there is a unique linear map

$$\mu : \mathcal{LP}_c(V, \mathrm{Symm}^n(V)) \to \mathrm{Symm}^n(V)$$

determined by the following condition: for all linear functionals $\lambda : \mathrm{Symm}^n(V) \to \mathbb{Q}$ and all $\varphi \in \mathcal{LP}_c(V, \mathrm{Symm}^n(V))$,

$$\lambda \circ \mu(\varphi) = \mu(\lambda \circ \varphi) \tag{2.5}$$

Thus, if $\mu \in \mathcal{D}_{poly}(V)$ is a locally polynomial distribution, the map

$$f \mapsto \int_V f(v)v^n d\mu(v) \in \mathrm{Symm}^n(v)$$

is a naive distribution in $\mathcal{D}_{naive}(V, \mathrm{Symm}^n(V))$. If G is a group acting on V, then G acts on $\mathrm{Symm}^n(V)$ and

$$\int_V (\gamma v)^n f|\gamma d\mu = \gamma \int_V v^n f|\gamma d\mu$$

Writing $\mathbb{Q}[[V]] = \prod_{n\geq 0} \mathrm{Symm}^n(V)$, we define a map $\mathcal{F} : \mathcal{D}_{poly}(V) \to \mathcal{D}_{naive}(V, \mathbb{Q}[[V]])$

$$\mathcal{F}(\mu)(f) := \sum_{n\geq 0} \int_V f(v)\frac{v^n}{n!} d\mu(v), \tag{2.6}$$

and call it the *Fourier transform.* Hopefully without causing too much confusion, we write

$$\mathcal{F}(\mu)(f) = \int_V f(v)e^v d\mu(v),$$

with the understanding that the right hand side is defined as the right hand side of (2.6).

In order to go from a naive distribution valued in $\mathbb{Q}[[V]]$ to a a locally polynomial distribution, we need a way of extracting coefficients from elements of $\mathbb{Q}[[V]]$. The ring

$\mathbb{Q}[[V]]$ is local with maximal ideal $V\mathbb{Q}[[V]]$ and residue field $\mathbb{Q}$. We write

$$ev : \mathbb{Q}[[V]] \longrightarrow \mathbb{Q}[[V]]/V = \mathbb{Q}$$

for the quotient map, or "evaluating at 0".

Replacing V with V^*, in §3.2, we have for each $\lambda \in V^*$ a $\mathbb{Q}$-derivation $D_\lambda : \mathbb{Q}[V] \to \mathbb{Q}[V]$ and this extends to $D_\lambda : \mathbb{Q}[[V]] \to \mathbb{Q}[[V]]$.

Lemma 2.1.10. *For all $n \in \mathbb{N}$, $v \in V$, and $\lambda \in V^*$, $ev((D_{\lambda^n} v^n) = n!\lambda(v)$.*

Proof. From the definition of D_λ,

$$D_\lambda v^n = \sum_{i=1}^{n} \lambda(n) v^{n-1} = n\lambda(n) v^{n-1},$$

so by induction

$$D_{\lambda^n} v^n = D_\lambda^n v^n = n!\lambda(v) \in \mathrm{Symm}^0(V),$$

and it follows $ev(D_{\lambda^n} v^n) = n!\lambda(v)$. □

Lemma 2.1.11. *For all $\lambda \in V^*$ and $v \in V$, $D_\lambda e^v = \lambda(v) e^v$.*

Proof. We compute

$$D_\lambda \sum_{n\geq 0} \frac{v^n}{n!} = \sum_{n\geq 0} \frac{D_\lambda v^n}{n!} = \sum_{n\geq 0} \lambda(v) \frac{n v^{n-1}}{n!} = \lambda(v) \sum_{n\geq 0} \frac{v^n}{n!} = \lambda(v) e^v$$

□

Lemma 2.1.12. *For all $v \in V$, $ev(e^v) = 1$.*

Proof. Obvious. □

Putting it all together, we have

Lemma 2.1.13. *For all $P \in \mathbb{Q}[V^*]$ and $v \in V$, $ev(D_P e^v) = P(v)$.*

The inverse of the Fourier transform, as we will see, is the map $\mathcal{F}^{-1} : \mathcal{D}_{naive}(V, \mathbb{Q}[[V]]) \to \mathcal{D}_{poly}(V, \mathbb{Q})$ sending a distribution ν to the functional $\mathcal{F}^{-1}\nu : \mathcal{S}(V) \otimes \mathbb{Q}[V^*] \to \mathbb{Q}$,

$$\int_V f \otimes P d\mathcal{F}^{-1}\nu(v) = ev\left(D_P \int_V f(v)d\nu(v)\right).$$

Observe $\mathcal{F}^{-1}\nu$ is the composition of linear maps

$$\mathcal{S}(V) \otimes \mathbb{Q}[V^*] \xrightarrow{\nu \otimes Id} \mathbb{Q}[[V]] \otimes \mathbb{Q}[V^*] \xrightarrow{Q \otimes P \mapsto D_P Q} \mathbb{Q}[[V]] \xrightarrow{ev} \mathbb{Q},$$

so that $\mathcal{F}^{-1}\nu$ is indeed a locally polynomial distribution on V.

Example 2.1.14. Let $\delta_w \in \mathcal{D}_{poly}(V, \mathbb{Q})$ be the (locally polynomial) Dirac delta: $\delta_w(f \otimes P) := f(w)P(w)$. The Fourier transform of δ_w is $e^w \delta_w \in \mathcal{D}_{naive}(V, \mathbb{Q}[[V]])$. Going backwards,

$$\int_V f(v) P d\mathcal{F}^{-1} \circ \mathcal{F}(\delta)(v) = ev(D_P e^w f(w)) = P(w)f(w) = \delta_w(f \otimes P)$$

Proposition 2.1.15. *The Fourier transform*

$$\mathcal{F} : \mathcal{D}_{poly}(V, \mathbb{Q}) \to \mathcal{D}_{naive}(V, \mathbb{Q}[[V]])$$

is an isomorphism of $\mathbb{Q}$-vector spaces. Moreover, $\mathcal{F}$ commutes with the action of G and with the action of $\mathbb{Q}[V]$:

$$\mathcal{F}(\gamma \cdot \mu) = \gamma \cdot \mathcal{F}(\mu) \text{ and } \mathcal{F}(D_Q \mu) = Q\mathcal{F}(\mu)$$

Proof. For the sake of clarity, it is convenient to pick a basis $w_1, \ldots, w_n$ of V, and write $x_1 = w_1^*, \ldots, x_n = w_n^*$ for the dual basis of V^*. The basis identifies $\mathbb{Q}[V]$ with $\mathbb{Q}[w_1, \ldots, w_n]$, $\mathbb{Q}[[V]]$ with $\mathbb{Q}[[w_1, \ldots, w_n]]$, and $\mathcal{F} : \mathcal{D}_{poly}(V, \mathbb{Q}) \to \mathcal{D}_{naive}(V, \mathbb{Q}[[w_1, \ldots, w_n]])$ as the map sending μ to the distribution

$$\mathcal{F}(\mu)(f) = \int_V f(x) e^{x_1(v)w_1 + \cdots x_n(v)w_n} d\mu$$
$$= \sum_{k_1,\ldots,k_n \geq 0} \left(\int_V x_1(v)^{k_1} \cdots x_n(v)^{k_n} f(v) d\mu(v) \right) \frac{w_1^{k_1} \cdots w_n^{k_n}}{k_1! \cdots k_n!}$$

First, we show the $\mathbb{Q}[V]$-equivariance: To compute $\mathcal{F}(D_{w_1}\mu)$, observe that $D_{w_1} x_1^{k_1} \cdots x_n^{k_n} = k_1 x_1^{k_1 - 1} \cdots x_n^{k_n}$ for all $k_1, \ldots, k_n \geq 0$. Therefore

$$\begin{aligned}\mathcal{F}(D_{w_1})(f) &= \sum_{k_1,\ldots,k_n \geq 0} \left(\int_V k_1 x_1(v)^{k_1 - 1} \cdots x_n(v)^{k_n} f(v) d\mu(v) \right) \frac{w_1^{k_1} \cdots w_n^{k_n}}{k_1! \cdots k_n!} \\ &= \sum_{k_1,\ldots,k_n \geq 0} \left(\int_V x_1(v)^{k_1} \cdots x_n(v)^{k_n} f(v) d\mu(v) \right) \frac{w_1^{k_1+1} \cdots w_n^{k_n}}{k_1! \cdots k_n!} \\ &= w_1 \mathcal{F}(\mu)(f)\end{aligned}$$

for all test functions f, so $\mathcal{F}(D_{w_1}\mu) = w_1 \mathcal{F}(\mu)$. Similarly, $\mathcal{F}(D_{w_i}\mu) = w_i \mathcal{F}(\mu)$ and hence $\mathcal{F}(D_{Q(w_1,\ldots,w_n)}\mu) = Q(w_1, \ldots, w_n)\mathcal{F}(\mu)$ for all $Q \in \mathbb{Q}[w_1, \ldots, w_n] = \mathbb{Q}[V]$.

To see the $\mathrm{GL}(V)$-equivariance, note

$$\gamma \cdot v = x_1(v)\gamma \cdot w_1 + \cdots x_n(v)\gamma \cdot w_n = x_1(\gamma \cdot v)w_1 + \cdots x_n(\gamma \cdot v)w_n.$$

Therefore,

$$\mathcal{F}(\gamma \cdot \mu)(f) = \int_V f(\gamma v) e^{\gamma v} d\mu(v) = \gamma \cdot \int_V f(\gamma v) e^v d\mu(v) = (\gamma \cdot \mathcal{F}(\mu))(f). \tag{2.7}$$

Lastly we show that $\mathcal{F}$ is an isomorphism. To check that $\mathcal{F}^{-1} \circ \mathcal{F} = Id$, observe that, for each $P \in \mathbb{Q}[V^*]$, the map $ev \circ D_P$ is a linear functional on $\mathbb{Q}[[V]]$, so equation (2.5) tells us

$$\int_V f \otimes P \; d\mathcal{F}^{-1} \circ \mathcal{F}(\mu) = ev\left(D_P \int_V f(v) e^v d\mu \right) = \int_V f(v) ev(D_P e^v) d\mu(v) = \int_V f \otimes P \; d\mu(v)$$

for all $f \in \mathcal{S}(V)$ and $P \in \mathbb{Q}[V^*]$. That is, $\mu = \mathcal{F}^{-1} \circ \mathcal{F}(\mu)$. Showing that $\mathcal{F} \circ \mathcal{F}^{-1} = Id$ follows from simply comparing coefficients of $\nu(f)$ and $(\mathcal{F} \circ \mathcal{F}^{-1}\nu)(f)$. □

2.2 p-adic measures and distributions

Let us fix p a prime for the remainder of this chapter. Recall $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The lattice L_p endows V_p with the structure of a $\mathbb{Q}_p$-Banach space.

We denote by $\mathcal{C}_c(V_p)$ the $\mathbb{Q}_p$-vector space of continuous functions $V_p \longrightarrow \mathbb{Q}_p$ with compact support. In addition to the compactly supported continuous functions on V_p, we will need to recall the spaces of locally polynomial and locally analytic functions (of compact support). First, we recall the notions of rigid analytic functions:

For each compact open subset $U \subset V_p$ and positive real number $r > 0$, define

- $B[U, r] := \{z \in V \widehat{\otimes} \mathbb{C}_p \mid \exists a \in U \text{ s.t.} |z - a| \leq r\}$.
- $A[U, r] :=$ the $\mathbb{Q}_p$-Banach algebra of rigid analytic functions on $B[U, r]$ whose taylor expansions on U have $\mathbb{Q}_p$-coefficients.
- $P[U, r] :=$ the subspace in $A[U, r]$ of polynomial functions.
- $D[U, r] :=$ the Banach dual of $A[U, r]$.

The norm on $A[U, r]$ is the supremum norm, and the dual norm on $D[U, r]$ is defined by

$$||\mu||_{U,r} = \sup_{\substack{f \in A[U,r] \\ f \neq 0}} \frac{|\mu(f)|}{||f||}. \tag{2.8}$$

Proposition 2.2.1. *Let U be a compact open and $r > s > 0$. The restriction map $A[U, r] \to A[U, s]$ is completely continuous.*

Proof. This is a well-known property of rigid analytic functions. □

The space of locally analytic functions on U is naturally identified with the injective limit

$$\mathcal{A}(U) = \varinjlim_{r>0} A[U,r] \tag{2.9}$$

endowed with the inductive limit topology. We also define, for fixed s, the overconvergent analytic functions

$$\mathcal{A}^{\dagger}(U,s) = \varinjlim_{r>s} A[U,r] \tag{2.10}$$

and endow it with the inductive limit topology.

The locally analytic functions are naturally embedded in the continuous functions, $\mathcal{A}(U) \subset \mathcal{C}(U)$. Within the space of locally analytic functions, we have the subspace of *locally polynomial* functions.

$$\mathcal{LP}(U) = \varinjlim_{r>0} P[U,r]. \tag{2.11}$$

Since $P[U,r] \subset A[U,r]$ is dense, $\mathcal{LP}(U) \subset \mathcal{A}(U)$ is dense with respect to the inductive limit topology.

Definition 2.2.2. The $\mathbb{Q}_p$-vector space of measures, distributions, and locally polynomial distributions on U are defined as

$$\mathcal{M}(U) := \mathrm{Hom}_{cts}(\mathcal{C}(U), \mathbb{Q}_p),\ \mathcal{D}(U) := \mathrm{Hom}_{cts}(\mathcal{A}(U), \mathbb{Q}_p),\ \text{and } \mathcal{D}_{poly}(U) := (\mathcal{LP}(U), \mathbb{Q}_p), \tag{2.12}$$

respectively. We endow $\mathcal{M}(U)$ and $\mathcal{D}(U)$ with the strong topology.

Now let $U \subset W$ be two compact opens of V_p.

Lemma 2.2.3. *Extension by zero induces a continuous map $\mathcal{A}(U) \hookrightarrow \mathcal{A}(W)$*

Proof. A locally analytic function $f \in \mathcal{A}(U)$ is the restriction to U, for some $r > 0$, of a rigid analytic function $\varphi \in A[U,r]$. Since $U \subset W$, we know $B[U,r] \subset B[W,r]$. Now $B[W,r]$ and $B[U,r]$ are each the disjoint union of finitely many closed balls of radius r, and by the ultrametric property the complement of $B[U,r]$ in $B[W,r]$ is a (possibly empty) union of disjoint open balls of radius r. Thus a rigid analytic function $\varphi \in A[U,r]$ extends by zero to a rigid analytic function in $A[W,r]$. The map $A[U,r] \longrightarrow A[W,r]$ is continuous and injective, so we have a continuous inclusion $\mathcal{A}(U) \hookrightarrow \mathcal{A}(W)$. $\square$

Dually, a locally analytic distribution $\mu \in \mathcal{D}(W)$ restricts to a locally analytic distribution $\mu|_U \in \mathcal{D}(U)$. The restriction maps are continuous, surjective, and compatible, so we form the projective limits:

Definition 2.2.4. The measures, distributions, and locally polynomial distributions on V_p are defined as

$$\mathcal{M}(V_p) := \varprojlim_U \mathcal{M}(U), \tag{2.13}$$

$$\mathcal{D}(V_p) := \varprojlim_U \mathcal{D}(U), \tag{2.14}$$

$$\mathcal{D}_{poly}(V_p) := \varprojlim_U \mathcal{D}_{poly}(U). \tag{2.15}$$

If μ is a distribution (or measure, or locally polynomial distribution) on V_p, we write $\mu|_U$ for the natural restriction to $U \subset V_p$.

The Amice transform

After choosing coordinates, all computations will reduce to the case of measures on $\mathbb{Z}_p^n$. The space of measures on $\mathbb{Z}_p^n$ can be explicitly described in terms of power series over $\mathbb{Z}_p$. We refer to [1] or [10] for proofs. When $n = 1$, Mahler's theorem gives an ON-basis of $\mathcal{C}(\mathbb{Z}_p)$ via the generalized binomial coefficients

$$\binom{x}{k} := \begin{cases} \frac{x(x-1)\cdots(x-k+1)}{k!} & \text{if } k \geq 1; \\ 1 & \text{if } k = 0. \end{cases}$$

This generalizes in a straightforward way to the case of several variables.

Theorem 2.2.5 (Mahler)**.** *The functions* $\left\{\binom{x_1}{k_1}\cdots\binom{x_n}{k_n}\right\}$ *form an ON-basis of* $\mathcal{C}(\mathbb{Z}_p^n)$*. Con-*

cretely, every $f \in \mathcal{C}(\mathbb{Z}_p^n)$ *can be written uniquely as*

$$f(x_1, \dots, x_n) = \sum_{k_1,\dots,k_n \geq 0} a_{k_1,\dots,k_n} \binom{x_1}{k_1} \cdots \binom{x_n}{k_n}$$

with $|a_{k_1,\dots,k_n}|_p \to 0$ *and* $||f|| = \sup |a_{k_1,\dots,k_n}|_p$.

Proof. See, for example, the remarks following Proposition 7, Section 7 in [1]. □

Theorem 2.2.5 implies measure μ is uniquely determined by the moments $\mu\left(\binom{x_1}{k_1}\cdots\binom{x_n}{k_n}\right)$, and that $||\mu|| = \sup_k \left|\binom{x}{k}\right|_p$. To ease notation, our convention will be to write x and k for the vectors $x = (x_1, \dots, x_n)$ and $k = (k_1, \dots, k_n)$. We simply write $\binom{x}{k} = \binom{x_1}{k_1}\cdots\binom{x_n}{k_n}$ when no confusion may arise.

These moments can be conveniently packaged into rigid analytic functions on an appropriate p-adic space. For each $r \in \mathbb{R}_+$, let us write $B(a, r) \subset \mathbb{C}_p$ for the open disc $B(1, r) = \{z \in \mathbb{C}_p : |z - a|_p < r\}$. We will write $\mathcal{B}_r$ for the open polydisk $B(1, r)^n$, and abbreviate $\mathcal{B}_1$ to $\mathcal{B}$. Denote by $(q_1, \dots, q_n)$ parameters on $\mathcal{B}$. The space of rigid analytic functions on $\mathcal{B}$, defined over $\mathbb{Q}_p$, is denoted by $A_{\mathbb{Q}_p}(\mathcal{B})$. It is the space of power series, in $q_1 - 1, \dots, q_n - 1$ over $\mathbb{Q}_p$ with bounded coefficients. For example, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_p^n$, we will write q^α for the function $q \mapsto q_1^{\alpha_1} \cdots q_n^{\alpha_n}$. By the binomial theorem, $q_i^{\alpha_i}$ is analytic in $q_i - 1$, so q^α is analytic.

Definition 2.2.6. We define the *Amice transform* as the homomorphism map $\mathcal{D}(\mathbb{Z}_p^n) \longrightarrow \mathbb{Q}_p[[q_1 - 1, \dots, q_n - 1]]$,

$$\mathcal{A}(\mu)(q_1, \dots, q_n) = \int_{\mathbb{Z}_p^n} q^x d\mu(x) = \sum_{k \in \mathbb{Z}_{\geq 0}^n} \left(\int_{\mathbb{Z}_p^n} \binom{x}{k} d\mu(x) \right) (q_1 - 1)^{k_1} \cdots (q_n - 1)^{k_n}. \quad (2.16)$$

Two fundamental theorems, due to Amice and Amice-Velu, describe the image of this integral transform.

Theorem 2.2.7 (Amice)**.** *The map* $\mathcal{A}$ *is an isomorphism of* $\mathbb{Q}_p$*-Banach algebra* $\mathcal{M}(\mathbb{Z}_p^n, \mathbb{Q}_p)$ *and* $\mathbb{Z}_p[[q_1 - 1, \dots, q_n - 1]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Theorem 2.2.8 (Amice-Velu, [2]). *The map $\mathcal{A} : \mathcal{D}(\mathbb{Z}_p^n) \longrightarrow A_{\mathbb{Q}_p}(\mathcal{B}))$, is an isomorphism of Fréchet spaces.*

The Amice transform is nothing more than a local version of the Fourier transform. Indeed, the p-adic exponential function identifies $\mathcal{B}$ with the polydisk $\mathbb{B}(0, p^{-1/p-1})^n$ with via $q_i = \exp_p(X_i)$. Then

$$\mathcal{A}(\mu) = \int_{\mathbb{Z}_p^n} \exp_p(x_1 X_1 + \cdots x_n X_n) d\mu(x) = \sum_{k_1, \ldots k_n \geq 0} \left(\int_{\mathbb{Z}_p^n} x_1^{k_1} \cdots x_n^{k_n} d\mu(x) \right) \frac{X_1^{k_1} \cdots X_n^{k_n}}{k_1! \cdots k_n!}. \tag{2.17}$$

It will be convenient to have notation for this version of the Amice transform, so we define

Definition 2.2.9. $\mathcal{F} : \mathcal{D}_{poly}(V_p, \mathbb{Q}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V_p), \mathbb{Q}_p[[V_p]])$ by

$$\mathcal{F}(\mu)(f_p) = \int_{V_p} f_p(x_1, \ldots, x_n)) e^v d\mu(v) = \int_{V_p} f_p(v) e^{x_1 X_1 + \cdots + x_n X_n} d\mu(x_1, \ldots, x_n). \tag{2.18}$$

Lemma 2.2.10. *For all $\gamma \in \mathrm{GL}(V)$, $(\gamma \cdot \mathcal{F})(\mu) = \mathcal{F}(\gamma \cdot \mu)$.*

2.3 Rational poles

The differential operators D_P, $P \in \mathbb{Q}[V]$, act on the function space $\mathcal{A}_c(V_p)$. (This follows, for example, from Proposition). Dually, $\mathcal{D}(V_p)$ and $\mathcal{D}_{poly}(V_p)$ are equipped with $\mathbb{Q}[V]$-actions, and we localize both with respect to $S = \langle V \backslash \{0\} \rangle \subset \mathbb{Q}[V]$.

Definition 2.3.1. Let $U \subset V_p$ be a compact open, and let $\mathcal{D}$ be one of the distribution spaces $\mathcal{D}_{poly}(U)$, $\mathcal{D}_{poly}(V_p)$, $\mathcal{D}(U)$ or $\mathcal{D}(V_p)$. The corresponding distribution space with rational poles is the localization $S^{-1}\mathcal{D} = \mathcal{D} \otimes_{\mathbb{Q}[V]} \mathbb{Q}[V]_S$.

Proposition 2.3.2. *The natural map $\mathcal{D}_{poly}(V_p) \to \widetilde{\mathcal{D}}_{poly}(V_p)$ is an injection.*

Proof. The proof is identical to the proof of Proposition 2.1.9 ∎

Lemma 2.3.3. *The restriction map $\mathcal{D}(V_p) \longrightarrow \mathcal{D}(U)$, $\mu \mapsto \mu|_U$, induces an injective homomorphism $\widetilde{\mathcal{D}}(V_p) \longrightarrow \widetilde{\mathcal{D}}(U)$, which we also refer to as the restriction map.*

Proof. The restriction map respects the $\mathbb{Q}[V]$-module structure of $\mathcal{D}(V_p)$ and $\mathcal{D}(U)$, so it extends to the tensor product $\mathcal{D}(V_p) \otimes_{\mathbb{Q}[V]} \mathbb{Q}[V]_S \longrightarrow \mathcal{D}(U) \otimes_{\mathbb{Q}[V]} \mathbb{Q}[V]_S$ by $\mu \otimes P \mapsto \mu|_U \otimes P$. □

We mimic our global constructions and extend, in the natural way, the Fourier transform

$$\mathcal{F} : \widetilde{\mathcal{D}}_{poly}(V_p, \mathbb{Q}_p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V_p), S^{-1}\mathbb{Q}_p[[V_p]]). \tag{2.19}$$

In the same way, we can extend the Amice transform $\mathcal{A}$ from locally analytic distributions to allow for rational poles. Inverting the linear forms $a_1X_1 + \cdots a_nX_n$ corresponds, on the polydisk $\mathcal{B}$, to inverting the linear form of logarithms $a_1 \log(q_1) + \cdots + a_n \log(q_n)$. Let us write $K(\mathcal{B})$ for the ring of meromorphic functions on $\mathcal{B}$. Then the Amice transform extends to an injective map of $\mathbb{Q}_p$ vector spaces

$$\mathcal{A} : \widetilde{\mathcal{D}}(L_p, \mathbb{Q}_p) \longrightarrow K(\mathcal{B}). \tag{2.20}$$

Making the obvious change of variables,

Lemma 2.3.4. *For all $\mu \in \widetilde{\mathcal{D}}(V_p, \mathbb{Q}_p)$, $\mathcal{A}(\mu|_{L_p}) = \mathcal{F}(\mu)([L_p])$.*

2.4 p-adic distributions from global distributions

Proposition 2.4.1. *The natural map $\mathcal{S}(V_p) \otimes \mathbb{Q}_p[V^*] \to \mathcal{LP}_c(V_p)$ is an isomorphism of $\mathbb{Q}_p$-vector spaces.*

Proof. The product of a polynomial function and a locally constant function is To see that the map is surjective, suppose $f \in \mathcal{LP}_c(V_p)$. Let $W \subset V_p$ be a compact open containing the support of f. At each point $w \in W$, there exists a real number r such that f, restricted to the closed polydisk $B[w, r_w]$, is a polynomial function $P_w : B[w, r_w] \to \mathbb{Q}_p$. Since W

is compact, there is a finite collection of $w \in W$ such that $U_w := B(w, r_w) \cap W$ cover W, and in fact we can take the U_w to be disjoint. Now it is easy to see that f is the image of $\sum [U_w] \otimes P_w \in \mathcal{S}(V_p) \otimes \mathbb{Q}_p[V^*]$ $\square$

Given a prime p, an element $f' \in \mathcal{S}(V^{(p)})$, and a locally polynomial distribution $\mu \in \mathcal{D}_{poly}(V, W)$, there is a unique linear map $\mu_{f'} : \mathcal{LP}_c(V_p) \longrightarrow \mathbb{Q}_p$ making the following diagram commute:

$$\begin{array}{ccccccc} \mathcal{LP}_c(V_p) & \xrightarrow{\sim} & \mathcal{S}(V_p) \otimes \mathbb{Q}_p[V^*] & \xrightarrow{\otimes f'} & \mathcal{S}(V) \otimes \mathbb{Q}[V^*] \otimes_{\mathbb{Q}} \mathbb{Q}_p & \xrightarrow{\sim} & \mathcal{LP}_c(V) \otimes_{\mathbb{Q}} \mathbb{Q}_p \\ & \searrow^{\mu_{f'}} & & & & & \downarrow \mu \\ & & & & & & \mathbb{Q}_p \end{array} \tag{2.21}$$

It is clear that the above diagram gives us a homomorphism $\mathcal{D}_{poly}(V) \longrightarrow \mathcal{D}_{poly}(V_p)$. After tensoring each $\mathbb{Q}[V]$-module by $\mathbb{Q}[V]_S$, we have proved

Lemma 2.4.2. *Fixing $f' \in \mathcal{S}(V^{(p)})$, we get a $\mathbb{Q}$-linear map $\widetilde{\mathcal{D}}_{poly}(V, \mathbb{Q}) \longrightarrow \widetilde{\mathcal{D}}_{poly}(V_p, \mathbb{Q}_p)$. Given $\mu \in \widetilde{\mathcal{D}}_{poly}(V, \mathbb{Q})$, we will write $\mu_{f'} \in \widetilde{\mathcal{D}}_{poly}(V_p, \mathbb{Q}_p)$ for the image under this map.*

In fact, since $\mathcal{S}(V) = \mathcal{S}(V^{(p)}) \otimes_{\mathbb{Z}} \mathcal{S}(V_p)$, the tensor-hom adjunction identifies a homomorpism $\mu : \mathcal{S}(V_p) \otimes \mathcal{S}(V^{(p)}) \longrightarrow S^{-1}\mathbb{Q}[[V]]$ with a homomorphism $\varphi_\mu \in \mathrm{Hom}(\mathcal{S}(V^{(p)}), \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V_p), S^{-1}\mathbb{Q}[[V]])$. Taking the Fourier transform, we have a useful lemma:

Lemma 2.4.3. *The Fourier transform induces a homomorphism of $\mathbb{Z}[\mathrm{GL}(V)]$-modules, $\mathcal{D}_{poly}(V, S^{-1}\mathbb{Q}[[V]]) \longrightarrow Hom_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}_{poly}}(V_p))$.*

Proof. To make this absolutely clear, recall the Fourier transform induces an isomorphism $\widetilde{\mathcal{D}}_{poly}(V_p, \mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V_p), S^{-1}\mathbb{Q}[[V]])$. The tensor-hom adjunction gives

$$\mathcal{D}(V, S^{-1}\mathbb{Q}[[V]]) := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}) \otimes \mathcal{S}(V_p), S^{-1}\mathbb{Q}[[V]]) \tag{2.22}$$

$$= \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V_p), S^{-1}\mathbb{Q}[[V]]) \tag{2.23}$$

$$\cong \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}}(V_p, \mathbb{Q})) \tag{2.24}$$

We just need to check that the homomorphism respects the $\mathrm{GL}(V)$-actions. The natural $\mathrm{GL}(V)$ action on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}}_{poly}(V_p)$ is given by $(\gamma \cdot \varphi)(f') = \gamma\varphi(f'|\gamma)$. $\square$

Chapter 3

The Shintani cocycle

3.1 Background: Shintani zeta functions

If C is a pointed simplicial cone in $V_{\mathbb{R}}^{+}$ and $f \in \mathcal{S}(V)$ is a test function (a finite $\mathbb{Z}$-linear combination of indicator functions of affine lattices), the *Shintani zeta function* $\zeta_{Sh}(f, C; s)$ is defined, for $\mathrm{Re}(s) \gg 0$, as the sum

$$\zeta_{Sh}(f, C; s) := \sum_{v \in C} \frac{f(v)}{\mathrm{N}(v)^s}$$

where $\mathrm{N}(v) = e_1^*(v) \cdots e_n^*(v)$ is the product of the coordinates. One can show that the sum converges for $\mathrm{Re}(s) \gg 0$ (see, for example, [12]), and Shintani showed the these have meromorphic continuation to $s \in \mathbb{C}$. Moreover, the values $\zeta_{SH}(f, C; -k)$ can be expressed in terms of Bernoulli polynomials.

Shintani used these results to study the special values of Hecke L-functions of totally real fields. Let us suppose that F is a totally real field of degree n. The Hecke L-functions of F decompose as sums of partial zeta functions, sometimes called *ray class zeta functions.* If $\mathfrak{f}$ is an integral ideal of F and $\mathfrak{a}$ is a fractional ideal relatively prime to $\mathfrak{f}$, then the ray class zeta function for $[\mathfrak{a}]_{\mathfrak{f}}$ is defined by

$$\zeta([\mathfrak{a}]_{\mathfrak{f}}, s) := \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_F \\ \mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{a}}} \frac{1}{\mathrm{N}\,\mathfrak{b}^s} \text{ for } \mathrm{Re}(s) \gg 0$$

where the sum is over all integral ideals representing $[\mathfrak{a}]_{\mathfrak{f}}$ in the narrow ray class group.

Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent in the narrow ray class group if and only if there exists a totally positive $\alpha \equiv 1 \pmod{\mathfrak{f}}$ such that $(\alpha) = \mathfrak{b}\mathfrak{a}^{-1}$. Put

$$E(\mathfrak{f}) = \{u \in \mathcal{O}_F^\times | u \gg 0 \text{ and } u \equiv 1 \pmod{\mathfrak{f}}\},$$

so that $(\alpha) \sim_{\mathfrak{f}} (\beta)$ if and only if $\alpha\beta^{-1} \in E(\mathfrak{f})$. We may rewrite the sum as

$$\zeta([\mathfrak{a}]_{\mathfrak{f}}, s) = \sum_{\substack{\beta \in (a+\mathfrak{a}^{-1}\mathfrak{f})/E(\mathfrak{f}) \\ \beta \gg 0}} \frac{1}{\mathrm{N}(\mathfrak{a}\beta)^s} = \mathrm{N}\,\mathfrak{a}^{-s} \sum_{\substack{\beta \in (a+\mathfrak{a}^{-1}\mathfrak{f})/E(\mathfrak{f}) \\ \beta \gg 0}} \frac{1}{\mathrm{N}\,\beta^s} \tag{3.1}$$

where $a \in \mathfrak{a}^{-1}$ is any fixed element congruent to $1 \pmod{\mathfrak{f}}$. If $\mathfrak{a}$ is integral, then it suffices to take $a = 1$.

In order to interpret these ray class zeta functions as Shintani zeta functions, we embed F in $\mathbb{R}^n$. Write $\tau_1, \ldots, \tau_n$ for the n embeddings of F into $\mathbb{R}$ and $F \hookrightarrow \mathbb{R}^n$ by $\alpha \mapsto (\tau_1(\alpha), \ldots, \tau_n(\alpha))$. The norm $N = e_1^* \cdots e_n^*$ on $\mathbb{R}^n$ extends the usual norm on F to $\mathbb{R}^n$. Shintani's insight was to construct a fundamental domain for the action of $E(\mathfrak{f})$ by decomposing $\mathbb{R}^n_+$ (where the totally positive elements live) into disjoint polyhedral cones. For example, if F is a real quadratic field and ε is a totally positive unit generating $E(\mathfrak{f})$, then the polyhedral cone $C^o(1, \varepsilon) \cup C^o(1)$ forms a fundamental domain for the action of $\mathcal{O}_F^\times$ (extended continuously to $\mathbb{R}^n$). More generally, Shintani proved

Proposition 3.1.1 (Proposition 4 of [21])**.** *Let $E \subset (\mathcal{O}_F^\times)_+$ be a finite index subgroup of totally positive units. Then there exists a disjoint union of simplical cones, C, such that $\varepsilon C \cap C = \emptyset$ for all $\varepsilon \in E$ and*

$$\mathbb{R}^n_+ = \coprod_{\varepsilon \in E} \varepsilon C.$$

Such a collection of cones will be called a *Shintani domain* for E. A Shintani domain lets us decompose the ray class zeta functions as

$$\zeta([\mathfrak{a}]_{\mathfrak{f}}, s) = (\mathrm{N}\,\mathfrak{a})^{-s} \sum_{\alpha \in (a+\mathfrak{a}^{-1}\mathfrak{f} \cap C)} \frac{1}{\mathrm{N}\,\alpha^s} = (\mathrm{N}\,\mathfrak{a})^{-s} \zeta_{Sh}([a + \mathfrak{a}^{-1}\mathfrak{f}], C; s) \tag{3.2}$$

reducing the study of special values of Hecke L-functions to the study of Shintani zeta functions.

Special values

Let us now suppose that C is the "open" cone $C = C^o(v_1, \ldots, v_r)$, with $v_1, \ldots, v_r \in V_{\mathbb{R},+}$. We remark that any cone can be written as a disjoint union of these "open" cones (and possibly the origin), hence it suffices to treat only this case. Shintani's meromorphic continuation of

$$\zeta_{SH}(f, C; s) = \sum_{v \in C} \frac{f(v)}{N(v)^s}$$

generalizes Riemann's arguments for the meromorphic continuation of $\zeta(s)$. First, one expresses $\zeta_{SH}(f, C; s)$ as Mellin-transform by

$$\sum_{v \in C} \frac{f(v)}{N(v)^s} = \sum_{v \in C} f(v) \frac{1}{\Gamma(s)^n} \int_{(0,\ldots,0)}^{(\infty,\ldots,\infty)} e^{-(e_1^*(v)x_1 + \cdots e_n^*(v)x_n)} x^s \frac{dx}{x} =$$
$$\frac{1}{\Gamma(s)^n} \int_{(0,\ldots,)}^{(\infty,\ldots,\infty)} \sum_{v \in C} f(v) e^{-(e_1^*(v)x_1 + \cdots e_n^*(v)x_n)} x^s \frac{dx}{x}.$$

To simplify notation, write $v \cdot x$ for $e_1^*(v)x_1 + \cdots e_n^*(v)x_n$. With the hypothesis that f is rational with respect to C, there exist $a_1, \ldots, a_r \in \mathbb{Q}$ such that f is periodic with respect to the lattice $a_1 v_1 \mathbb{Z} + \cdots + a_r v_r \mathbb{Z}$. After rescaling $v_1, \ldots, v_r$, we may assume f is periodic with respect to the lattice $v_1 \mathbb{Z} + \cdots v_r \mathbb{Z}$, then we can rewrite $\sum_{v \in C} f(v) e^{-v \cdot x}$ as the "rational function"

$$\sum_{v \in C} f(v) e^{-v \cdot x} = \frac{1}{1 - e^{-v_1 \cdot x}} \cdots \frac{1}{1 - e^{-v_r \cdot x}} \sum_{v \in \mathcal{P}} f(v) e^{-v \cdot x},$$

where $\mathcal{P} \subset C$ is fundamental domain for translation by $v_1 \mathbb{Z}_{\geq 0} + \cdots v_r \mathbb{Z}_{\geq 0}$. Switching signs, write

$$G(x_1, \ldots, x_n) = \frac{1}{1 - e^{v_1 \cdot x}} \cdots \frac{1}{1 - e^{v_r \cdot x}} \sum_{v \in \mathcal{P}} f(v) e^{v \cdot x}.$$

The function $\frac{1}{e^z - 1}$ has a simple pole at $z = 0$ with residue 1, so $G(x_1, \ldots, x_n)$ potentially has simple poles along the hyperplanes $v_1 \cdot x = 0, \ldots, v_r \cdot x = 0$.

Next, one would like to find the Mellin-transform of $G(-x)$ as a term in the complex contour integral

$$(1-e^{2\pi isn})\int_{(0,\dots,)}^{(\infty,\dots,\infty)} G(-x)x^s\frac{dx}{x} = \int_C G(-z)e^{(s-1)\log(z_1)+\cdots(s-1)\log(z_n)}dz,$$

where C is a product of keyhole contours $+\infty \to +\infty$ around 0. However, when $n>1$, the poles of G will intersect any sphere about the origin, hence our contour C, and we come to an impasse. Shintani managed to circumvent these problems by cleverly decomposing the domain of the Mellin transform. For details, we refer the reader to Shintani's original paper [21] or the notes of Greenberg and Dasgupta [11] for very readable accounts.

The following theorem is a reformulation of Proposition 1 of [21].

Theorem 3.1.2 (Shintani)**.** *The function $\zeta_{SH}(f,C;s)$ has meromorphic continuation to the whole complex plane with at most a simple pole at $s=1$. Moreover, the special values are given by*

$$\zeta_{SH}(f,C;-k) = \frac{1}{nk!^n}\left(\sum_{i=1}^{n}\mathrm{Coeff}(G(ux_1,ux_2,\dots,ux_n)\mid_{x_i=1};u^{nk}x_2^k\cdots x_n^k)\right) \qquad (3.3)$$

where $\mathrm{Coeff}(F(x_1,\dots,x_n),x_1^{k_1}\cdots x_n^{k_1})$ denotes the coefficient of $x_1^{k_1}\cdots x_n^{k_1}$ in the Laurent series of F about the origin.

Note that $G(x_1,\dots,x_n)$ is not necessarily a sum of monomials $x_1^{k_1}\cdots x_n^{k_n}$ near 0 (consider $\frac{e^{x+y}}{e^{x+y}-1} = \frac{1}{x+y}\sum_{n,m\geq 0}B_{n+m}\frac{x^ny^m}{n!m!}$). However, it's not hard to see that $G(ux_1,\dots,ux_n)|_{x_1=1}$ has a well-defined Laurent series in powers of $u,x_2,\dots,x_n$. If G happens to be holomorphic at the origin, then equation (3.3) simplifies to

$$\zeta_{SH}(f,C;-k) = \frac{1}{k!^n}\mathrm{Coeff}(G(x_1,\dots,x_n),x_1^k\cdots x_n^k) \qquad (3.4)$$

$$= \mathrm{Coeff}(\frac{\partial^{nk}}{\partial^k x_1\cdots\partial^k x_n}G(x_1,\dots,x_n);x_1^0\cdots x_n^0) \qquad (3.5)$$

$$= \frac{\partial^{nk}}{\partial^k x_1\cdots\partial^k x_n}G(x_1,\dots,x_n)\mid_{x=0} \qquad (3.6)$$

3.2 Cones and Distributions with rational poles

In this section, we construct distributions with rational poles from the data of a test function $f' \in \mathcal{S}(V^{(p)})$ and a pointed simplicial cone C. Before getting to the construction, we illustrate the ideas with a familiar example.

Let V be a finite dimensional $\mathbb{Q}$-vector space with basis $e_1, \ldots, e_n$. Let us fix a pointed open cone $C = C^o(v_1, \ldots, v_n)$, (here $v_1, \ldots, v_n$ linearly independent) and a test function $f \in \mathcal{S}(V)$. Consider the *formal sum*

$$\sum_{v \in C} f(v) e^{x \cdot v} = \sum_{v \in C \cap L} e^{(e_1^*(v)x_1 + \cdots e_n^*(v)x_n)}. \tag{3.7}$$

As we argued in the introduction, this sum represents a rational function; it converges absolutely to a rational function of exponentials for $x = (x_1, \ldots, x_n)$ in the *polar cone* $C^* := \{x \in V^* : x \cdot v < 0 \text{ for all } v \in C\}$. Indeed, after possibly scaling the extremal vectors $v_1, \ldots, v_n$ by positive rational numbers, we may assume f is periodic with respect to $v_1, \ldots, v_n$. Let $\mathcal{P} = \{\sum_{i=1}^n \lambda_i v_i : \lambda_i \in (0,1] \cap \mathbb{Q}\}$, which is a fundamental domain for translation by the half-lattice $\mathbb{Z}_{\geq 0} v_1 + \cdots \mathbb{Z}_{\geq 0} v_n$. For x in the polar cone, we can rearrange the absolutely convergent sum to get

$$\sum_{v \in C} f(v) e^{x \cdot v} = \sum_{v \in \mathcal{P}} \left(f(v) e^{x \cdot v} \sum_{m_1, \ldots, m_n \geq 0} e^{x \cdot (m_1 v_1 + \cdots m_n v_n)} \right) \tag{3.8}$$

$$= \left(\sum_{m_1, \ldots, m_n \geq 0} e^{x \cdot (m_1 v_1 + \cdots m_n v_n)} \right) \left(\sum_{v \in \mathcal{P}} f(v) e^{x \cdot v} \right) \tag{3.9}$$

$$= \frac{1}{1 - e^{x \cdot v_1}} \cdots \frac{1}{1 - e^{x \cdot v_n}} \sum_{v \in \mathcal{P}} f(v) e^{x \cdot v}, \tag{3.10}$$

This last sum is a finite linear combination of products of terms of the form $\frac{e^{\alpha x \cdot v}}{1 - e^{x \cdot v}}$, where $\alpha \in \mathbb{Q}$ and $v \in V$.

Remark 3.2.1. The rational function $\frac{e^{\alpha x \cdot v}}{1 - e^{x \cdot v}}$, which has Laurent series expansion about the origin

$$\frac{e^{\alpha x\cdot v}}{1-e^{x\cdot v}} = \frac{1}{x\cdot v}\sum_{n\geq 0} B_n(\alpha)\frac{(x\cdot v)^n}{n!}, \tag{3.11}$$

where B_n is the n-th Bernoulli polynomial. If we identify the symmetric algebra $\mathbb{Q}[[V]]$ with $\mathbb{Q}[[x_1,\ldots,x_n]]$ via the map $v\mapsto x\cdot v$, then the above Laurent series identifies $\frac{e^{\alpha x\cdot v}}{1-e^{x\cdot v}}$ with an element of $S^{-1}\mathbb{Q}[[V]]$.

We will tend to suppress coordinates from our notation and note that our constructions are independent of choice of basis. For any vector $v\in V$, we will write e^v for $e^{e_1^*(v)x_1+\cdots e_n*(v)x_n}$. Informally, a pointed cone C and a test function f give us a Laurent series in $S^{-1}\mathbb{Q}[[V]]$ which represents the rational function with formal expansion $\sum_{v\in C} f(v)e^v$. A pointed cone C corresponds to a unique distribution $\mu_C\in\mathcal{D}(V,S^{-1}\mathbb{Q}[[V]])$ characterized by

Cone functions

It will be convenient to work with weighted cones, or *cone functions*. Following Hill, let us write $\mathcal{K}_V^o$ for the abelian group of functions from $V_\mathbb{R}$ minus the origin to $\mathbb{Z}$, generated by the characteristic function of rational open cones, restricted to $V_\mathbb{R}\backslash\{0\}$. We write $\mathcal{K}_V$ for the group of functions $V_\mathbb{R}\to\mathbb{Z}$ whose restrictions to $V_\mathbb{R}\backslash\{0\}$ are in $\mathcal{K}_V^o$. The group $\mathrm{GL}(V)$ acts on $\mathcal{K}_V$ by

$$(\gamma\cdot\kappa)(v) = \mathrm{sign}(\det\gamma)\kappa(\gamma^{-1}v). \tag{3.12}$$

If κ_1,κ_2 are cone functions, then we will say $\kappa_1\leq\kappa_2$ if the support of κ_1 is contained in the support of κ_2.

The constant functions on $V_\mathbb{R}$ (minus the origin) $V_\mathbb{R}$ form a submodule of $\mathcal{K}_V$, and we write $\mathcal{L}_V$ for the quotient $\mathcal{K}_V/\mathbb{Z}$.

For example, if $v_1,\ldots,v_n$ are linearly independent vectors of V, the rational open cone $C^o(v_1,\ldots,v_n)$ is the set $\{\sum_{i=1}^n \alpha_i v_i : \alpha_i\in\mathbb{R}_+\}$. Then the characteristic function of this open cone, denoted $[C^o(v_1,\ldots,v_n)]$, is an element of $\mathcal{K}_V$.

The following proposition is a well-known result in the theory of lattice point enumeration in rational polytopes. Following Hill, we refer to the result as the "Solomon-Hu" pairing, which appeared in this context first in [14]. We refer to Hill's article [13] for a clear summary.

Proposition 3.2.2. *There is a non-degenerate bilinear pairing $\mathcal{K}_V \times \mathcal{S}(V) \longrightarrow S^{-1}\mathbb{Q}[[V]]$ characterized as follows: if κ is the characteristic function of a pointed cone C and $f \in \mathcal{S}(V)$, then*

$$\langle \kappa, f \rangle \text{“}=\text{”} \sum_{v \in C} f(v) e^{v}, \tag{3.13}$$

where the right hand side is understood as the (coordinate free) Laurent series representing the rational function represented by the right hand side.

We omit the proof. The following is an immediate corollary:

Corollary 3.2.3. *A cone function κ corresponds to a unique distribution $\mu_\kappa \in \mathcal{D}(V, S^{-1}$ $\mathbb{Q}[[V]])$ characterized by*

$$\mu_\kappa(f) = \sum_{v \in V} f(v)\kappa(v)e^{v}. \tag{3.14}$$

Moreover, the map $\kappa \mapsto \mu_\kappa$ is a $\mathbb{Z}[\mathrm{GL}^+(V)]$-homomorphism: for all $\gamma \in \mathrm{GL}^+(V)$, $\mu_\kappa|\gamma = \mu_{\gamma\cdot\kappa}$ and $\mu_\kappa + \mu_{\kappa'} = \mu_{\kappa+\kappa'}$.

Proof. The only thing left to check is the $\mathrm{GL}^+(V)$-equivariance. Fix $\gamma \in \mathrm{GL}^+(V)$ and $f \in \mathcal{S}(V)$. Using the formal expansion of $\mu_\kappa(f)$, we have

$$\mu_\kappa|\gamma(f) = \gamma \cdot \sum_{v \in V} \kappa(v) f(\gamma v) e^{v} \tag{3.15}$$

$$= \sum_{v \in V} \kappa(v) f(\gamma v) e^{\gamma v} \tag{3.16}$$

$$= \sum_{w = \gamma v \in V} \kappa(\gamma^{-1} w) f(w) e^{w} \tag{3.17}$$

$$= \sum_{w \in V} \gamma \cdot \kappa(w) f(w) e^{w} \tag{3.18}$$

$$=\mu_{\gamma\cdot\kappa}(f). \tag{3.19}$$

Since f was arbitrary, we conclude $\mu_\kappa|\gamma = \mu_{\gamma\cdot\kappa}$. □

Locally polynomial p-adic distributions with poles

Let us fix p prime for the remainder of this chapter. The Fourier transform identifies $\mathcal{D}(V, S^{-1}\mathbb{Q}[[V]])$ with $\widetilde{\mathcal{D}}_{poly}(V,\mathbb{Q})$, so a cone function κ gives rise to a unique locally polynomial distribution (with rational poles) with Fourier transform equal to μ_κ. Fixing $f' \in \mathcal{S}(V^{(p)})$ away from p. Lemma 2.4.2 gives us the following:

Proposition 3.2.4. *A cone function κ and a test function $f' \in \mathcal{S}(V^{(p)})$ away from p correspond to a unique locally polynomial distribution with rational poles $\mu_{\kappa,f'} \in \widetilde{\mathcal{D}}_{poly}(V_p,\mathbb{Q})$ characterized by*

$$\mathcal{F}(\mu_{\kappa,f'})(f_p) = \sum_{v\in V}(f'\otimes f_p)(v)\kappa(v)e^v. \tag{3.20}$$

Moreover, for all $\gamma \in \mathrm{GL}(V)$,

$$\mu_{\kappa,f'}|\gamma = \pm\mu_{\gamma\cdot\kappa,f'|\gamma^{-1}}, \tag{3.21}$$

where the sign is equal to the sign of $\det(\gamma)$.

Proof. It remains to verify the $\mathrm{GL}(V)$-action. Fixing $\kappa \in \mathcal{K}_V$, $f' \in \mathcal{S}(V^{(p)})$, it suffices to show that $\mathcal{F}(\mu_{\kappa,f'})|\gamma = \mathcal{F}(\mu_{\gamma\cdot\kappa,f'|\gamma^{-1}})$. So, let $f_p \in \mathcal{S}(V_p)$ be fixed but arbitrary, and take $\gamma \in \mathrm{GL}^+(V)$. Then $\mathcal{F}(\mu_{\kappa,f'})|\gamma(f_p) = \gamma\mathcal{F}(\mu_{\kappa,f'})(f_p|\gamma)$. By the definition of $\mu_{\kappa,f'}$,

$$\mathcal{F}(\mu_{\kappa,f'})|\gamma(f_p) = \gamma\cdot\sum_{v\in V}(f'\otimes(f_p|\gamma))(v)\kappa(v)e^v \tag{3.22}$$

$$= \gamma\cdot\sum_{v\in V}(f'|\gamma^{-1}\otimes f_p)(\gamma v)\kappa(v)e^v \tag{3.23}$$

$$= \pm\sum_{v\in V}(f'|\gamma^{-1}\otimes f_p)(\gamma v)\kappa(v)e^{\gamma v} \tag{3.24}$$

$$= \pm\sum_{w\in V}(f'|\gamma^{-1}\otimes f_p)(w)\gamma\cdot\kappa(w)e^w \tag{3.25}$$

$$= \pm\mathcal{F}(\mu_{\gamma\cdot\kappa,f'|\gamma^{-1}})(f_p). \tag{3.26}$$

Since f_p was arbitrary, we conclude $\mu_{\kappa,f'}|\gamma = \pm\mu_{\gamma\cdot\kappa,f'|\gamma^{-1}}$. □

Our first theorem shows that this pseudo-distribution extends, after clearing poles, to a unique linear functional on locally analytic functions.

Theorem 3.2.5. *For each $\kappa \in \mathcal{K}_V$ and $f' \in \mathcal{S}(V^{(p)})$, $\mu_{\kappa,f'} \in \widetilde{\mathcal{D}}(V_p, \mathbb{Q}_p)$. If $\kappa = [C^o(v_1, \ldots, v_r)]$ with $v_1, \ldots, v_r$ linearly indendent, then $D_{v_1} \cdots D_{v_n}\mu_{\kappa,f'} \in \mathcal{D}(V_p, \mathbb{Q}_p)$. In other words, $\mu_{\kappa,f'}$ is represented by*

$$\mu_{\kappa,f'} = (D_{v_1} \cdots D_{v_n}\mu_{\kappa,f'}) \otimes \frac{1}{v_1 \cdots v_n} \in \mathcal{D}(V_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}[V]} \mathbb{Q}[V]_S. \tag{3.27}$$

Proof. Since every cone function can be written as a finite linear combination of the characteristic functions of open cones, it suffices to consider the case $\kappa = [C^o(v_1, \ldots, v_r)]$ with $v_1, \ldots, v_r$ linearly independent vectors. After possibly rescaling $v_1, \ldots, v_r$ by positive scalars, we may assume without loss of generality that $v_1, \ldots, v_r$ are period vectors for f'. If $r < n = \dim_{\mathbb{Q}} V$, we extend to a maximal set of linearly independent vectors $v_1, \ldots, v_r$ by $v_{r+1}, \ldots, v_n$. Since $\widetilde{\mathcal{D}}(V_p) \subset \widetilde{\mathcal{D}}_{poly}(V_p)$ are equal to the projective limit, over all compact opens U, of $\widetilde{\mathcal{D}}(U) \subset \widetilde{\mathcal{D}}_{poly}(U)$, it suffices to show $\mu_{\kappa,f'} \in \mathcal{D}(U)$ for all compact opens U. Now each compact open U is contained in a lattice $L_m = p^{-m}\mathbb{Z}_p v_1 + \cdots p^{-m}\mathbb{Z}_p v_n$, for some integer m, so it suffices to show that $\mu_{\kappa,f'}|_{L_m} \in \widetilde{\mathcal{D}}(L_m)$ for an arbitrary m. Fix such an m, and identify V_p with $\mathbb{Q}_p^n$ and L_m with $\mathbb{Z}_p^n$ via the basis $\{p^{-m}v_1, \ldots, p^{-m}v_n\}$. To simplify notation, let us write μ for the restriction of $\mu_{\kappa,f'}$ to L_m. By computing the Amice transform of the, a priori, locally polynomial distribution $D_{v_1} \cdots D_{v_r}\mu$, we recognize it as a locally analytic distribution on L_m. Our choice of basis gives us coordinates $q_1, \cdots q_n$ on $\mathcal{B}$ and $(X_1, \cdots, X_n)$ on $B(0, p^{-1/p-1})^n$, with $q_i = \exp_p(X_i)$. By the comparison of Fourier and Amice transforms, $\mathcal{A}(D_{v_1}\cdots D_{v_r}\mu)=\mathcal{F}(D_{v_1}\cdots D_{v_r}\mu)([\mathbb{Z}_p^n])=v_1\cdots v_r\mathcal{F}(\mu)([\mathbb{Z}_p^n])$.

$$\mathcal{A}(D_{v_1}\cdots D_{v_r}\mu) = v_1\cdots v_r\sum_{v\in V}(f'\otimes[L_m])(v)\kappa(v)e^V \tag{3.28}$$

$$= X_1\cdots X_r\sum_{x_1,\dots x_r\in\mathbb{Q}^+}(f'\otimes[L_m])(x_1p^{-m}v_1+\cdots+x_np^{-m}v_n)e^{x_1X_1+\cdots x_nX_n}. \tag{3.29}$$

Now we claim $f'\otimes[L_m]$ is periodic with respect to the basis vectors $p^{-m}v_1,\dots,p^{-m}v_n$. Certainly $[L_m]$ is periodic, and we assumed f' was periodic with respect to $v_1,\dots,v_n$, so f' is periodic with respect to the basis. Therefore,

$$\mathcal{A}(D_{v_1}\cdots D_{v_r}\mu) \tag{3.30}$$

$$= X_1\cdots X_r\frac{1}{1-e^{X_1}}\cdots\frac{1}{1-e^{X_r}}\sum_{(x_1,\dots,x_n)\in((0,1]\cap\mathbb{Q})^n}(f'\otimes[\mathbb{Z}_p^n])(x)e^{x_1X_1+\cdots x_nX_n} \tag{3.31}$$

In the q-parameter, the Amice transform of $D_{v_1}\cdots D_{v_r}\mu$ is

$$\mathcal{A}(D_{v_1}\cdots D_{v_r}\mu)=\frac{\log(q_1)}{1-q_1}\cdot\frac{\log(q_r)}{1-q^r}\sum_{v\in\mathcal{P}}(f'\otimes[L_m])(v)q^v \tag{3.32}$$

where $\mathcal{P}=\{\sum_{i=1}^r x_ip^{-m}v_i|x_i\in(0,1]\cap\mathbb{Q}\}$.

We claim that $\mathcal{A}(D_{v_1}\cdots D_{v_r}\mu)$ is analytic on $\mathcal{B}$. First, observe that the sum is a finite linear combination of $q^v=q_1^{x_1}\cdots q_n^{x_n}$ with $x_1,\dots,x_n$ p-integral rational numbers. Each summand is thus an analytic function on the open polydisk $\mathcal{B}$. We are reduced to showing the that $\log(q_i)/(1-q_i)$ define analytic functions on $\mathcal{B}$. Observe

$$\log(q)/(1-q)=\frac{1}{q-1}\sum_{n\geq1}\frac{(-1)^n}{n}(q-1)^n\frac{(-1)^n}{n}=-\sum_{n\geq0}\frac{(-1)^{n+1}}{n+1}(q-1)^n. \tag{3.33}$$

The above power series converges for all $q\in\mathbb{C}_p$ such that $|q-1|_p<1$, so $\log(q)/1-q$

is an analytic function on the open unit ball $B(1,1)$. Thus $\log(q_i)/(1-q_i)$ is an analytic function on the polydisk $\mathcal{B}$, and $\mathcal{A}(D_{v_1}\cdots D_{v_r}\mu)$ is analytic. We conclude, by the theorem of Amice-Velu, that $D_{v_1}\cdots D_{v_r}\mu_{\kappa,f'}|_{L_m}$ defines a locally analytic distribution on L_m, and hence $D_{v_1}\cdots D_{v_r}\mu_{\kappa,f'}$ is a locally analytic distribution on V_p. □

As we will soon see, certain choices of f' and κ define honest p-adic analytic distribution. This happens, for example, when we write down the distributions coming from the "smoothed" partial zeta functions of Deligne, Ribet and Cassou-Noguès. When this is the case, the moments of $\mu_{\kappa,f'}$ are essentially special values of Shintani zeta functions. Suppose we pick a basis $e_1,\ldots,e_n$ of $V_{\mathbb{R}}$. Recall that this basis gives us a "norm function" $N : V \longrightarrow \mathbb{Q}$ which is the product of linear forms $N = e_1^*\cdots e_n^*$. This norm function is non-zero on the positive octant $(V_{\mathbb{R}})^+ = \{\sum_{i=1}^n \lambda_i v_i | \lambda_i > 0\}$

Proposition 3.2.6. *Suppose $\kappa = [C]$ for a pointed cone $C \subset (V_{\mathbb{R}})^+$, and suppose $\mu_{\kappa,f'} \in \mathcal{D}(V_p)$. Then, for any compact open $U \subset V_p$*

$$\int_U N^k(v)d\mu_{\kappa,f'} = \text{ the value at } s=-k \text{ of the analytic continuation of}$$

$$\zeta(f'\otimes[U],C;s) = \sum_{v\in C}\frac{f'\otimes[U](v)}{N(v)^s}$$

Proof. Fix C, f' as above and abbreviate $\mu_{\kappa,f'}$ to μ. Let $D_N : A_{\mathbb{Q}_p}(\mathcal{B}) \to A_{\mathbb{Q}_p}(\mathcal{B})$ be the differential operator $D_N q^v = N(v)q^v$. After rearranging a uniformly convergent series, we have

$$\int_U N^k(v)d\mu = \left(\int_U D_N^k q^v d\mu\right)\Big|_{q=1} = D_N^k\mathcal{A}(\mu)\mid_{q=1}.$$

We make the change of variables $q_i = e^{x_i}$, so $q^v = e^{e_1^*(v)x_1+\cdots e_n^*(v)x_n}$. Under this change of variables, $D_N^k = \frac{\partial^{nk}}{\partial^k x_1\cdots\partial^k x_n}$ and $\mathcal{A}(\mu)$ becomes the function $G(x_1,\ldots,x_n)$ represented by

$$\sum_{v\in C} f'\otimes[U]e^{e_1^*(v)x_1+\cdots+e_n^*(v)x_n}. \tag{3.34}$$

Since $\mathcal{A}(\mu)$ is holomorphic at $q=1$, G is holomorphic at $x_1,\ldots,x_n=0$. Shintani's theorem then implies

$$D_N^k \mathcal{A}(\mu) \mid_{q=1} = \frac{\partial^{nk}}{\partial^k x_1 \cdots \partial^k x_n} G(x_1, \ldots, x_n)|_{x_1,\ldots,x_n=0} = \zeta_{SH}(f' \otimes [U], C; -k).$$

□

3.3 Vanishing Hypothesis

In light of Proposition 3.2.6 it is natural to ask, "when is $\mu_{\kappa,f'}$ a distribution?" The main result of this section is an exact criterion for $\mu_{\kappa,f'}$ to be a distribution, and this happens if and only if $\mu_{\kappa,f'}$ is a *measure*. Roughly speaking, this happens whenever the test function f' has vanishing average in the directions of the extremal rays of C. To make precise this vague statement, we introduce some notation.

For each non-zero $w \in V$ and any $v \in V$, write $\pi_{v,w} : \mathcal{S}(V_\ell) \to \mathcal{S}(\mathbb{Q}_\ell)$ for the map which sends a test function $f \in \mathcal{S}(V_\ell)$ to the function $\pi_{v,w} f : \mathbb{Q}_\ell \to \mathbb{Z}$

$$(\pi_{v,w} f)(x) := f(v + xw), \text{ for all } x \in \mathbb{Q}_\ell. \tag{3.35}$$

A fortiori, $\pi_{v,w} f$ is indeed a test function on $\mathbb{Q}_\ell$. Similarly, we define $\pi_{v,w} : \mathcal{S}(V^{(p)}) \to \mathcal{S}(\mathbb{Q}^{(p)})$ and $\pi_{v,w} : \mathcal{S}(V) \to \mathcal{S}(\mathbb{Q})$.

Definition 3.3.1 (Vanishing Hypothesis). Let $w \in V$ be a non-zero vector. We will say a test function f' *satisfies the Vanishing Hypothesis for* w if $h^{(p)}(\pi_{v,w} f') = 0$ for all $v \in V$.

We remark that $h^{(p)}(\pi_{v,w} f')$ depends only on $v \mod \langle w \rangle$, since $h^{(p)}$ is the product of translation invariant local Haar measures.

Lemma 3.3.2. *If a test function $f' \in \mathcal{S}(V^{(p)})$ satisfies the vanishing hypothesis for w, then for all $U \subset V_p$ and $v \in V$,*

$$h_{\mathbb{Q}}(\pi_{v,w}(f' \otimes [U])) = 0. \tag{3.36}$$

Proof. The vectors v, w embed $\mathbb{Q} \hookrightarrow V$ and $\mathbb{Q}_p \hookrightarrow V_p$ via the inclusion $\lambda \mapsto v + \lambda w$. Let $W \subset \mathbb{Q}_p$ denote the projection of U to the line $v + \lambda w$. Then $\pi_{v,w}(f' \otimes [U]) = (\pi_{v,w} f') \otimes (\pi_{v,w}[U]) = (\pi_{v,w} f') \otimes [W]$, and $h_V(\pi_{v,w}(f' \otimes [U])) = h^{(p)}(\pi_{v,w} f') h_p([W]) = 0$. □

One may interpret this lemma as saying a test function f' satisfies the vanishing hypothesis for w if the average value of $f' \otimes f_p$ is 0 along all lines parallel to w, for all $f_p \in \mathcal{S}(V_p)$. While this hypothesis may seem odd, it is in fact easy to verify in important cases. Indeed, we will show that the vanishing hypothesis is satisfied when f' comes from the data of a "smoothed" ray class zeta function of a totally real field. In the next chapter, we show how the construction of p-adic L-functions of totally real fields is a corollary of our main theorem.

Theorem 3.3.3. *Suppose κ is the characteristic function of a pointed cone C with linearly independent extremal rays $v_1, \ldots, v_r$. The distribution with rational poles $\mu_{\kappa,f'} \in \widetilde{\mathcal{D}}(V_p)$ is a distribution if and only if f' satisfies the vanishing hypothesis for $v_1, \ldots, v_r$. Moreover, if $\mu_{\kappa,f'}$ is a distribution, it is a measure.*

Proof. Let us first describe the method of proof. After some simple reductions, we reduce to the case of analyzing $\mu_{\kappa,f'}$ restricted to a lattice. We show the Amice transform, a priori a meromorphic function on $\mathcal{B}_r$ for some $r > 0$, is *analytic* if and only if f' satisfies the correct vanishing hypothesis. Finally, we observe that if $\mathbb{A}(\mu_{\kappa,f'})$ is analytic on $\mathcal{B}$, it converges for free on the closure $\overline{\mathcal{B}}$. By Amice's theorem, we conclude $\mu_{\kappa,f'}$ is a measure when $\mu_{\kappa,f'}$ is a distribution.

We begin with the same reductions from Theorem 3.2.5. After possibly rescaling $v_1, \ldots, v_r$ by positive scalars, we may assume without loss of generality that $v_1, \ldots, v_r$ are period vectors for f'. If $r < n = \dim_{\mathbb{Q}} V$, we extend to a maximal set of linearly independent vectors $v_1, \ldots, v_r$ by $v_{r+1}, \ldots, v_n$. Since the distribution spaces $\mathcal{D}(V_p) \subset \widetilde{\mathcal{D}}(V_p)$ are equal to the projective limit, over all compact opens U, of $\mathcal{D}(U) \subset \widetilde{\mathcal{D}}(U)$, it suffices to show $\mu_{\kappa,f'} \in \mathcal{M}(U) \subset \mathcal{D}(U)$ for each compact open U if and only if f' satisfies the vanishing hypothesis for $v_1, \ldots, v_r$. Now each compact open U is contained in a lattice

$L_m = p^{-m}\mathbb{Z}_p v_1 + \cdots p^{-m}\mathbb{Z}_p v_n$, for some integer m, so it suffices to show, for each arbitrary m, that $\mu_{\kappa,f'}|_{L_m} \in \widetilde{\mathcal{M}}(L_m) \subset \mathcal{D}(L_m) \subset \widetilde{\mathcal{D}}(L_m)$ if and only if f' satisfies the right vanishing hypothesis. Without loss of generality, we may rescale $v_1, \ldots, v_n$ by p-th powers and take $m = 0$. The basis $\{v_1, \ldots, v_n\}$ identifies V_p with $\mathbb{Q}_p^n$ and L_m with $\mathbb{Z}_p^n$. To simplify notation, let us write μ for the restriction of $\mu_{\kappa,f'}$ to L_m.

We know, by the arguments of Theorem 3.2.5, that

$$\mathcal{A}(\mu) = \sum_{v \in C} f' \otimes [L_m](v) q^v = \frac{1}{1-q_1} \cdot \frac{1}{1-q^r} \sum_{v \in \mathcal{P}} (f' \otimes [L_m])(v) q^v \tag{3.37}$$

which we claim is analytic on $\mathcal{B}$ if and only if f' satisfies the vanishing hypothesis. To see this, we change coordinates, putting $(1+T_i) = q_i$ for $i \in \{1, \ldots, n\}$. The function becomes

$$\frac{1}{1-(1+T_1)} \cdots \frac{1}{1-(1+T_r)} \sum_{v \in \mathcal{P}} f' \otimes [L_m](v)(1+T_1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)}. \tag{3.38}$$

Thus, this function is holomorphic if and only if $T_1 \cdots T_r$ divides

$$F(T_1, \ldots, T_n) := \sum_{v \in \mathcal{P}} f' \otimes [L_m](v)(1+T_1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)} \in \mathbb{Z}_p[[T_1, \ldots, T_n]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \tag{3.39}$$

which is equivalent to each of $T_1, \ldots, T_r$ dividing F.

Claim: For each $i \in 1, \ldots, r$, $T_i | F$ if and only if f' satisfies the vanishing hypothesis for v_i.

For notational simplicity, we focus on the case $i = 1$. It is easy to see that T_1 divides $F(T_1, \ldots, T_r) \in \mathbb{Z}_p[[T_1, \ldots, T_n]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ if and only if $F(0, T_2, \ldots, T_n) = 0$, i.e.

$$F(0, T_2, \ldots, T_n) = \sum_{v \in \mathcal{P}} f' \otimes [L_m](v)(1)^{v_1^*(v)} \cdots (1+T_n)^{v_n^*(v)} = 0. \tag{3.40}$$

We carefully rearrange the sum as

$$F(0,T_2,\ldots,T_n)=\sum_{v\in\mathcal{P}\cap v_1^\perp}\left(\sum_{x\in(0,1]} f'\otimes[L_m](v+xv_1)\right)(1+T_2)^{v_2^*(v)}\cdots(1+T_n)^{v_n^*(v)}. \tag{3.41}$$

The coefficient in the parenthesis can be rewritten as

$$\sum_{x\in(0,1]} f'\otimes[L_m](v+xv_1)=\sum_{x\in(0,1]} f'\otimes[L_m](v+xv_1)=h_{\mathbb{Q}}(\pi_{v,v_1}(f'\otimes[L_m])). \tag{3.42}$$

This shows that $F(0,T_2,\ldots,T_n)=0$ if and only if $h_{\mathbb{Q}}(\pi_{v,v_1}f'\otimes[L_m])=0$ for all $v\in V$. Lemma 3.3.2 tells this is equivalent to the vanishing hypothesis for v_1, so T_1 divides F if and only if f' satisfies the vanishing hypothesis for v_1.. Similarly, we conclude T_i divides F if and only if f' satisfies the vanishing hypothesis for v_i giving the result.

Finally, let us notice that if $\mu_{\kappa,f'}$ is a distribution, then the Amice transforms $\mathcal{A}(\mu_{\kappa,f'}|L_m)$ belong to the subring $\mathbb{Z}_p[[q_1-1,\ldots,q_n-1]]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p\subset A_{\mathbb{Q}_p}(\mathcal{B})$, and thus $\mu_{\kappa,f'}$ is a measure. □

3.4 The GL_1 cocycle

Let us record a very simple cocycle for $\mathrm{GL}_1(\mathbb{Q})=\mathbb{Q}^\times$. This will be a useful example, and in fact the GL_2 cocycle is, in some sense, a product of the GL_1 cocycles. This connection will be made clear in Chapter 5.

A 0-cocycle $\varphi\in Z^0(\mathrm{GL}_1,\mathcal{K}_{\mathbb{Q}}/\mathbb{Z})$, is a $\mathbb{Q}^\times$-equivariant group homomorphism $\varphi:\mathbb{Q}^\times\longrightarrow\mathcal{K}_{\mathbb{Q}}/\mathbb{Z}$. We remark that the characteristic function of the open cone $C^o(1)$ is invariant under the action of $\mathrm{GL}_1(\mathbb{Q})$:

$$\lambda\cdot[C^o(1)]=\mathrm{sign}(\lambda)[C^o(\lambda^{-1})]=\mathrm{sign}(\lambda)[C^o(\mathrm{sign}(\lambda))]. \tag{3.43}$$

Since $[C^o(-1)]+[C^o(1)]\equiv 0 \pmod{\mathbb{Z}}$, we conclude $\mathrm{sign}(\lambda)[C^o(\mathrm{sign}(\lambda))]=[C^o(1)]$. By the Solomon-Hu pairing and the Fourier transform, we have a $\mathrm{GL}_1(\mathbb{Q})$-cocycle $\varphi\in\widetilde{\mathcal{D}}_{poly}(\mathbb{Q})^{\mathrm{GL}_1^+(\mathbb{Q})/\delta_0}$, characterized by

$$\mathcal{F}(\varphi)(f) = \sum_{v \in \mathbb{Q}_+} f(v) e^{vx}. \tag{3.44}$$

Definition 3.4.1. Fixing f' away from p, we define $\xi \in \widetilde{\mathcal{D}}(V_p)^{\Gamma^+_{f'}}/\delta_0$ to be the sum of $\Gamma^+_{f'}$-invariant distributions

$$\xi_{f'} := \varphi_{f'} + \frac{1}{2}\delta_0, \tag{3.45}$$

which is essentially the Kubota-Leopoldt distribution.

Our theorems states that if $D_x \xi_{f'}$ is a distribution and if f' satisfies the vanishing hypothesis, then $\xi_{f'}$ is a p-adic measure.

We define the p-adic L-function of a distribution $\mu \in \mathcal{D}(V_p)$ to be

$$L_p(\mu, s) := \int_{\mathbb{Z}_p^\times} x^{s-1} d\mu(x). \tag{3.46}$$

In fact, one can extend L_p to include distributions with poles, using the identity $D_x x^s = s x^{s-1}$. We define, for $s \in \mathbb{Z}_p \hookrightarrow \mathcal{X}(\mathbb{Q}_p)$,

$$L_p(\mu \otimes \frac{1}{D_x}, s) := \frac{1}{s} \int_{\mathbb{Z}_p^\times} x^s d\mu(x). \tag{3.47}$$

Taking $f' = \otimes_{q \neq p} [\mathbb{Z}_q]$, we have for all $n \geq 0$

$$L_p(\xi_{f'}, n) = \frac{1}{n} \int_{\mathbb{Z}_p^\times} x^n dD_x \xi_{f'} \tag{3.48}$$

$$= \frac{1}{n} \operatorname{Coeff}\left(\frac{Xe^X}{1-e^X} - \frac{Xe^{pX}}{1-e^{pX}}, \frac{X^n}{n!}\right) \tag{3.49}$$

$$= \frac{1}{n}(-B_n)(1-p^{n-1}) \tag{3.50}$$

By the density of $\mathbb{N}$ in $\mathbb{Z}_p$, we conclude

Lemma 3.4.2. *With f' as above,*

$$L_p(\xi_{f'}, s) = \zeta_p(1-s). \tag{3.51}$$

Here we are using the normalization $\zeta_p(1-k) = (p^{k-1}-1)\frac{B_k}{k}$.

3.5 Solomon's GL_2 cocycle

In the following sections, we will fix an arbitrary basis $\{w_1, \ldots, w_n\}$ for V.

Definition 3.5.1. If $\alpha_1 w_1, \ldots, \alpha_n w_1$ are linearly (in-) dependent, we will say $(\alpha_1, \ldots, \alpha_n)$ is (non-) degenerate.

Naively, one might try to define a $\mathrm{GL}(V)$ cocycle by sending the tuple $(\alpha_1, \ldots, \alpha_n)$ to the cone function $[C^o(\alpha_1 w_1, \ldots, \alpha_n w_1)]$. However, one must decided what to do in the degenerate cases. Even after solving this problem, the resulting cocycle will no longer satisfying the cocycle condition (3.53): the "edges" of cones are missing, so they do not glue together. In the case of $V = \mathbb{Q}^2$, Solomon [22] solves these problems by giving the edges weight 1/2. His cocycle (in Hill's language) is defined by

$$\sigma_{Solomon}(\alpha, \beta) = \operatorname{sign}\det(\alpha w_1, \beta w_1)\left([C^o(\alpha w_1, \beta w_1)] + \frac{1}{2}[C^o(\alpha w_1)] + \frac{1}{2}[C^o(\beta w_1)]\right) \tag{3.52}$$

with the convention that $\operatorname{sign} 0 = 0$. Observe that this actually descends to a cocycle on the quotient $\mathrm{PGL}_2(\mathbb{Q})$, since $\sigma_{Solomon}(\lambda\alpha, \lambda\beta) = \sigma_{Solomon}(\alpha, \beta)$ for all λ in the center of $\mathrm{GL}_2(\mathbb{Q})$. However, it's not clear how to extend this to higher dimensions. Solomon-Hu [14] define a cocycle on $\mathrm{PGL}_3(\mathbb{Q})$, but their methods to not extend to higher dimension. Hill's construction, which we briefly recall, elegantly side-steps these problems.

3.6 Hill's GL_n cocycle

We recall §3 of [13], slightly modifying Hill's conventions and construction . Hill's construction takes as input a choice of basis for V, so fix $\{w_1, \ldots, w_n\}$ a basis.

Hill's cocycle is a $\mathrm{GL}(V)$-equivariant map $\sigma_{Hill} : \mathrm{GL}(V)^n \to \mathcal{K}_V$ which, after quotienting out the constant functions, satisfies

$$\sum_{i=0}^{n}(-1)^n\sigma_{Hill}(\alpha_1,\ldots,\widehat{\alpha_i},\ldots,\alpha_n)=0. \tag{3.53}$$

First, Hill notes that if $\{v_1,\ldots,v_n\}$ is a basis of V, then the cone function $[C^o(v_1,\ldots,v_n)]$ is given by

$$[C^o(v_1,\ldots,v_n)](w)=\begin{cases}1 & \text{if } v_1^*(w),\ldots,v_n^*(w)>0\\ 0 & \text{otherwise}\end{cases}$$

Next, Hill "deforms" $\alpha_1 w_1,\ldots,\alpha_n w_1$ to a linearly independent set of vectors. Let $\varepsilon_1,\ldots,\varepsilon_n$ be indeterminates, and $\mathbb{F}=\mathbb{Q}((\varepsilon_1))\cdots((\varepsilon_n))$. Every element of $f\in F$ can be expressed as a sum of monomials

$$f=\sum_{\mathbf{r}=(r_1,\ldots,r_n)\in\mathbb{Z}^n} a_{\mathbf{r}}\varepsilon_1^{r_1}\cdots\varepsilon_n^{r_n}. \tag{3.54}$$

Ordering the indices $\mathbf{r}\in\mathbb{Z}^n$ lexicographically, Hill defines the leading term of (a non-zero) f to be the non-zero monomial $a_{\mathbf{r}}\varepsilon^{\mathbf{r}}$ for which $\mathbf{r}$ is smallest. For distinct $f,g\in\mathbb{F}$, Hill declares $f>g$ if the leading term of $f-g$ has positive coefficient, thus endowing $\mathbb{F}$ with the structure of an ordered field. Under this ordering, every positive power of ε_j is smaller than every positive power of ε_{j-1}, and every positive power of ε_1 is smaller than every rational number.

Now consider the vector space $V_{\mathbb{F}}:=V\otimes_{\mathbb{Q}}\mathbb{F}$ over $\mathbb{F}$. For each $i\in\{1,\ldots,n\}$, define $b_i=w_1+\varepsilon_i w_2+\cdots+\varepsilon_i^{n-1}w_n$. This forms an $\mathbb{F}$-basis for $V_{\mathbb{F}}$ over $\mathbb{F}$, and in fact:

Lemma 3.6.1 ([13], Lemma 1)**.** *For any $\alpha_1,\ldots,\alpha_n\in\mathrm{GL}(V)$ the vectors $\alpha_1 b_1,\ldots,\alpha_n b_n$ form a basis of $V\otimes_{\mathbb{Q}}\mathbb{F}$ over $\mathbb{F}$.*

Thus, for any $\alpha_1,\ldots,\alpha_n\in\mathrm{GL}(V)$, we have a natural cone function (on $V_{\mathbb{F}}$) by putting

$$[C^o(\alpha_1 b_1,\ldots,\alpha_n b_n)](w)=\begin{cases}1 & \text{if } (\alpha_1 b_1)^*(w),\ldots,(\alpha_n b_n)^*(w)>0\\ 0 & \text{otherwise}\end{cases}$$

The key result is that this cone function on $V_{\mathbb{F}}$ restricts to a cone function on $V \subset V_{\mathbb{F}}$.

Theorem 3.6.2 ([13], Theorem 1)**.** *The cone function* $[C^o(\alpha_1 b_1, \ldots, \alpha_n b_n)] : V_{\mathbb{F}} \to \mathbb{Z}$ *restricts to a rational cone function* $[C^o(\alpha_1 b_1, \ldots, \alpha_n b_n)] : V \to \mathbb{Z}$.

Definition 3.6.3. Let $\sigma_{Hill}(\alpha_1, \ldots, \alpha_n) = \operatorname{sign}\det(\alpha_1 b_1, \ldots, \alpha_n b_n)[C^o(\alpha_1 b_1, \ldots \alpha_n b_n)]|_V$.

By Theorem 3.6.2, σ_{Hill} is valued the module $\mathcal{K}_V$. It is not hard to see that it is $\mathrm{GL}(V)$-equivariant, but moreover it satisfies the cocycle condition (22).

Theorem 3.6.4 (Hill)**.** *The map* $\sigma_{Hill} : \mathrm{GL}(V)^n \to \mathcal{K}_V$ *is, modulo constant functions, an* $n-1$ *cocycle for* $\mathrm{GL}(V)$*. Moreover, if* $(\alpha_1, \ldots, \alpha_n)$ *is non-degenerate, there exists a simplicial cone*

$$C^o(\alpha_1 w_1, \ldots, \alpha_n w_1) \subset C \subset C(\alpha_1 w_1, \ldots \alpha_n w_1) \tag{3.55}$$

such that $\sigma_{Hill}(\alpha_1, \ldots, \alpha_n) = \pm[C]$.

Proof. A calculation shows (3.55)– for details see [24], Lemma 3.5. □

3.7 The p-adic Shintani cocycle

Fix p prime. Proposition 3.2.4 and Theorem 3.2.5 tell us a cone function κ and a test function f' give us a pseudo-distribution $\mu_{\kappa,f'} \in \widetilde{\mathcal{D}}(V_p)$. Put another way, κ gives a group homomorphism $\mathcal{S}(V^{(p)}) \longrightarrow \widetilde{\mathcal{D}}(V_p)$, $f' \mapsto \mu_{\kappa,f'}$. By abuse of notation we write μ_κ for this homomorphism.

Proposition 3.7.1. *The map* $\kappa \mapsto \mu_\kappa : \mathcal{S}(V^{(p)}) \longrightarrow \widetilde{\mathcal{D}}(V_p, \mathbb{Q}_p)$ *is a* $\mathrm{GL}^+(V)$*-equivariant homomorphism* $\mathcal{K}_V \longrightarrow Hom_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}}(V_p, \mathbb{Q}_p))$.

Proof. We see $\mu_\kappa + \mu_{\kappa'} = \mu_{\kappa+\kappa'}$, and in fact it is equivariant under the action of $\mathrm{GL}^+(V)$: $\gamma \cdot \mu_\kappa(f'|\gamma) = \mu_{\gamma\cdot\kappa}(f')$ by the second part of Proposition 3.2.4. □

We will use Hill's cocycle to construct a cocycle valued in p-adic distributions with rational poles. Since Hill's cocycle takes values in the quotient $\mathcal{L}_V = \mathcal{K}_V/\mathbb{Z}$, we need to compute the image of the constant functions under the map $\mathcal{K}_V \longrightarrow \mathrm{Hom}_{\mathcal{S}}(V^{(p)}, \widetilde{\mathcal{D}}(V_p))$. A calculation shows

Lemma 3.7.2. *Let* $[1] : V/\{0\} \longrightarrow \mathbb{Z}$ *be the constant cone function* $[1](v) = 1$ *for all* $v \neq 0$. *Then* $\mu_{[1],f'} = -f'(0)\delta_0$.

Proof. Recall that [1] corresponds to a naive distribution $\mu_{[1]} \in \mathcal{D}(V, S^{-1}\mathbb{Q}[[V]])$, and that by taking the inverse Fourier transform and composing with f', we arrive at the locally analytic distribution $\mu_{[1],f'} \in \mathcal{D}(V_p)$. The lemma is equivalent to the claim that $\mu_[1] = -\delta_0$, since $\mathcal{F}(\delta_0) = \delta_0$ and $\delta_0(f' \otimes f_p) = f'(0)\delta_0(f_p)$. This is proved in §2 of [13], but let us give a geometric argument. From the perspective of cone functions, $\mu_{[1]}$ is the characteristic function of the punctured space $V/\{0\}$, and $\mu_{[1]} + \delta_0$ is the characteristic function of the space V. Therefore, $\mu_{[1]} + \delta_0$ is invariant under convolution by all Dirac delta δ_v. Convolution by δ_v is equivalent to multiplication by e^v, so $(\mu_{[1]} + \delta_0)(f) + e^v(\mu_1 + \delta_0)(f) = 0$ for all $f \in \mathcal{S}(V)$. Since $S^{-1}\mathbb{Q}[[V]]$ is a domain, we conclude $\mu_1 + \delta_0 = 0$. □

Therefore, the homomorphism $\mathcal{K} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}, \widetilde{\mathcal{D}}(V_p))$ realizes the quotient $\mathcal{K}_V/\mathbb{Z}$ in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}}(V_p)/\delta_0)$.

Definition 3.7.3. We define the *p-adic Shintani cocycle* as the composition

$$\Phi : \mathrm{GL}^+(V)^n \xrightarrow{\sigma_{Hill}} \mathcal{K}_V/\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}}(V_p)/\delta_0) \tag{3.56}$$

If we fix $f' \in \mathcal{S}(V^{(p)})$ and write $\Gamma_{f'} \subset \mathrm{GL}(V)$ for the stabilizer of f', we get via restriction an evaluation a cocycle

$$\Phi_{f'} : \Gamma_{f'}^n \xrightarrow{Res} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}}(V_p)/\delta_0) \xrightarrow{\circ f'} \widetilde{\mathcal{D}}(V_p)/\delta_0. \tag{3.57}$$

3.8 Measure-valued cocycles

If V is an n-dimensional vector space, let us say that an n-tuple of elements $(\alpha_1, \ldots, \alpha_n) \in \mathrm{GL}(V)^n$ is *non-degenerate* if $\alpha_1 w_1, \ldots, \alpha_n w_1$ are linearly independent.

Theorem 3.8.1. *Suppose f' satisfies the vanishing hypothesis for w_1. Then $\Phi_{f'}(\alpha_1, \dots, \alpha_n)$ is a measure on V_p for all non-degenerate $(\alpha_1, \dots, \alpha_n) \in \Gamma_{f'}^n$.*

Before proceeding the the proof, we record an elementary lemma.

Lemma 3.8.2. *For all non-zero $w \in V$, $\gamma \in \Gamma_f$, and $v \in V$:*

$$\pi_{v,\gamma w} f(x) = f(v + x\gamma w) = f(\gamma(\gamma^{-1} v + xw)) = \pi_{\gamma^{-1}v,w} f|\gamma = \pi_{\gamma^{-1}v,w} f. \tag{3.58}$$

Now we are ready to prove the main result.

Proof. If f' satisfies the vanishing hypothesis for w_1, then by Lemma 3.8.2 it satisfies the vanishing hypothesis for all $v \in \Gamma w_1$. If $(\alpha_1, \dots, \alpha_n) \in \Gamma_{f'}^n$ is non-degenerate, then Equation (3.55) implies $\sigma_{Hill}(\alpha_1, \dots, \alpha_n)$ is $\pm$ the characteristic function of a cone C sandwiched between $C^o(\alpha_1 w_1, \dots, \alpha_n w_n) \subset C \subset C(\alpha_1 w_1, \dots, \alpha_n w_1)$. The cone C decomposes as a disjoint union of open cones $\{C_i\}$, each generated by a subset of the extremal rays $\alpha_1 w_1, \dots, \alpha_n w_1$. The vanishing criterion implies μ_{f',C_i} is a measure, so

$$\Phi_{f'}(\alpha_1, \dots, \alpha_n) = \pm \sum \mu_{f',C_i} \tag{3.59}$$

is a measure. □

Corollary 3.8.3. *Suppose* $\dim_{\mathbb{Q}}(V) = 2$, *and f' satisfies the vanishing hypothesis for w_1. Then $\Phi_{f'}$ is a measure-valued cocycle for $\Gamma_{f'}$*

Proof. Thanks to Theorem 3.8.1, we only have to verify that $\Phi_{f'}(\alpha_1, \alpha_2)$ is a measure in the degenerate case. Since $\Gamma_{f'}$ acts transitively on V, we can find $\gamma \in \Gamma$ such that γw_1 is not in the line spanned by $\alpha_1 w_1, \alpha_2 w_1$. The cocycle condition tells us

$$\Phi_{f'}(\alpha_1, \alpha_2) - \Phi_{f'}(\alpha_1, \gamma) + \Phi_{f'}(\alpha_2, \gamma) \equiv 0 \pmod{\delta_0}.$$

Our choice of γ implies $\Phi_{f'}(\alpha_1, \gamma)$ and $\Phi_{f'}(\alpha_2, \gamma)$ are measures, again by Theorem 3.8.1. Thus $\Phi_{f'}(\alpha_1, \alpha_2)$ is a measure. □

While this proof does not generalize to higher dimension, we believe that the conclusion should hold. That is, we believe the Shintani cocycle $\Phi_{f'}$ should be *measure-valued* whenever f' satisfies the vanishing hypothesis for w_1. However, Hill's cocycle becomes unwieldy in higher dimensional degenerate cases and our methods depend on knowledge of the generators of the cones. Even though we cannot conclude all specializations are p-adic measures, all cases of arithmetic interest are non-degenerate and fit within the framework of our results. Of particular interest is the case $V = F$, a totally real field of degree n, the subject of our next chapter.

Chapter 4

Totally real fields

4.1 Partial zeta functions

Let F be a totally real field, $\mathfrak{f}$ is an integral ideal (prime to p), $\mathfrak{c} \nmid \mathfrak{f}$ a prime ideal of degree 1, and m a nonnegative integer. For all fractional ideals $\mathfrak{a}$ prime to $\mathfrak{f}$, define for $s \in \mathbb{C}$

$$\zeta^*([\mathfrak{a}]_{\mathfrak{f}p^m}, s) = \sum_{\substack{0 \neq \mathfrak{b} \subset \mathcal{O}_F \\ [\mathfrak{b}]_{\mathfrak{f}p^m} = [\mathfrak{a}]_{\mathfrak{f}p^m} \\ (\mathfrak{b}, p) = 1}} \frac{1}{\mathrm{N}(\mathfrak{b})^s}.$$

Note that we have removed from the sum the ideals divisible by p. If $m \neq 0$, $\zeta^*([\mathfrak{a}]_{\mathfrak{f}p^m}, s) = \zeta([\mathfrak{a}]_{\mathfrak{f}p^m}, s)$. We also define

$$\zeta^*_{\mathfrak{c}}([\mathfrak{a}]_{\mathfrak{f}p^m}, s) := \zeta([\mathfrak{a}]_{\mathfrak{f}p^m}, s) - \mathrm{N}(\mathfrak{c})^{1-s} \zeta^*([\mathfrak{a}\mathfrak{c}^{-1}]_{\mathfrak{f}p^m}, s).$$

By Cebotarev, we may assume, without loss of generality, that $\mathfrak{a}$ is relatively prime to p and $\mathfrak{c}$.

Let $\mathcal{X}$ denote weight space, the rigid analytic variety $\mathcal{X} := \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{G}_m)$. We embed $\mathbb{Z} \hookrightarrow \mathcal{X}(\mathbb{Q}_p)$ by $k \mapsto (t \mapsto t^k)$ (note that we do not project t to $1 + p\mathbb{Z}_p$). For arbitrary elements $s \in \mathcal{X}(\mathbb{C}_p)$, $t \in \mathbb{Z}_p^\times$, we will write t^s for the image $s(t)$.

Theorem 4.1.1 (Deligne-Ribet, Cassou Noguès, Barsky)**.** *There exists a p-adic analytic function* $\zeta_{\ell,p}([\mathfrak{a}]_{\mathfrak{f}p^m}, s)$*,* $s \in \mathcal{X}(\mathbb{C}_p)$*, such that*

$$\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}p^m}, -k) = \zeta^*_{\mathfrak{c}}([\mathfrak{a}]_{\mathfrak{f}p^m}, -k)$$

for all integers $k \geq 0$*.*

The theorem will follow by taking $V = F$ and considering the Schwartz function

$$f' = \bigotimes_{q \nmid p\ell} [1 + \mathfrak{a}^{-1}\mathfrak{f}\mathcal{O}_{F,q}] \bigotimes ([\mathcal{O}_{F,\ell}] - \ell[\mathfrak{c}\mathcal{O}_{F,\ell}]) \,. \tag{4.1}$$

Let us write $\mathcal{O}_{F,p}$ for the lattice $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Note that $f' \otimes [1 + p^m\mathcal{O}_{F,p}] = [1 + \mathfrak{a}^{-1}\mathfrak{f}p^m\mathcal{O}_F] - \ell[c + \mathfrak{a}^{-1}\mathfrak{f}p^m\mathfrak{c}\mathcal{O}_F]$, where $c \in \mathfrak{c}$ is prime to $\mathfrak{f}$ and is $\equiv 1 \pmod{\mathfrak{f}}$. In what follows, it will be convenient to take $c \in \mathbb{Q}$. First, we verify the vanishing hypothesis for f':

Lemma 4.1.2. *The test function* f' *satisfies the vanishing hypothesis for* $w_1 = 1$*.*

Proof. Fix $\alpha \in F$. The projection $\pi_{\alpha,1}f' \in \mathcal{S}(\mathbb{Q}^{(p)})$ factors as

$$\pi_{\alpha,1}f' = \bigotimes_{q \nmid p\ell} \pi_{\alpha,1}[1 + \mathfrak{a}^{-1}\mathfrak{f}\mathcal{O}_{F,q}](\alpha + x) \bigotimes \pi_{\alpha,1}([\mathcal{O}_{F,\ell}] - \ell[\mathfrak{c}\mathcal{O}_{F,\ell}]) \,, \tag{4.2}$$

and so it suffices to show $h_\ell(\pi_{\alpha,1}[\mathcal{O}_{F,\ell}] - \pi_{\alpha,1}\ell[\mathfrak{c}\mathcal{O}_{F,\ell}]) = 0$. Since ℓ splits completely in F, we may choose coordinates identifying $\mathcal{O}_{F,\ell}$ with $\mathbb{Z}_\ell^n$, and $\mathfrak{c}\mathcal{O}_{F,\ell}$ with $\ell\mathbb{Z}_\ell \times \mathbb{Z}_\ell^{n-1}$. Then, if $\alpha \in \mathcal{O}_{F,\ell}$,

$$\pi_{\alpha,1}([\mathbb{Z}_\ell^n] - \ell[\ell\mathbb{Z}_\ell \times \mathbb{Z}_\ell^{n-1}]) = [-a + \mathbb{Z}_\ell] - \ell[-a + \ell\mathbb{Z}_\ell] \tag{4.3}$$

where $\alpha \equiv a \pmod{\mathfrak{c}}$, which clearly has Haar measure 0. If $\alpha \notin \mathcal{O}_{F,\ell}$, then the projection is 0, which also has Haar measure 0. Thus f' satisfies the vanishing hypothesis for 1. □

Since $E(\mathfrak{fc}) \subset \Gamma$, pairing our cocycle with non-degenerate elements of $H_{n-1}(E(\mathfrak{fc}), \mathbb{Z})$ gives us measures, and by picking out the right units we can recover zeta values as moments of our measure. The exact element we need to pair our cocycle is provided by Lemme 2.2 of [9], but it is not a priori clear that this will give us the correct zeta values. The problem is that Hill's cocycle, a priori, does not evaluate to Shintani domains when the degree of the field is greater than 2. However, Spiess has shown that Hill's construction does indeed recover Shintani domains:

Proposition 4.1.3 (Spiess)**.** *Let $\eta \in \mathbb{Z}[E(\mathfrak{fc})^n]$ be a generator of $H_{n-1}(E(\mathfrak{fc}), \mathbb{Z}) \cong \mathbb{Z}$. Then the cone function $\sigma_{Hill}(\eta)$ is $\pm$ the characteristic function of a Shintani domain.*

Proof. This is Proposition 3.7 of [24]. □

Proposition 4.1.4. *$\Phi_{f'} \cap \eta$ is a measure.*

Proof. Let $\varepsilon_1, \ldots, \varepsilon_{n-1}$ be fundamental units of $E(\mathfrak{fc})$. From Remark 2.1(c) of [24], $\eta = \pm\sum_{\tau \in S_{n-1}} \text{sign}(\tau)[\varepsilon_{\tau(1)}|\cdots|\varepsilon_{\tau(n-1)}]$, where $[\varepsilon_{\tau(1)}|\ldots|\varepsilon_{\tau(n-1)}]$ represents the cycle

$$(1, \varepsilon_{\tau(1)}, \varepsilon_{\tau(1)}\varepsilon_{\tau(2)}, \ldots, \varepsilon_{\tau(1)} \cdots \varepsilon_{\tau(n-1)}) \in \mathbb{Z}[\Gamma^n].$$

By Lemma 2.1 of [9], this is non-degenerate. Using our Lemma 4.1.2 and Theorem 3.8.1, we deduce that $\Phi_{f'} \cap \eta$ is a measure. □

Now we are ready to prove Theorem 4.1.1.

Proof. Fix $k \geq 0$ an integer, and let $\kappa = \sigma_{Hill}(\eta)$. By Proposition 4.1.3, $\sigma_{Hill}(\eta)$ is $\pm$ the characteristic function of a Shintani domain for $E(\mathfrak{f})$. In particular, κ is supported on the positive orthant $\mathbb{R}^n_+$. Let μ be the measure $\mu = \pm\Phi_{f'}(\eta)$, where the sign is the sign of κ. By Proposition 3.2.6, the moments of μ are given by

$$\int_{1+p^m\mathcal{O}_{F,p}} \mathrm{N}(\alpha)^k d\mu(\alpha) = \zeta_{SH}(f' \otimes [1 + p^m\mathcal{O}_{F,p}], \kappa; -k)$$

and

$$\int_{\mathrm{N}^{-1}(\mathbb{Z}_p^\times)} \mathrm{N}(\alpha)^k d\mu(\alpha) = \zeta_{SH}(f' \otimes [\mathrm{N}^{-1}(\mathbb{Z}_p^\times)], \kappa; -k).$$

By Equation 3.2,

$$\int_{1+p^m\mathcal{O}_{F,p}} \mathrm{N}(\alpha)^k d\mu(\alpha) = \mathrm{N}(\mathfrak{a})^k \zeta_{\mathfrak{c}}([\mathfrak{a}]_{\mathfrak{f}p^m}, -k) \tag{4.4}$$

and

$$\int_{\mathrm{N}^{-1}(\mathbb{Z}_p^\times)} \mathrm{N}(\alpha)^k d\mu(\alpha) = \mathrm{N}(\mathfrak{a})^k \zeta^*_{\mathfrak{c}}([\mathfrak{a}]_{\mathfrak{f}}, -k) \tag{4.5}$$

If $m > 0$, we define $\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}p^m}, s)$ to be the analytic function

$$\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}p^m}, s) := \mathrm{N}(\mathfrak{a})^s \int_{1+p^m\mathcal{O}_{F,p}} N(\alpha)^{-s} d\mu(\alpha), \tag{4.6}$$

where $\mathrm{N}(\alpha)^{-s} := s(\mathrm{N}(\alpha)^{-1})$. If $m = 0$,

$$\zeta_{\mathfrak{c},p}([\mathfrak{a}]_{\mathfrak{f}}, s) := \mathrm{N}(\mathfrak{a})^s \int_{\mathrm{N}^{-1}(\mathbb{Z}_p^\times)} N(\alpha)^{-s} d\mu(\alpha). \tag{4.7}$$

By equations 4.4 and 4.5, $\zeta_{\mathfrak{c},p}$ has the correct interpolation property. $\square$

Chapter 5

Critical slope Eisenstein series

In this chapter, we construct p-adic analytic families of modular eigensymbols containing overconvergent modular symbols attached to critical slope Eisenstein series. Concretely, we construct modular symbols valued in power series (or differential forms) with coefficients analytically varying analytically in a weight parameter $\kappa \in \mathcal{X}(\mathbb{Q}_p)$. At positive integers $\kappa = k$, the modular symbol will correspond to a non-zero overconvergent modular symbol in $\mathrm{Symb}_\Gamma(\mathcal{D}_k(\mathbb{Z}_p))$ with the eigenvalues of a critical slope Eisenstein series. This will let us construct the p-adic L-function of critical slope (née evil) Eisenstein series.

5.1 Modular Symbols

Notation

In this chapter, we run the risk of overwhelming the reader with our cumbersome notational conventions. To remedy this situation, we set $V = \mathbb{Q}^2$ equipped with the basis e_1, e_2. This identifies $\mathbb{Q}[V]$ with $\mathbb{Q}[X, Y]$ by $v \mapsto e_1^*(v)X + e_2^*(v)Y$. We put

$$R = \mathbb{Q}[[X, Y]] \qquad \widetilde{R} = S^{-1}\mathbb{Q}[[X, Y]]$$

$$\mathcal{S}^{(p)} = \mathcal{S}(V^{(p)}), \qquad \mathcal{D}_p = \mathcal{D}(V_p)$$

$$G = \mathrm{GL}_2(\mathbb{Q}), \qquad G^+ = \mathrm{GL}_2^+(\mathbb{Q})$$

For a $\mathbb{Q}[V]$ module M, we will write $\widetilde{M}$ for the localization $M \otimes_{\mathbb{Q}[V]} S^{-1}\mathbb{Q}[V]$ with respect to the multiplicative set $S = \langle V - \{0\}\rangle$. A generic element $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ will always have coordinates $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let Γ be a subgroup of $\mathrm{GL}_2(\mathbb{Q})$. We let Γ act on $\mathbb{P}^1(\mathbb{Q})$ via fractional linear transformations $\gamma \cdot s = \frac{as+b}{cs+d}$. Let Δ_0 denote the group of degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$ with the induced Γ-action. If M is a *right* Γ-module, a M-valued modular symbol is a Γ-invariant homomorphism $\varphi : \Delta_0 \longrightarrow M$. Here we are using the convention that $\mathrm{Hom}_{\mathbb{Z}}(\Delta_0, M)$ is a *right* Γ-module under the action:

$$(\varphi|\gamma)(D) := \varphi(\gamma D)|\gamma. \tag{5.1}$$

Definition 5.1.1. The module of M-valued modular symbols is denoted

$$\mathrm{Symb}_{\Gamma}(M) := \mathrm{Hom}_{\mathbb{Z}}(\Delta_0, M)^{\Gamma}. \tag{5.2}$$

In the previous chapters, we worked with G-*equivariant* maps of left G-modules. To stay consistent with the literature, we will now want to use *invariant* maps to right modules. This unfortunate switch of convention will likely lead to confusion, so we ease into the change by adopting the an unsightly notational convention, then drop it once no confusion may arise.

Definition 5.1.2. Let G be a group and M a left G-module. For each isomorphism $\varphi : G \xrightarrow{\sim} G^{op}$, we will write $M^{(\varphi)}$ for M equipped with the *right* G action $m|g = \varphi(g)m$. When no confusion may arise, we will simply write M with the understood right action.

We will primarily use the adjugate map $G \xrightarrow{\sim} G^{op}$, $\gamma \mapsto \gamma^* := \det(\gamma)\gamma^{-1}$, or the inversion map, $\gamma \mapsto \gamma^{-1}$.

A G^+-modular symbol is completely determined by its value on $\{\infty, 0\} \in \Delta_0$, since for all distinct cusps $r, s \in \mathbb{P}^1(\mathbb{Q})$, there exists α such that $\{r, s\} = \alpha \cdot \{\infty, 0\}$. Therefore, $\varphi\{r, s\} = \varphi\{\alpha\infty, \alpha 0\} = \varphi\{\infty, 0\}|\alpha^{-1}$.Conversely, one can define a G^+-modular symbol $\varphi \in \mathrm{Symb}_{G^+}(M)$ by specifying an element $m \in M$ and checking the necessary relations are satisfied.

Lemma 5.1.3. *Let M be a right $\mathbb{Z}[G^+]$ module, and let $m \in M$ be an element such that*

1. *$m|t = m$ for all $t \in T^+ \subset G^+$, the subgroup of positive determinant diagonal matrices,*

2. *$m + m|\sigma^{-1} = 0$, for all $\sigma \in G^+$ such that $\sigma\{\infty, 0\} = \{0, \infty\}$, and*

3. *$m + m|\tau^{-1} + m|\tau^{-2} = 0$, for all $\tau \in G^+$ such that $\tau\{\infty, 0\} = \{0, 1\}$ and $\tau\{0, 1\} = \{1, \infty\}$.*

Then there is a unique modular symbol $\varphi \in \mathrm{Symb}_{G^+}(M)$ such that $\varphi\{\infty, 0\} = m$.

Proof. This is nothing other than the Manin relations for $\mathrm{SL}_2(\mathbb{Z})$. $\square$

The Shintani modular symbol

In chapter 3, we used Hill's cone function cocycle and the Solomon-Hu pairing $\mathcal{K}_V \times \mathcal{S}(V) \longrightarrow S^{-1}\mathbb{Q}[[V]])$ to construct a distribution valued cocycle, which gave rise to the cocycle valued in p-adic distributions (with rational poles). In this section, we will execute the same strategy using Solomon's $\mathrm{GL}_2(\mathbb{Q})$ cocycle $\sigma_{Solomon}$ to obtain a $\mathrm{GL}_2^+(\mathbb{Q})$ modular symbol $\Phi \in \mathrm{Symb}_{G^+}(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}^{(p)}, \widetilde{\mathcal{D}_p}/\delta_0)^{(*)})$

Recall $\sigma_{Solomon}$ was defined by sending (α, β) to the cone function

$$\sigma_{Solomon}(\alpha, \beta) = \mathrm{sign}\det(\alpha e_1, \beta e_1)\left([C^o(\alpha e_1, \beta e_1)] + \frac{1}{2}[C^o(\alpha e_1)] + \frac{1}{2}[C^o(\beta e_1)]\right).$$

The Solomon-Hu pairing gives us a $\mathrm{GL}(V)$-equivariant homomorphism $\mathcal{K}_{\mathbb{Q}^2}/\mathbb{Z} \longrightarrow \mathcal{D}(\mathbb{Q}^2, \widetilde{R})/\delta_0$ and hence a homogenous 1-cocycle

$$\psi : \mathrm{GL}_2(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{Q}) \longrightarrow \mathcal{D}(V, \widetilde{R})/\delta_0. \tag{5.3}$$

Lemma 5.1.4. *The cocycle Ψ factors through $\mathrm{PGL}_2(\mathbb{Q})$.*

Proof. It should be clear that the cone function is invariant with respect to rescaling α, β by *positive* scalars. Moreover, a calculation (see Proposition 2.1 and Theorem 4.1 of [22]) shows

$$\langle \sigma_{Solomon}(-\alpha, -\beta), f \rangle = \langle \sigma_{Solomon}(\alpha, \beta), f \rangle \tag{5.4}$$

□

Let us fix $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ with the property $\alpha \cdot \infty = 0$. We will show the distribution $\psi(1, \alpha) \in \mathcal{D}(V, \widetilde{R})^{(*)}/\delta_0$ satisfies conditions (1)-(3) of the lemma, so we can define a modular symbol (also denoted Ψ), by specifying $\Psi\{\infty, 0\} := \psi(1, \alpha)$.

Proposition 5.1.5. *There is a unique modular symbol $\Psi \in \mathrm{Symb}_{\mathrm{GL}_2(\mathbb{Q})}(\mathcal{D}(V, \widetilde{R})^{(*)})$ such that,*

$$\Psi\{\infty, \cdot 0\} = \psi(1, \alpha) \tag{5.5}$$

Proof. First, we show that $\psi(1, \alpha)$ is invariant under the action of T^+. This follows from the above Lemma and the fact that $\sigma_{Solomon}(\alpha, \beta)$ only depends on the first columns of α and β.

Now we claim that conditions 2 and 3 of Lemma 5.1.3 are automatic from the cocycle condition. Indeed, if $\sigma \in \mathrm{GL}_2(\mathbb{Q})$ swaps the cusps ∞ and 0, then, modulo T, σ acts by $\sigma \cdot e_1 = e_2$, and $\sigma \cdot e_2 = e_1$, then $\psi\{\infty, 0\}|\sigma^{-1} = \sigma^{*-1} \cdot \psi(1, \alpha) = \psi(\sigma^{*-1}, \sigma^{*-1}\alpha) = \psi(\frac{1}{\det(\sigma)}\sigma, \frac{1}{\det(\sigma)}\alpha)$. Since Solomon's cocycle factors through $\mathrm{PGL}_2(\mathbb{Q})$, (and only depends on the first columns of the arguments) this is equal to $\psi(\alpha, 1)$. The cocycle property implies $\psi(\alpha, 1) + \psi(1, \alpha) = \psi(1, 1) = 0$, so $\Psi\{\infty, 0\}|\sigma^{-1} = -\Psi\{\infty, 0\}$. A similar argument shows condition 3 is satisfied. □

Remark 5.1.6. With the adjugate action on $\mathcal{D}(V, \widetilde{R})/\delta_0$, $\psi(1, \alpha)|\gamma^{-1} = \gamma^{*-1} \cdot \psi(1, \alpha) = \psi\left(\frac{1}{\det \gamma}\gamma, \frac{1}{\det \gamma}\gamma\alpha\right) = \psi(\gamma, \gamma\alpha)$. Thus, we have shown

$$\Psi\{\gamma \cdot \infty, \gamma \cdot 0\} = \psi(\gamma, \gamma\alpha). \tag{5.6}$$

By taking the inverse Fourier transform (which is $\mathrm{GL}_2(\mathbb{Q})$-equivariant), we get a modular symbol valued in locally polynomial distributions with rational poles. By Lemma 2.4.3, we have a $\mathrm{GL}_2(\mathbb{Q})$-equivariant homomorphism $\mathcal{D}(V,\widetilde{R}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}(V^{(p)}), \widetilde{\mathcal{D}_{poly}}(V_p))$. Theorem 3.2.5 shows that this modular symbol is valued in locally analytic distributions with rational poles.

The modular symbol

$$\Phi \in \mathrm{Symb}_{G^+}(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}^{(p)}, \widetilde{\mathcal{D}_p}/\delta_0) \tag{5.7}$$

is the unique modular symbol characterized by (abusing notation)

$$\mathcal{F}(\Phi(f'))(f_p) = \Psi(f' \otimes f_p). \tag{5.8}$$

In other words, for all $D \in \Delta_0$ and $f_p \in \mathcal{S}(V_p)$,

$$\int_{V_p} f_p(v) e^v d\Phi(D)(f') = \Psi(f' \otimes f_p). \tag{5.9}$$

This is a wonderful modular symbol for its G^+-invariance. However, we will need to restrict to much smaller subgroups in order to get arithmetically interesting modular symbols. So, we pick an auxiliary test function $f' \in \mathcal{S}(V^{(p)})$ and let $\Gamma_{f'}^+ \subset \mathrm{GL}_2^+(\mathbb{Q})$ denote the stabilizer of f'.

Definition 5.1.7 (**The Shintani Modular Symbol** attached to f')**.** $\Phi_{f'}$ is the modular symbol

$$\Phi_{f'} = \Phi \circ f' \in \mathrm{Symb}_{\Gamma_{f'}^+}(\mathcal{D}(V_p)^{(*)}/\delta_0). \tag{5.10}$$

When we compute the action of the Hecke operators on $\Phi_{f'}$, we will constantly use the G^+-invariance property of Φ. This translates to the following easy lemma:

Lemma 5.1.8. *For all $\gamma \in G^+$ and $f' \in \mathcal{S}(V^{(p)})$, $\Phi_{f'}|\gamma = \Phi_{f'|\gamma^{*-1}}$*

Proof. The right G^+ action on $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}^p, \widetilde{\mathcal{D}}_p/\delta_0)^{(*)}$ is described by $(\varphi|\gamma)(f) = \varphi(f|\gamma^*)|\gamma$

(viewing $\widetilde{\mathcal{D}}_p$ as a right $\mathrm{GL}_2^+(\mathbb{Q})$-module). Since $\Phi|\gamma = \Phi$ for each $\gamma \in G^+$, we have for all $D \in \Delta_0$,

$$\begin{aligned}
(\Phi_{f'}|\gamma)(D) &= \Phi_{f'}(\gamma D)|\gamma && (5.11)\\
&= (\Phi(\gamma D)(f'))|\gamma && (5.12)\\
&= \Phi(\gamma D)(f'|\gamma^{*-1}|\gamma^*)|\gamma && (5.13)\\
&= \Phi(D)(f'|\gamma^{*-1}) && (5.14)\\
&= \Phi_{f'|\gamma^{*-1}}(D). && (5.15)
\end{aligned}$$

Thus $\Phi_{f'}|\gamma = \Phi_{f'|\gamma^{*-1}}$, as claimed. □

The Vanishing Hypothesis

In Section 3.3, we introduced the vanishing hypothesis to describe which specialization of the Shintani cocycle produce measures. The vanishing hypothesis, stated in terms of lines in V can be reformulated in terms of the cusps $\mathbb{P}^1(\mathbb{Q}) = \mathbb{P}(V)$.

Definition 5.1.9. We say $f' \in \mathcal{S}^{(p)}$ satisfies the vanishing hypothesis for a cusp $\frac{a}{b} \in \mathbb{P}^1(\mathbb{Q})$, if it satisfies the vanishing hypothesis for the vector $\binom{a}{b} \in \mathbb{Q}^2$, which is equivalent to f' satisfying the vanishing hypothesis for all non-zero scalars of $\binom{a}{b}$. For a fixed f', we will say a cusp $r \in \mathbb{P}^1(\mathbb{Q})$ is *good* for f' if f' satisfies the vanishing hypothesis for s.

In this language, Theorems 3.2.5 and 3.3.3 can be summarized as

Theorem 5.1.10. *Let $r, s \in \mathbb{P}^1(\mathbb{Q})$ be cusps, represented by the vectors $v_r, v_s \in \mathbb{Q}^2$. Then there exists $\mu\{r,s\} \in \mathcal{D}(V_p)$ such that distribution with rational poles $\Phi_{f'}\{r,s\}$ is represented by*

$$\Phi_{f'}\{r,s\} = \mu\{r,s\} \otimes \frac{1}{D_{v_r} D_{v_s}}. \quad (5.16)$$

Moreover,

- $\Phi_{f'}\{r,s\} = \mu' \otimes \frac{1}{D_{v_r}}$ *for some $\mu' \in \mathcal{D}(V_p)$ if f' satisfies the vanishing hypothesis for* *s and*

- $\Phi_{f'}\{r,s\} = \mu' \in \mathcal{M}(V_p) \subset \mathcal{D}(V_p)$ *if* f' *satisfies the vanishing hypothesis for* r, s.

5.2 The specialization maps

The classical specialization map

Let μ be a distribution supported on $\mathbb{Z}_p^2$. To simplify notation, let us write $\mathcal{F}(\mu)$ for

$$\mathcal{F}(\mu) = \int_{\mathbb{Z}_p\times\mathbb{Z}_p} \exp(xX+yY)d\mu(x,y) = \sum_{k\geq} \frac{1}{k!}\int_{\mathbb{Z}_p\times\mathbb{Z}_p}(xX+yY)^k d\mu(x,y) \tag{5.17}$$

when the support of μ is clear.

For a fixed integer $k \geq 0$, we have the usual specialization map $\mathcal{D}(\mathbb{Z}_p^2) \longrightarrow \mathrm{Symm}^k(\mathbb{Q}_p)^2 \cong \mathbb{Q}[X,Y]_k$ by taking the k-th homogeneous part of the Fourier transform:

$$\mathcal{F}(\mu)_k = \frac{1}{k!}\sum_{n=0}^{k}\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p} x^n y^{k-n} d\mu(x,y)\right) X^n Y^{k-n}. \tag{5.18}$$

Remark 5.2.1. One can think of this weight k specialization as sending μ its restriction to the homogeneous degree k polynomials in $A[\mathbb{Z}_p^2, 1] \subset \mathcal{A}(\mathbb{Z}_p^2)$. Equivalently, the weight k specialization gives a linear functional on the 1-variable polynomials of degree $\leq k$.

A problem occurs when pass to the realm of distributions with rational poles. The Fourier transform of distribution with poles has a well defined homogeneous k-part, but in general this pieces of the Fourier transform will be a rational function rather than a polynomial. We can salvage this by by passing to negative weights, which are dual to the positive weights in a sense we will soon make precise.

Algebraic differential forms

Unfortunately, we must make a brief digression. Let $\Omega^1_{\widetilde{R}/\mathbb{Q}}$ denote the $\widetilde{R}$-module of Kahler differentials of $\widetilde{R}/\mathbb{Q}$ and $\Omega^2_{\widetilde{R}/\mathbb{Q}} := \bigwedge^2 \Omega^1_{\widetilde{R}/\mathbb{Q}}$. Observe that $\Omega^2_{\widetilde{R}/\mathbb{Q}}$ is a free $\widetilde{R}$-module generated by $dX \wedge dY$. When confusion is unlikely to arise, we will write $\Omega^i := \Omega^i_{\widetilde{R}/\mathbb{Q}}$ for $i = 1, 2$.

The rings R and $\widetilde{R}$ are equipped with the degree grading and this induces a grading on the modules Ω^1 and Ω^2 in the obvious way: $\deg(df) := \deg(f)$. For each integer k, we will denote by Ω^i_k the homogeneous differentials of degree k. The total derivative can easily be computed on homogeneous differentials of a certain form.

Lemma 5.2.2. *Let $F(X,Y) \in (S^{-1}\mathbb{Q}[X,Y])_k$ be a homogeneous rational polynomial of degree k. The total derivative of $F(X,Y)\frac{XdY-YdX}{XY} \in \Omega^1_k$ is equal to*

$$d\left(F(X,Y)\frac{XdY - YdX}{XY}\right) = -kF(X,Y)\frac{dX \wedge dY}{XY}. \tag{5.19}$$

Proof. First, if $F \in \mathbb{Q}[X,Y]_k$ is homogeneous of degree k, then

$$d\left(F\frac{XdY - YdX}{XY}\right) = F_X\frac{dY \wedge dX}{Y} - F_Y\frac{dX \wedge dY}{X} \tag{5.20}$$

$$= -(XF_X + YF_Y)\frac{dX \wedge dY}{XY}. \tag{5.21}$$

Since F is homogenous of degree k, it's easy to see that $XF_X + YF_Y = kF$. Now suppose $F = \frac{P}{Q}$, with P, Q homogeneous polynomials of degrees k_1, k_2, respectively. Then, by the quotient rule,

$$d\left(\frac{P}{Q}\frac{XdY - YdX}{XY}\right) = \frac{X(QP_X - PQ_X)}{Q^2}\frac{dY \wedge dX}{XY} - \frac{Y(QP_Y - PQ_Y)}{Q^2}\frac{dX \wedge dY}{XY}. \tag{5.22}$$

Collecting terms and rearranging, this becomes

$$\frac{(-XP_X - YP_X)}{Q}\frac{dX \wedge dY}{XY} + \frac{P(XQ_Y + YQ_Y)}{Q^2}\frac{dX \wedge dY}{XY} \tag{5.23}$$

$$= \frac{-k_1 P}{Q}\frac{dX \wedge dY}{XY} + \frac{k_2 F}{Q}\frac{dX \wedge dY}{XY} \tag{5.24}$$

$$= -kF\frac{dX \wedge dY}{XY}. \tag{5.25}$$

□

Remark 5.2.3. More generally, if L_1 and L_2 are linearly independent linear forms and F is homogenous of degree k,

$$d\left(F\left(\frac{dL_1}{L_1}-\frac{dL_1}{L_2}\right)\right)=kF\frac{dL_1\wedge dL_2}{L_1L_2}. \tag{5.26}$$

For each integer k, let $\Omega^1_{\log,k}\subset\Omega^1_k$ denote the subspace of differential forms

$$\{F(X,Y)\left(\frac{dL_1}{L_1}-\frac{dL_2}{L_2}\right):F(X,Y)\in\widetilde{\mathbb{R}}_k,\text{ and }L_1,L_2\text{ linearly independent.}\} \tag{5.27}$$

We have a map π from $\Omega_{\log,k}$ to differential forms on $\mathbb{P}^1$ (with logarithmic poles) via the change of variables $(X,Y)\mapsto(1,Z)$. For example,

$$\pi:F(X,Y)\frac{Xdy-YdX}{XY}\mapsto F(1,Z)\frac{dZ}{Z}\in\Omega^1_{\mathbb{Q}(Z)/\mathbb{Q}} \tag{5.28}$$

One can define the residue at $Z=Y/X=0$ of $\omega(X,Y)=F(X,Y)\frac{XdY-YdX}{XY}$ by writing $\omega(1,Z)$ as a Laurent polynomial $\sum a_nZ^ndZ$ and putting $\mathrm{Res}_{Z=0}(\omega)=a_{-1}$.

Starting with a differential of the form $\omega=F\otimes\frac{XdY-YdX}{XY}$, $F\in\widetilde{R}_{-k}$, we get a homomorphism $\mu_\omega:\mathrm{Symm}^k(\mathbb{Q}_p^2)\longrightarrow\mathbb{Q}_p$ by $\mu_\omega(P)=\mathrm{Res}(P\omega)$. Put another way,

Proposition 5.2.4. *The residue map induces a bilinear pairing $\widetilde{R}_{-k}\times\mathrm{Symm}^k(\mathbb{Q}_p)^2\longrightarrow\mathbb{Q}_p$ by*

$$\langle F,P\rangle\mapsto\mathrm{Res}\left(FP\otimes\frac{XdY-YdX}{XY}\right) \tag{5.29}$$

Our goal is to specialize to the dual $\mathrm{Symm}^k(\mathbb{Q}_p^2)^*\cong\mathrm{Symm}^k(\mathbb{Q}_p^2)(k)$ by producing homogenous Kahler differentials of degree $-k$. As we remarked earlier, $\mathrm{Symm}^k(\mathbb{Q}_p^2)$ may be though of as the linear functionals on homogeneous degree k polynomials in X,Y, or on (inhomogeneous) polynomials of degree $\leq k$ in $Z=Y/X$. We will in fact specialize to overconvergent distributions on $\mathbb{Z}_p$ by passing to the realm of differential forms on annuli in $\mathbb{P}^1(\mathbb{C}_p)$.

Differential forms on wide opens

In this section, we recall some material on differential forms on the p-adic upper half plane and wide opens in $\mathbb{P}^1(\mathbb{C}_p)$. This material is drawn from unpublished notes of Stevens; for a published reference we refer the reader to [19].

For any p-adic field K, let $\mathbb{P}^1(K)$ denote the projective line. Viewing K^2 as row vectors (x, y), the coordinate $z := \frac{y}{x}$ identifies $\mathbb{P}^1(K)$ with $K \cup \{\infty\}$, and $\mathrm{Aut}(\mathbb{P}^1(K))$ with $\mathrm{PGL}_2(K)$. Concretely, if $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, then

$$z|\gamma = [(x, y)\gamma] = \frac{d + cz}{b + az}. \tag{5.30}$$

The adjugate map gives us a left action by $\gamma \cdot z = [(x, y)\gamma] = \frac{-b+az}{d-cz}$.

For each $s \in \mathbb{P}^1(\mathbb{Q}_p)$, fix the following choice of uniformizer at s: $w_s(z) = z - s$ if $s \in \mathbb{Z}_p$, $w_s(z) = \frac{1}{z} - \frac{1}{s}$ otherwise. For each compact $S \subset \mathbb{P}^1(\mathbb{Q}_p)$ and $r \in \mathbb{R}_+$, we define the closed ball

$$B[S, r] := \{z \in \mathbb{P}^1(\mathbb{C}_p) : \exists s \in S \text{ s.t. } |w_s(z)| \leq r\}. \tag{5.31}$$

Write $\mathcal{W}(S, r)$ for the wide open

$$\mathcal{W}(S, r) := \mathbb{P}^1(\mathbb{C}_p) - B[S, r]. \tag{5.32}$$

The p-adic upper half plane, $\mathcal{H}_p(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$, is admissibly covered by the wide opens $\mathcal{W}(\mathbb{P}^1(\mathbb{Q}_p), r)$ and thus is a rigid analytic variety.

For a wide open $\mathcal{W}$, denote by $\Omega^1(\mathcal{W})$ the $\mathbb{Q}_p$-vector space of Kahler differentials over the ring of $\mathbb{Q}_p$-rigid analytic functions $A(\mathcal{W})$. This space is slightly too small for our purposes, so we enlarge our space by adding "logarithmic differentials"

Definition 5.2.5. Let $\mathcal{W} = \mathbb{P}^1(\mathbb{C}_p) - \bigcup_{i=1}^n B[s_i, r_i]$. The logarithmic differential forms $\omega \in \Omega_{\log}(\mathcal{W})$ are the forms

$$\omega = \sum_{i=1}^n \sum_{n \geq 1} a_n(i) w_{s_i}^n \frac{dw_{s_i}}{w_{s_i}} \tag{5.33}$$

where, for all $t_i > r_i$, $|a_n(i)| = o(t_i^n)$ as $n \to \infty$.

The choice of notation is justified by the following observation. If we had a logarithm function $\log(w_s)$ in $A(\mathcal{W})$, then $d\log(w_s) = \frac{dw_s}{w_s}$. However, the small radius of convergence for the logarithm series means that, in general, no antiderivative of $\frac{dw_s}{w_s}$ exists on $\mathcal{W}$.

Proposition 5.2.6. *Let ω be as above. If $\sum_{i=1}^n a_0(i) = 0$, then $\omega \in \Omega^1(\mathcal{W})$.*

Covering $\mathcal{H}_p$ by $\mathcal{W}(\mathbb{P}^1(\mathbb{Q}_p, r))$ and taking the projective limit, we get the locally analytic logarithmic differential forms

$$\Omega_{\log}(\mathcal{H}_p) := \varprojlim_{r>0} \Omega_{\log}(\mathcal{W}(\mathbb{P}^1(\mathbb{Q}_p), r)). \tag{5.34}$$

The weight-k action

The wide opens we will primarily work with are

$$\mathcal{W}_0 := \mathcal{W}(\mathbb{Z}_p, 1)\ \mathcal{W}_\infty := \mathcal{W}(\mathbf{x}_\infty, 1/p),\ \text{and}\ \mathcal{Z} := \mathcal{W}_0 \cap \mathcal{W}_\infty = \{z \in \mathbb{C}_p : 1 < |z| < p\}. \tag{5.35}$$

The key point is that $\Omega_{\log}(\mathcal{W}_0)$ is isomorphic, as a $\mathbb{Q}_p$-Frechèt space, to $\mathcal{D}^\dagger(\mathbb{Z}_p, 1)$, which is where we will find our p-adic L-functions. However, it will be more convenient to perform our constructions in $\Omega_{\log}(\mathcal{W}_\infty)$. These two spaces can be interchanged via the Atkin-Lehner involution W_{pN} (for modular symbols of level pN)

$$W_N = \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}. \tag{5.36}$$

Put

$$S_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0, (a,p) = 1,\ \text{and}\ p|c \right\} \tag{5.37}$$

$$S_0(p) = WS_0(p)W^{-1} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0, (d,p) = 1, \text{ and } p|c \right\} \tag{5.38}$$

Definition 5.2.7. For each integer k, the (standard) weight k-action of $S_0(p)$ on $\Omega^1_{\log}(\mathcal{H}_p)$ is defined, for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$(\omega|_k\gamma)(z) = (d - cz)^k \omega(\gamma \cdot z) = (d - cz)^k \omega\left(\frac{-b + az}{d - cz}\right). \tag{5.39}$$

When $k \leq 0$, this extends to a well-defined action on $\Omega^1_{\log}(\mathcal{W}_0)$, since, for all $\gamma \in S_0(p)$, $\gamma\mathcal{W}_0$ containes $\mathcal{W}_0$.

The (conjugated) weight k-action of $S_0(p)$ on $\Omega^1_{\log}(\mathcal{H}_p)$ is defined by

$$(\omega|^W_k\gamma)(z) = \omega|_k(W_p\gamma W_p^{-1}) = (a + bpz)^k \omega\left(\frac{c/p + dz}{a + bpz}\right). \tag{5.40}$$

When $k \geq 0$, this extends to a well-defined action on $\Omega_{\log}(\mathcal{W}_\infty)$. To indicate which action we are using, we put

$$\Omega_{\log,k}(\mathcal{H}_0) := \Omega_{\log}(\mathcal{H}_p) \text{ equipped the standard weight } k \text{ action.} \tag{5.41}$$

and

$$\Omega_{\log,k}(\mathcal{H}_\infty) := \Omega_{\log}(\mathcal{H}_p) \text{ with the conjugate weight } k \text{ action.} \tag{5.42}$$

Remark 5.2.8. The Atkin-Lehner involution W_{Np} preserves $\mathcal{Z}$, and for each integer k the map $\Omega_{\log}(\mathcal{Z}_0) \longrightarrow \Omega_{log}(\mathcal{Z}_\infty)$, $\omega \mapsto \omega|_k W$ intertwines the action of $S_0(p)$.

Lemma 5.2.9. *For each integer k, the map $\pi : \Omega^1_k \longrightarrow \Omega_{\log,k}(\mathcal{Z}) \subset \Omega_{\log,k}(\mathcal{H}_p)$ given by*

$$\pi(\omega(X,Y)) = \omega(1, z) \tag{5.43}$$

is $\Sigma_0(p)$-equivariant.

Proof. Let $\omega \in \Omega^1_k$:

$$\pi(\omega(X,Y)|\gamma) = \pi(\omega(dX - bY, -cX + aY)) \tag{5.44}$$

$$= \pi\left((-cX + aY)^k \omega\left(\frac{dX - bY}{-cX + aY}, 1\right)\right) \tag{5.45}$$

$$= (-c + az)^k \omega\left(\frac{d - bz}{-c + az}, 1\right) \tag{5.46}$$

$$= (\pi\omega)|_k \gamma \tag{5.47}$$

□

The Residue map

Suppose $S \subset \mathbb{P}^1(\mathbb{Q}_p)$ is a compact open, and $r \in \mathbb{R}$ is a positive real number such that $B[S,r]$ is the finite disjoint union of balls $B[S,r] = \bigcup_{i=1}^n B[s_i, r]$. Suppose $G \subset \mathrm{GL}_2(\mathbb{Q})$ stabilizes each of the balls $B[s_i, r]$, under the left action ($\gamma \cdot z \mapsto \frac{-b+az}{d-cz}$).

Letting G acton the $\Omega_{\log}(\mathcal{W})$ via the weight 0 naive action,

Lemma 5.2.10. *There is a canonical* G*-invariant homomorphism, the residue map,* Res : $\Omega_{\log}(\mathcal{W}) \longrightarrow \mathbb{Q}_p$, *defined by*

$$\mathrm{Res} : \sum_{i=1}^{n} \sum_{k \geq 0} a_k(s_i) w_{s_i}^k \frac{dw_{s_i}}{w_{s_i}} \mapsto \sum_{i=1}^{n} a_0(s_i). \tag{5.48}$$

Proof. See the Remark on pg. 223 of [19]. □

In this language, Proposition 5.2.6 states that the logarithmic differential forms with residue 0 are exactly the Kahler differentials on $A(\mathcal{W})$.

Let us now fix a differential form $\omega \in \Omega_{\log}(\mathcal{W}_0)$. If f is an overconvergent function $f \in \mathcal{A}^\dagger(\mathbb{Z}_p, 1)$, then f is the restriction of a rigid analytic function on an open ball $B[\mathbb{Z}_p, r]$ of radius $r > 1$. The ball $B[\mathbb{Z}_p, r]$ intersects the wide open $\mathcal{W}_0$, and the intersection contains a nonempty open oriented annulus $\mathcal{Z}$. Taking the residue in the obvious way, we see ω

gives us a continuous linear functional $\mu_\omega : \mathcal{A}^\dagger(\mathbb{Z}_p, 1) \longrightarrow \mathbb{Q}_p$ by

$$\mu_\omega(f) = \mathrm{Res}(f\omega). \tag{5.49}$$

Thus,

Proposition 5.2.11. *The residue map induces a $\Sigma_0(p)$-equivariant isomorphism of $\mathbb{Q}_p$ topological vector spaces $\Omega_{\log,-k}(\mathcal{W}_0)(k) \longrightarrow \mathcal{D}_k^\dagger(\mathbb{Z}_p, 1)$.*

Here there (k) denotes the action of G twisted by $\det^k$.

Proof. It follows from a result of Vishik that there is an exact sequence

$$0 \longrightarrow \Omega(\mathcal{W}_0) \xrightarrow{\mathrm{Res}^*} \mathcal{D}^\dagger(\mathbb{Z}_p, 1) \xrightarrow{\rho} \mathbb{Q}_p \longrightarrow 0 \tag{5.50}$$

where $\rho(\mu) := \int_{\mathbb{Z}_p} 1 d\mu$. The dirac delta δ_0 gives us a splitting of the exact sequence via the section $\lambda \mapsto \lambda\delta_0$. Since $\mathrm{Res}(f\frac{dz}{z}) = f(0)$, we can identify δ_0 and $\frac{dz}{z}$, extending the isomorphism to $\Omega_{\log}(\mathcal{W}) \cong \mathcal{D}^\dagger(\mathbb{Z}_p, 1)$.

The claim of $\Sigma_0(p)$-equivariance is easy, but since we have so many group actions to keep track of, we carefully record this simple calculation.

Let us fix $\omega \in \Omega^1_{\log}(\mathcal{W}_0)$ and $f \in \mathcal{A}^\dagger(\mathbb{Z}_p, 1)$. An automorphism $\gamma \in \Sigma_0(p)$ acts by

$$\omega|_{-k}\gamma(z) = (d - cz)^{-k}\omega(z|\gamma^*) \text{ and } \gamma \cdot_k f(z) = (a + cz)^k f(z|\gamma) \tag{5.51}$$

Applying γ^* to f, we get $\gamma^* \cdot_k f(z) = (d - cz)^k f(z|\gamma^*)$. Multiplying these together, the automorphy factor disappears

$$(\omega|_{-k}\gamma)(\gamma^* \cdot_k f) = (\omega f)(z|\gamma^*). \tag{5.52}$$

Since $\Sigma_0(p)$ preserves the disks $B[\mathbb{Z}_p, 1]$ and $B[\mathbf{x}_\infty, 1/p]$, Lemma 5.2.10 shows $\mathrm{Res}(\omega f)(z|\gamma^*) = \mathrm{Res}(\omega f)$. Therefore,

$$\mu_{\omega|_{-k}\gamma}(f) = \mu_\omega(\gamma^{*-1} \cdot_k f) \tag{5.53}$$

Since $\gamma^{*-1} = \det^{-1}(\gamma)\gamma$ and $(\lambda I) \cdot_k f = \lambda^k f$, we get

$$\mu_\omega(\gamma^{*-1} \cdot_k f) = \mu_\omega(\det^{-k}(\gamma)\gamma \cdot_k f) = \det^{-k}(\gamma)(\mu_\omega|\gamma)(f). \tag{5.54}$$

□

The weight k specialization map to differential forms

Definition 5.2.12. Let $\mathcal{F}_k^* : \widetilde{\mathcal{D}}(V_p) \longrightarrow (\Omega^2)_k$ be the map $\mathcal{F}_k^*(\mu) = k!(\mathcal{F}(\mu)([\mathbb{Z}_p \times \mathbb{Z}_p^\times]))_{k-2} dX \wedge dY$. We will write

$$\mathcal{F}_k^*(\mu) = k! \left(\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \exp(xX + yY)\mu(x,y) \right)_{k-2} dX \wedge dY \in \Omega_k^2 \tag{5.55}$$

understanding that the term in parenthesis is again the homogeneous degree $k-2$ part of $\mathcal{F}(\mu)([\mathbb{Z}_p \times \mathbb{Z}_p^\times])$. The apparent shift from $(k-2)!$ to $k!$ allows us to talk about the homogeneous degree -2 part, which, after multiplying by $dX \wedge dY$, gives us a degree 0 differential form. Note that the Shintani modular symbol will typically have non-zero terms in degree -2.

Lemma 5.2.13. *$\mathcal{F}_k^*$ is a $\mathrm{Stab}_{S_0(p)}([\mathbb{Z}_p \times \mathbb{Z}_p^\times])$-equivariant map*

$$\mathcal{F}_k^* : \widetilde{\mathcal{D}}(V_p)(1) \longrightarrow \Omega_k^2. \tag{5.56}$$

That is, if $\gamma \in \Sigma_0(p)$ fixes $[\mathbb{Z}_p \times \mathbb{Z}_p^\times]$, then $\mathcal{F}_k^*(\mu)|\gamma = det(\gamma)\mathcal{F}_k^*(\mu|\gamma)$. Furthermore, $\mathcal{F}_k^*(\delta_0) = 0$, so $\mathcal{F}_k^*$ factors through the quotient

$$\mathcal{F}_k^* : \widetilde{\mathcal{D}}(V_p)(1)/\delta_0 \longrightarrow \Omega_k^2. \tag{5.57}$$

Proof. First, let us observe that, for $\gamma \in G$, $dX \wedge dY|\gamma = \det(\gamma) dX \wedge dY$. Since the Fourier transform is equivariant with respect to G, the result follows. The claim that $\mathcal{F}_k^*(\delta_0) = 0$ follows from the fact that $0 \notin \mathbb{Z}_p \times \mathbb{Z}_p^\times$. □

Corollary 5.2.14. *Applying $\mathcal{F}_k^*$ to $\Phi_{f'}$, we get a $\Gamma_0(f') := \Gamma_{f'} \cap \Gamma_0(p)$-modular symbol $\mathcal{F}_k^*\Phi_{f'} \in \mathrm{Symb}_{\Gamma_0(f')}(\Omega_k^2)$.*

Now let us compute $\mathcal{F}_k^*\Phi_{f'}\{\infty, 0\}$, using Theorem 5.1.10. We know $\Phi_{f'}\{\infty, 0\}$ is represented by $\mu \otimes \frac{1}{D_{e_1}D_{e_2}}$, with μ now an honest distribution, so

$$\mathcal{F}(\Phi_{f'}\{r,s\})([\mathbb{Z}_p \times \mathbb{Z}_p^\times]) = \mathcal{F}(\mu)([\mathbb{Z}_p \times \mathbb{Z}_p^\times])\frac{1}{XY}. \tag{5.58}$$

Multiplying by $dX \wedge dY$, this becomes

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \exp(xX + yY) d\mu(x,y) \frac{dX \wedge dY}{XY} \tag{5.59}$$

$$= \sum_{k \geq 0} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} \left(\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^n y^{k-n} d\mu(x,y) \right) X^n Y^{k-n} \frac{dX \wedge dY}{XY}. \tag{5.60}$$

Therefore, $\mathcal{F}_k^*\Phi_{f'}\{\infty, 0\}$ is equal to

$$\sum_{n=0}^{k} \left(\binom{k}{n} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^n y^{k-n} d\mu(xy) \right) X^n Y^{k-n} \frac{dX \wedge dY}{XY} \tag{5.61}$$

Observe that, for $k \neq 0$, it's possible to *integrate* $\mathcal{F}_k^*\Phi_{f'}\{r,s\}$ to a 1-form by dividing by k.

Proposition 5.2.15. *For each integer $k > 0$ and unimodular pair of cusps $\{r,s\}$ (in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of $\{\infty, 0\}$),*

$$\mathcal{F}_k^*\Phi_{f'}\{r,s\} = d\left(\frac{1}{k} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} (xX + yY)^k d\mu\{r,s\}(x,y) \left(\frac{dL_r}{L_r} - \frac{dL_s}{L_s} \right) \right), \tag{5.62}$$

where L_r and L_s are the linear forms corresponding to the differential operators D_{v_r} and D_{v_s}.

Proof. Since $\mu\{r,s\}$ is a distribution, $\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} (xX + yY)^k d\mu$ is a homogeneous polynomial of degree k. Thus

$$d\left(\frac{1}{k}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}(xX+yY)^k d\mu\left(\frac{dL_r}{L_r}-\frac{dL_s}{L_s}\right)\right)=\frac{k}{k}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}(xX+yY)^k d\mu\frac{dL_r\wedge dL_s}{L_rL_s}, \tag{5.63}$$

$$=\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}(xX+yY)^k d\mu\frac{dX\wedge dY}{L_rL_s} \tag{5.64}$$

where the last step is justified by the unimodularity of $\{r,s\}$ (this is only a convenient hypothesis. We can remove it by keeping track of the determinant). The right hand side is exactly $\mathcal{F}_k^*(\Phi_{f'}\{r,s\})$. □

Analytic families of modular symbols

The previous section begs the question of how to recover weight 0 1-forms. Observe that terms $\binom{k}{n}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}x^n y^{k-n}d\mu(x,y)$ vary *analytically* in $k\in\mathcal{X}(\mathbb{Q}_p)$, so expanding $\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}(xX+yY)^k$ we can, if f' satisfies the vanishing hypothesis for enough cusps, talk about analytic families of modular symbols. This will allow us to analytically continue the 1-forms to non-positive weights, where we will find the modular symbols we are looking for.

Definition 5.2.16. For each integer $k>0$ and pair of distinct cusps $\{r,s\}$, there is a unique 1-form $\omega_k\{r,s\}\in(\Omega^1_{\log})_k$ such that $d\omega_k\{r,s\}=\mathcal{F}_k^*\Phi_{f'}\{r,s\}$. We define

$$\Phi^k_{f'}\{r,s\}:=\omega_k\{r,s\}(1,z)\in\Omega_{\log}(\mathcal{Z})\subset\Omega_{\log}(\mathcal{H}_p). \tag{5.65}$$

We write $\Phi^k_{f'}$ for the $\Omega_{\log}(\mathcal{Z})$-valued modular symbol

$$\Phi^k_{f'}\in\mathrm{Symb}_{\Gamma_0(f')}(\Omega_{\log}(\mathcal{H}_p)), \tag{5.66}$$

remarking that $\Phi^k_{f'}$ actually takes values in $\Omega_{\log}(\mathcal{Z}))\subset\Omega_{\log}(\mathcal{H}_p)$. We have chosen to use the larger module $\Omega_{\log}(\mathcal{H}_p)$ for its natural action by $S_0(p)$.

Suppose $\Gamma_0(f') = \Gamma_0(pN)$, and recall the Atkin-Lehner involution $W := W_{Np}$. Since W normalizes $\Gamma_0(pN)$ we have a modular symbol

$$\widehat{\Phi}^k_{f'}\{r,s\} := (\Phi^k_{f'}\{r,s\})|_k W \in \mathrm{Symb}_{\Gamma_0(pN)}(\Omega^1_{\log}(\mathcal{H}_\infty)). \tag{5.67}$$

Remark 5.2.17. We could have also defined $\widehat{\Phi}_{f'}$ as $\Phi_{f'}|W$, but we want to emphasize the different $S_0(p)$ actions on $\Omega_{\log}(\mathcal{H}_p)$.

The following theorem is a generalization of results of Campell [6] and Kostadinov [16]:

Theorem 5.2.18. *Suppose f' satisfies the vanishing hypothesis 0 and for all other cusps not in $\{\Gamma_0(p)\cdot 0\} \cup \{\Gamma_0(pN)\cdot\infty\}$. Then for all $r,s \in \mathbb{P}^1(\mathbb{Q})$, there exist analytic functions $\alpha_i \in A(\mathcal{X}(\mathbb{Q}_p))$, $i \geq 0$, such that*

a) For all integers $k \geq 1$, $\widehat{\Phi}^k_{f'}\{r,s\} = \sum_{n\geq 0} \alpha_n(k) p^n z^n \frac{dz}{z}$

b) For all integers, $|\alpha_n(k)|_p$ is bounded as $n \to \infty$

In other words, $\widehat{\Phi}_{f'}$ varies in a family of differential forms on the annulus $\mathcal{Z}$, and and this family analytically continues to a family of differential forms on $\mathcal{W}_\infty$.

Proof. As a $\mathbb{Z}[\Gamma_0(pN)]$-module, Δ_0 is generated by $\{\infty, r\}$, $r \in \mathbb{P}^1(\mathbb{Q})$, so it suffices to consider divisors of this form. We first show that if r is a good cusp for f', then $\widehat{\Phi}^k_{f'}$ varies in a family on $\Omega_{\log}(\mathcal{W}_\infty)$. More generally, if r is not a good cusp, we will reduce to showing $\Phi_{f'}\{0,r\}$ varies in an an analytic family on $\mathcal{W}_\infty$ by the fact $\widehat{\Phi}^k_{f'}\{\infty,r\} = \widehat{\Phi}^k_{f'}\{\infty,0\} + \widehat{\Phi}^k_{f'}\{0,r\}$.

Let us first us consider the case that f' satisfies the vanishing hypothesis for the cusp $s \in \mathbb{P}^1(\mathbb{Q})$. Theorem 5.1.10 implies $\Phi_{f'}\{\infty,s\}$ is represented by $\mu \otimes \frac{1}{D_x}$ for some $\mu \in \mathcal{D}(\mathbb{Q}_p^2)$, so

$$(\mathcal{F}^*_k(\Phi_{f'}\{\infty,s\}) = k!\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \frac{(xX+yY)^{k-1}}{(k-1)!} d\mu(x,y) X^n Y^{k-n} \frac{dX\wedge dY}{XY}\right) \tag{5.68}$$

Integrating and projecting to $\Omega_{\log}(\mathcal{Z})$, we have

$$\widehat{\Phi}^k_{f'}\{\infty, s\} = \Phi^k_{f'}\{\infty, s\}|_k W \tag{5.69}$$

$$= \frac{k!}{k}\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \frac{(x+yz)^{k-1}}{(k-1)!} d\mu(x,y)\right) dz\Bigg|_k W \tag{5.70}$$

$$= \left(\sum_{n\geq 0}\binom{k-1}{n}\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^{k-n-1} d\mu(x,y)\right) z^{k-n}\frac{dz}{z}\right)\Bigg|_k W \tag{5.71}$$

$$= (-Npz)^k \sum_{n\geq 0}(k-1)\binom{k-2}{n}\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^{k-n-1} d\mu(x,y)\right)(-Npz)^{n-k}\frac{dz}{z} \tag{5.72}$$

$$= \sum_{n\geq 0}\binom{k-2}{n}\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^{k-n-1} d\mu(x,y)\right)(-Np)^n z^n \frac{dz}{z}. \tag{5.73}$$

We read off the coefficient $\alpha_n(k) = \binom{k-2}{n}\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^{k-n-1} d\mu(x,y)\right)(-N)^n$ for $n \geq 0$. The binomial coefficients $\binom{k-2}{n}$ extend to analytic functions on $\mathcal{X}(\mathbb{Q}_p)$ and are integral for $k \in \mathbb{Z}$. The moments $\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^{k-n} d\mu(x,y)$ extend to analytic functions on $\mathcal{X}(\mathbb{C}_p)$. To show that the coefficients are bounded, we must verify that $\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^{k-n} d\mu(x,y)$ is bounded as $n \to \infty$. (as, and since μ is a distributions, they are p-adically bounded for fixed k. Thus $\widehat{\Phi}_{f'}\{r, s\}$ is an analytic family of differential forms which extend to $\Omega_{\log}(\mathcal{W}_\infty)$.

Now we are left with the cases $\{0, s\}$ with $s \in \Gamma_0(pN)\cdot\infty$ or $s \in \Gamma_0(p)\cdot 0$. If $s \in \Gamma_0(pN)$, then $\{0, s\}$ is $\Gamma_0(pN)$ equivalent to $\{t, \infty\}$, where $t \in \Gamma_0(pN)\cdot 0$ is a good cusp for f' (since the vanishing hypothesis is invariant under $\Gamma_(pN))$, which we have already addressed. Since $\widehat{\Phi}^k_{f'}\{t, \infty\} = -\widehat{\Phi}^k_{f'}\{\infty, t\}$, we are reduced to the first case. Therefore, it suffices to consider the case that $s \in \Gamma_0(p)\cdot 0$.

If $s = \frac{a}{b} \in \Gamma_0(p)\cdot 0$, with $(a, b) = 1$, then $(b, p) = 1$. Theorem 5.1.10 says $\Phi_{f'}\{0, s\}$ is represented by $\mu \otimes \frac{1}{aD_x + bD_y}$, so

$$(\mathcal{F}^*_k(\Phi_{f'}\{0, s\}) = k!\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \frac{(xX+yY)^{k-1}}{(k-1)!} d\mu(x,y) X^{n+1}Y^{k-n-1}\frac{dX\wedge dY}{X(aX+bY)}\right) \tag{5.74}$$

(5.75)

This is the derivative of the 1-form

$$\frac{k!}{k}\left(\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\frac{(xX+yY)^{k-1}}{(k-1)!}d\mu(x,y)\frac{1}{b}\left(\frac{dX}{X}-\frac{d(aX+bY)}{(aX+bY)}\right)\right), \tag{5.76}$$

Projecting to $\Omega_{\log}(\mathcal{Z})$, observe $\frac{1}{b}(\frac{dX}{X}-\frac{d(aX+bY)}{aX+bY})\mapsto\frac{dz}{a+bz}=\frac{z}{a+bz}\frac{dz}{z}$, so

$$\Phi^k_{f'}\{0,s\}=\frac{1}{a+bz}\sum_{n\geq 0}\binom{k-1}{n}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}x^ny^{k-1-n}d\mu(x,y)z^{k-n}\frac{dz}{z}. \tag{5.77}$$

Applying W,

$$\widehat{\Phi}^k_{f'}\{0,s\}=\frac{-Npz}{b-aNpz}\sum_{n\geq 0}\binom{k-1}{n}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}x^ny^{k-1-n}d\mu(x,y)(-pNz)^n\frac{dz}{z} \tag{5.78}$$

$$=b\left(\sum_{r\geq 1}\left(\frac{aN}{b}\right)^n p^nz^n\right)\sum_{n\geq 0}\binom{k-1}{n}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}x^ny^{k-1-n}d\mu(x,y)(-pNz)^n\frac{dz}{z}$$

(5.79)

Since $(b,p)=1$ we see that the product of the series converges on $\mathcal{W}_\infty$, and the coefficients $\alpha_n(k)$ are finite sums of the analytic functions $\binom{k-1}{n}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}x^ny^{k-n-1}$ and are thus analytic.

□

Remark 5.2.19. As long as f' satisfies the vanishing hypothesis for a set of cusps $\Gamma_0(f')\cdot r$, the first part of the theorem remains true: i.e. we have an analytic family of modular symbols on $\Omega_{\log}(\mathcal{H}_p)$ (and in fact on $\Omega_{\log}(\mathcal{Z})$). However, without the stronger hypothesis stated in the theorem, we cannot guarantee that the modular symbols will analytically continue to the wide open $\mathcal{W}_\infty$.

The theorem states that, if f' satisfies the vanishing hypothesis for enough cusps, we have an analytic family of modular symbols $\widehat{\Phi}^k_{f'}\in\mathrm{Symb}_{\Gamma_0(f')}(\Omega_{\log,k}(\mathcal{W}_\infty))$. Since the positive integers are dense in $\mathcal{X}(\mathbb{Q}_p)$, we have a modular symbol

$$\widehat{\Phi}^{\kappa}_{f'} \in \mathrm{Symb}_{\Gamma_0(f')}(\Omega_{\log,\kappa}(\mathcal{W}_\infty)) \tag{5.80}$$

for all $\kappa \in \mathcal{X}(\mathbb{Q}_p)$. In particular, we can extend $\widehat{\Phi}^k_{f'} \in \mathrm{Symb}_{\Gamma_0 f'}(\Omega^1_{\log}(\mathcal{W}_\infty))$ to non-positive weights, and by taking the Atkin-Lehner involution, we recover, for arbitrary integers

$$\Phi^k_{f'} := \widehat{\Phi}^k_{f'}|_k W \in \mathrm{Symb}_{\Gamma_0(f')}(\Omega_{\log,k}(\mathcal{W}_0)). \tag{5.81}$$

5.3 Hecke operators

The Hecke module $\mathcal{S}(V)$

The groups of test functions $\mathcal{S}(V)$, $\mathcal{S}(V^{(p)})$ and $\mathcal{S}(V_p)$ can be equipped with an action of the Hecke algebra which specializes to the Hecke action on the Shintani modular symbols $\Phi_{f'}$ and $\Phi^k_{f'}$.

Definition 5.3.1. Let Γ be a congruence subgroup of $\mathrm{GL}_2(\mathbb{Q})$, and let $\mathcal{S}(V^*)^\Gamma$ denote the subgroup of test functions invariant with respect to Γ. Then $\mathcal{S}(V^*)^\Gamma$ has an action of the Hecke module $\mathcal{H}_\Gamma$ via the adjugate-inverse action: If T is represented by the coset $\bigcup_{i=1}^r \Gamma\beta_i$, then we write $f|T^{*-1} := \sum_{i=1}^r f|\beta_i^{*-1}$.

For each $\lambda \in \mathbb{Q}^\times$, we will write $\langle\lambda\rangle := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Observe that $\langle\lambda\rangle^* = \langle\lambda\rangle$ and $\langle\lambda\rangle^{-1} = \langle\lambda^{-1}\rangle$.

Lemma 5.3.2. *For all primes q, $[\mathbb{Z}_q^2]|T_q^{*-1} = q[\mathbb{Z}_q^2]|\langle q\rangle^{*-1} + [\mathbb{Z}_q^2]$.*

Proof.

$$[\mathbb{Z}_q^2]|T_q^{*-1} = \sum_{a=0}^{q-1} [\mathbb{Z}_q^2] \left| \begin{pmatrix} 1 & a \\ 0 & q \end{pmatrix}^{*-1} \right. + [\mathbb{Z}_q^2] \left| \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{*-1} \right. \tag{5.82}$$

$$= \sum_{a=0}^{q-1} \left[\begin{pmatrix} q & -a \\ 0 & 1 \end{pmatrix} \mathbb{Z}_q^2 \right] + \left[\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \mathbb{Z}_q^2 \right] \tag{5.83}$$

$$= \sum_{a=0}^{q-1} \sum_{i=0}^{q-1} \left[\begin{pmatrix} q & -a \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 0 \\ i \end{pmatrix} + \mathbb{Z}_q \times q\mathbb{Z}_q \right) \right] + [\mathbb{Z}_q \times q\mathbb{Z}_q] \tag{5.84}$$

$$= \sum_{a=0}^{q-1} \sum_{i=0}^{q-1} \left[\begin{pmatrix} -ai \\ i \end{pmatrix} + q\mathbb{Z}_q \times q\mathbb{Z}_q \right] + [\mathbb{Z}_q \times q\mathbb{Z}_q] \tag{5.85}$$

$$= q[q\mathbb{Z}_q^2] + [\mathbb{Z}_q] \tag{5.86}$$

$$= q[\mathbb{Z}_q^2]|\langle q \rangle^{*-1} + [\mathbb{Z}_q^2] \tag{5.87}$$

□

A similar calculation shows $[\mathbb{Z}_p \times p\mathbb{Z}_p]|U_p^{*-1} = p[(p\mathbb{Z}_p)^2]$ and $[\mathbb{Z}_p \times \mathbb{Z}_p]|U_p^{*-1} = p[(p\mathbb{Z}_p)^2] + [\mathbb{Z}_p \times \mathbb{Z}_p^\times]$. Together, these imply

Lemma 5.3.3. $[\mathbb{Z}_p \times \mathbb{Z}_p^\times]|U_p^{*-1} = [\mathbb{Z}_p \times \mathbb{Z}_p^\times]$.

Hecke eigenfamilies

Proposition 5.3.4. *For all integers k, $\Phi_{f'}^k|U_p = p\Phi_{f'}^k$.*

Proof. Let $\beta_0, \ldots, \beta_{p-1}$ be the usual representatives for U_p. From the definition of $\Phi_{f'}^k$,

$$d\Phi_{f'}^k|U_p = \mathcal{F}_k^*(\Phi_{f'}))|U_p. \tag{5.88}$$

Now $\mathcal{F}_k^*(\Phi_{f'})$ is $k!$ times the degree $k-2$ part of $\mathcal{F}(\Phi_{f'})([\mathbb{Z}_p \times \mathbb{Z}_p^\times])$, (which is equal to $\Psi(f' \times [\mathbb{Z}_p \times \mathbb{Z}_p^\times])$), multiplied by $dX \wedge dY$. Writing $\Psi(f)$ for the Γ_f modular symbol $\Psi(f)(D) := \Psi(D)(f)$, we compute

$$\Psi(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times])|U_p)(D) = \sum_{a=0}^{p-1} \Psi(\beta_a \cdot D)(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times]))|\beta_a \tag{5.89}$$

Since $\Psi \in \mathrm{Symb}_{\mathrm{GL}_2^+(\mathbb{Q})}(\mathcal{D}_{naive}(\mathbb{Q}^2, S^{-1}\mathbb{Q}[[X,Y]]))$, $\Psi|\gamma(D) = \Psi(D)$ for all $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$, i.e.

$$\Psi(\gamma \cdot D)(f|\gamma^*)|\gamma = \Psi(D)(f). \tag{5.90}$$

We conclude that

$$\Psi(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times])|U_p)(D) = \sum_{a=0}^{p-1} \Psi(D)(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times]|\beta_a^{*-1}) \tag{5.91}$$

$$= \Psi(D)\left(f' \otimes \left(\sum_{a=0}^{p-1} [\mathbb{Z}_p \times \mathbb{Z}_p^\times]|\beta_a^{*-1}\right)\right) \tag{5.92}$$

$$= \Psi(D)(f' \otimes ([\mathbb{Z}_p \times \mathbb{Z}_p^\times]|U_p^{*-1})) \tag{5.93}$$

$$= \Psi(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times])(D) \text{ by Lemma 5.3.3.} \tag{5.94}$$

Therefore,

$$\mathcal{F}_k^*(\Phi_{f'})|U_p(D) = k!(\Psi(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times])(D))_{k-2} dX \wedge dY)|U_p \tag{5.95}$$

$$= pk!(\Psi(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times](D)|U_p))_{k-2} dX \wedge dY \tag{5.96}$$

$$= pk!(\Psi(f' \otimes [\mathbb{Z}_p \times \mathbb{Z}_p^\times](D))_{k-2} dX \wedge dY \tag{5.97}$$

$$= p\mathcal{F}_k^*(\Phi_{f'})(D). \tag{5.98}$$

Now, since d is G-equivariant, we conclude $\Phi_{f'}^k|U_p = p\Phi_{f'}^k$. □

Proposition 5.3.5. *Let q be relatively prime to p. If $f'|T_q^{*-1} = \sum_{i=1}^n a_i f'|\langle\lambda\rangle^{*-1}$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}^\times$ and $a_1, \ldots, a_n \in \mathbb{Q}$, then*

$$\Phi_{f'}^k|T_q^{*-1} = (\sum_{i=1}^n q a_i \lambda_i^{k-2})\Phi_{f'}^k \tag{5.99}$$

Proof. Since the Hecke operators away from p can be written as a sum of matrices stabilizing $\mathbb{Z}_p \times \mathbb{Z}_p^\times$, Lemma 5.2.13 tells us

$$\Phi^k_{f'}|T_q = q\Phi^k_{f'|T_q^{*-1}}. \tag{5.100}$$

The proposition then follows immediately from the observation that if $\mu \in \widetilde{\mathcal{D}}(V_p)$, and $\lambda \in \mathbb{Q}^\times$ is a p-adic unit, $\mathcal{F}^*_k(\mu|\langle\lambda\rangle) = \lambda^{k-2}\mathcal{F}^*_k(\mu)$. □

5.4 Overconvergent modular symbols

Overconvergent distributions

The $\Sigma_0(p)$-isomorphism $\Omega_{\log,-k}(\mathcal{W}_0)(k) \xrightarrow{\mathrm{Res}^*} \mathcal{D}^\dagger_k(\mathbb{Z}_p, 1)$ induces a modular symbol isomorphism

$$\mathrm{Symb}_{\Gamma_0(f')}(\Omega_{\log,-k}(\mathcal{W}_0))(k) \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0(f')}(\Omega_{\log,-k}(\mathcal{W}_0)(k)) \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}^\dagger_k(\mathbb{Z}_p, 1))$$

The finite slope eigensymbols for U_p in $\mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}^\dagger(\mathbb{Z}_p, 1))$ in fact overconverge to modular symbols valued in $\mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}(\mathbb{Z}_p))$:

Lemma 5.4.1 (Lemma 5.3 of [17])**.** *The natural map*

$$\mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}(\mathbb{Z}_p))^{<h} \longrightarrow \mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}^\dagger(\mathbb{Z}_p, 1))^{<h}$$

is an isomorphism.

Remark 5.4.2. This is the modular symbol analogue of the fact "overconvergent eigenforms overconverge as far as possible."

Let

$$\nu^{-k}_{f'} := \mu_{\Phi^{-k}_{f'}} \in \mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}^\dagger_k(\mathbb{Z}_p, 1)), \tag{5.101}$$

keeping in mind that we have twisted the action of $\Sigma_0(p)$ by $\det^k$.

Since $\widehat{\Phi}_{f'}^{-k}|U_p = p\widehat{\Phi}_{f'}^{-k}$ in $\mathrm{Symb}_{\Gamma_0(f')}(\Omega_{\log,-k}(\mathcal{W}_\infty))$, $\Phi_{f'}^{-k}|U_p = p\Phi_{f'}{}^{-k}$, and after twisting by $\det{}^k$, $\nu_{f'}^{-k}|U_p = p^{k+1}\nu_{f'}^{-k}$. Thus $\nu_{f'}^{-k} \in \mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}^\dagger(\mathbb{Z}_p, 1))$ is (critical) finite slope eigensymbol for U_p. By the Lemma of Pollack-Stevens, we may consider $\nu_{f'}^{-k}$ as a modular symbol valued in $\mathcal{D}_k(\mathbb{Z}_p)$. Moreover, Proposition 5.3.5 says that if f' is an eigenvector for T_q with eigenvalue $\sum a_i[\lambda_i I]$, then $\nu_{f'}^{-k}$ is an eigenvector for T_q with eigenvalue $\sum a_i\lambda_i^{-k-2}q^{k+1}$. Therefore, we have

Theorem 5.4.3. *If f' satisfies the hypothesis of Theorem 5.2.18 and is a hecke eigenvector away from p, then for each $k \geq 0$ $\nu_{f'}^{-k} \in \mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}_k(\mathbb{Z}_p))$ is a critical slope eigensymbol with the above eigenvalues.*

p-adic L-functions

In order to explicitly compute p-adic L-functions, we will need a few easy lemmas:

Lemma 5.4.4. *Suppose $\varphi \in \mathrm{Symb}_{\Gamma_0(f')}(\mathcal{D}(\mathbb{Z}_p))$ is an eigenvector for U_p with eigenvalue α. Then*

$$\varphi\{\infty, 0\}|_{\mathbb{Z}_p^\times}(z^n) = \left(1 - \frac{p^n}{\alpha}\right)\varphi\{\infty, 0\}(z^n). \tag{5.102}$$

Proof. This is a standard computation with U_p and can be found, for example, in the proof of Propostion 6.3 in [18]. □

Before stating the next lemma, let us observe that some test functions on $\mathbb{Q}^2$ factorize (with respect to our chosen basis) as the product of test functions on $\mathbb{Q}$. For example, $[\mathbb{Z}\times\ell\mathbb{Z}]$ is the product of the test functions $[\mathbb{Z}]$ and $[\ell\mathbb{Z}]$: $[\mathbb{Z}\times\ell\mathbb{Z}](xe_1+ye_2) = [\mathbb{Z}](x)[\ell\mathbb{Z}](y)$. If $f_1, f_2 \in \mathcal{S}(\mathbb{Q})$ are test functions, we will write $f_1 \otimes f_2$ for the function

$$f_1 \otimes f_2(v) := f_1(e_1^*(v))f_2(e_2^*(v)). \tag{5.103}$$

Since f_1, f_2 are supported on latices in $\mathbb{Q}$, the product $f_1 \otimes f_2$ is supported on a lattice in $\mathbb{Q}^2$. Moreover, if f_1, f_2 are periodic with respect to the lattice $m\mathbb{Z}$, $n\mathbb{Z}$, then $f_1 \otimes f_2$ is periodic with respect to $m\mathbb{Z} \times n\mathbb{Z}$. Moreover, the distribution $\Psi\{\infty, 0\}$ factors as

$$\Psi\{\infty,0\}(f_1 \otimes f_2) = \frac{1}{1-e^{mX}} \frac{1}{1-e^{nY}} \sum_{(x,y)\in\mathcal{P}} f_1(x)e^{xX} f_2(y)e^{yY} \tag{5.104}$$

$$= \left(\frac{1}{1-e^{mX}} \sum_{0\leq x<m}{}' f_1(x)e^{xX} \right) \left(\frac{1}{1-e^{nY}} \sum_{0\leq y<n}{}' f_2(y)e^{yY} \right) \tag{5.105}$$

where $\mathcal{P}$ is the weighted parallelogram $\{(x,y) : 0 \leq x < n, 0 \leq y < m\}$ with the included boundaries weighted by $1/2$, and the prime on the sums indicates weighing $x = 0$ and $y = 0$ by $1/2$. In other words, $\Psi\{\infty, 0\}$ is the product of the GL_1 cocycle $\varphi + \frac{1}{2}\delta_0$. Fixing f' away from p and factorizable, we get the following useful lemma.

Lemma 5.4.5. *Suppose f' is factorizable, i.e. $f' = f_1' \otimes f_2'$. Then $\Phi_{f'}\{\infty, 0\}$ is the product of two 1-dimensional distributions $\Phi_{f'}\{\infty, 0\} = \xi_{f_1'}\ \xi_{f_2'}$ of §3.4.*

Finally, we will make frequent use the following trick.

Lemma 5.4.6. *Let $\mu \in \widetilde{\mathcal{D}}(\mathbb{Q}_p)$. For all $s \in \mathbb{Z}_p \subset \mathcal{X}(\mathbb{Q}_p)$, $L_p(D_x\mu, s+1) = sL_p(\mu, s)$.*

Proof. Let n be a non-negative integer. Then $L_p(D_x\mu, n+1) = \int_{\mathbb{Z}_p^\times} x^n d(D_x\mu) = \int_{\mathbb{Z}_p^\times} nx^{n-1} d\mu = nL_p(\mu, n)$. The result follows from the fact that the integers are dense in $\mathcal{X}(\mathbb{Q}_p)$. □

Now we come to the Main Theorem of Chapter 5.

Theorem 5.4.7. *Suppose $f' = f_1' \otimes f_2'$ and satisfies the hypothesis of Theorem 5.2.18. For all integers $k \geq 0$, the p-adic L-function of the modular symbol $\nu_{f'}^{-k} \in \mathrm{Symb}_\Gamma(\mathcal{D}_k(\mathbb{Z}_p))$ is given by*

$$L_p(\nu_{f'}^{-k}, s) = \frac{(-1)^{-k-1}}{k!}(s-1)\cdots(s-k-1)L_p(\xi_{f'}, s-k-1)L_p(\xi_{f_2'}, 1-s). \tag{5.106}$$

Proof. Observe, since f' satisfies the vanishing hypothesis for 0, $\Phi_{f'}\{\infty, 0\}$ is represented by $\mu \otimes \frac{1}{X}$, with $\mu = D_x\Phi_{f'}\{\infty, 0\} = (D_x\xi_{f_1'})\xi_{f_2'}$.

At a positive weight $k_0 > 0$

$$\widehat{\Phi}_{f'}^{k_0}\{\infty, 0\} = \left(\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} (xX + yY)^{k_0 - 1} d\mu(x, y) \frac{X dY - Y dX}{X} \right)_{(X,Y) = (-pNz, 1)} \tag{5.107}$$

$$= \sum_{n \geq 0} \binom{k_0 - 1}{n} \left(\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^n y^{k_0 - 1 - n} d\mu(x, y) \right) (-pz)^n \frac{dz}{z}. \tag{5.108}$$

Analytically continuing to negative weights, this becomes

$$\sum_{n \geq 0} \binom{-k - 1}{n} \left(\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^n y^{-k-1-n} d\mu(x, y) \right) (-Npz)^n \frac{dz}{z}. \tag{5.109}$$

Applying the Atkin-Lehner involution, at weight $-k$ (and remembering the twist by $\det{}^k$), we get

$$\Phi_{f'}^{-k} = \det(W)^k (pz)^{-k} \sum_{n \geq 0} \binom{-k - 1}{n} \left(\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^n y^{-k-1-n} d\mu(x, y) \right) z^{-n} \frac{-dz}{z} \tag{5.110}$$

$$= -\sum_{n \geq 0} \binom{-k - 1}{n} \left(\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^n y^{-k-1-n} d\mu(x, y) \right) z^{-n-k} \frac{dz}{z} \tag{5.111}$$

We can read off $\int_{\mathbb{Z}_p} x^m d\mu_{\Phi_{f'}^{-k}}(x)$ as the coefficient of $z^{-m} \frac{dz}{z}$ in the above equation, which corresponds to the index $n = m - k$. Thus, by the factorization of μ,

$$\int_{\mathbb{Z}_p} x^m d\mu_{\widehat{\Phi}_{f'}^{-k}}(x) = \binom{-k - 1}{m - k} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^{m-k} y^{-1-m} d\mu(x, y) \tag{5.112}$$

$$= -\binom{-k - 1}{m - k} \int_{\mathbb{Z}_p} x^{m-k} d(D_x \xi_{f_1'})(x) \int_{\mathbb{Z}_p^\times} y^{-1-m} d\xi_{f_2'}(y) \tag{5.113}$$

$$= -\binom{-k - 1}{m - k} \left(\int_{\mathbb{Z}_p} x^{m-k} d(D_x \xi_{f_1'}) \right) L_p(\xi_{f_2'}, -m) \tag{5.114}$$

Using Lemma 5.4.4, we get

$$\int_{\mathbb{Z}_p^\times} x^m d\mu_{\Phi_{f'}^{-k}} = -(1 - p^{m-k}) \binom{-k - 1}{m - k} \left(\int_{\mathbb{Z}_p} x^{m-k} d(D_x \xi_{f_1'}) \right) L_p(\xi_{f_2'}, -m) \tag{5.115}$$

Now $(1-p^{m-k})(\int_{\mathbb{Z}_p} x^{m-k} d(D_x\xi_{f_1'}) = \int_{\mathbb{Z}_p^\times} x^{m-k} d(D_x\xi_{f_1'})$, which is equal to $L_p(D_x\xi_{f_1'}, m-k+1) = (m-k)L_p(\xi_{f_1'}, m-k)$, so this turns into

$$= -\binom{-k-1}{m-k}(m-k)L_p(\xi_{f'}, m-k)L_p(\xi_{f_2'}, -m). \tag{5.116}$$

Using $\binom{-k-1}{m-k} = (-1)^{m-k}\binom{m}{m-k} = \frac{(-1)^{m-k}}{k!}m(m-1)\cdots(m-k+1)$, we deduce

$$\int_{\mathbb{Z}_p^\times} x^m d\mu_{\widehat{\Phi}_{f'}^{-k}} = \frac{(-1)^{m-k+1}}{k!}(m)\cdots(m-k)L_p(\xi_{f'}, m-k)L_p(\xi_{f_2'} - m). \tag{5.117}$$

Since the integers are dense in $\mathcal{X}(\mathbb{Q}_p)$, we get (for $s = m-1$, even)

$$L_p(\nu_{f'}^{-k}, s) = \frac{(-1)^{-k-1}}{k!}(s-1)\cdots(s-k-1)L_p(\xi_{f'}, s-k-1)L_p(\xi_{f_2'}, 1-s). \tag{5.118}$$

□

5.5 Critical slope Eisenstein series

We close this chapter by proving a conjecture of Pasol and Stevens concerning the p-adic L-function of the critical slope refinement of the Eisenstein series (for k even)

$$E_{k+2,\ell} = c(k) + \sum_{n\geq 1}\left(\sum_{\substack{d|n \\ (\ell,d)=1}} d^k\right) q^n \in M_{k+2}(\Gamma_0(\ell)), \tag{5.119}$$

with $k=0$ and $\ell = 11$. This Eisenstein series is a Hecke eigenform with eigenvalues $E_{k+2,\ell}|U_\ell = E_{k+2,\ell}$ and $E_{k+2,\ell}|T_q = (1+q^{k+1})E_{k+2,\ell}$ for $q \neq \ell$. For $p \neq \ell$, the critical p-stabilization of $E_{2,\ell}$ is a Hecke eigenform in $E_{k+2,\ell}^{crit} \in M_{k+2}(\Gamma_0(p\ell))$ with the same eigenvalues away from p, and $E_{k+2,\ell}^{crit}|U_p = p^{k+1}E_{k+2,\ell}^{crit}$.

When $k=2$ and $\ell = 11$ and $p=3$, Pasol and Stevens (using the MAGMA programs of R. Pollack) lifted the critical slope boundary symbol $\varphi \in \mathrm{Symb}_{\Gamma_0(33)}(\mathbb{Q}_p)$ (corresponding to $E_{2,11}^{crit}$) to an *approximation* of a modular symbol in $\mathrm{Symb}_{\Gamma_0(33)}(\mathcal{D}_0(\mathbb{Z}_3, 1))$, which seemed to be approximating an eigensymbol Φ with eigenvalues

- $\Phi|T_q = (1+q)\Phi$,
- $\Phi|U_\ell = \Phi$,
- $\Phi|U_p = p\Phi$,

and p-adic L-function

$$L_p(\Phi, s) = (1-s)\cdot(1-11^{1-s})\cdot\zeta_p(s)\cdot\zeta_p(2-s). \tag{5.120}$$

Stevens then conjectured that Φ fits into a p-adic analytic family $\Phi_{k,eis}$ corresponding to the higher weight Eisenstein series. By picking f' carefully, we can show $\Phi_{0,eis} = \nu_{f'}^{-k}$, and that at weight 0 we get the modular symbol conjecture by Pasol and Stevens.

Let us now fix p prime and $\ell \neq p$.

Theorem 5.5.1. *There exists a modular symbol $\nu \in \mathrm{Symb}_{\Gamma_0(p\ell)}(\mathcal{D}_0(\mathbb{Z}_p))$ such that*

- *$\nu|T_q = (1+q)\nu$ for $q \neq p, \ell$,*
- *$\nu||U_p = p\nu$.*

Moreover, the p-adic L-function of ν is equal to

$$L_p(\nu, s) = (1-s)(1-\ell^{1-s})\zeta_p(s)\zeta_p(2-s). \tag{5.121}$$

Proof. Let f' be the test function (away from p) given by

$$f' = \left(\bigotimes_{q\neq p,\ell}[\mathbb{Z}_q^2]\right)\otimes([\mathbb{Z}_\ell^2] - \ell[\mathbb{Z}_\ell\times\ell\mathbb{Z}_\ell]), \tag{5.122}$$

which factorizes as $f_1 \otimes f_2$, with

$$f_1' = \bigotimes_{q\neq p}[\mathbb{Z}_q] \text{ and } f_2' = \left(\bigotimes_{q\neq p,\ell}[\mathbb{Z}_q]\right)\otimes([\mathbb{Z}_\ell] - \ell[\ell\mathbb{Z}_\ell]). \tag{5.123}$$

Observe that $\Gamma_0(\ell)$ stabilizes f'. Furthermore, f' is a Hecke eigenvector away from ℓ, with $f'|T_q^{*-1} = qf'|(qI)^{*-1} + f'$ by Lemma 5.3.2. However, f' is *not* an eigenvector for U_ℓ!

Since $h(f_2') = 0$, f' satisfies the vanishing hypothesis for $\binom{0}{1}$, so f' satisfies the vanishing hypothesis for the cusps $\Gamma_0(p\ell)\cdot 0$, and $\Gamma_0(p\ell)\cdot\frac{1}{p}$, but not the cusps $\Gamma_(p\ell)\cdot\infty$ or $\Gamma_0(p\ell)\cdot\frac{1}{\ell} \subset \Gamma_0(p)\cdot 0$. By Theorem 5.2.18, $\widehat{\Phi}^k_{f'}$ varies in a family of analytic differential forms on $\mathcal{W}_\infty$, so we can compute the p-adic L-function of $\Phi^0_{f'}$. Theorem 5.4.7 tells us

$$L_p(\nu^0_{f'}, s) = -(s-1)L_p(\xi_{f_1'}, s-1)L_p(\xi_{f_2'}, 1-s). \tag{5.124}$$

We computed the first p-adic L-function, $L_p(\xi_{f_1'}, s) = \zeta_p(1-s)$ in Lemma 3.4.2. A similar calculation shows $L_p(\xi_{f_2'}, s) = (1-\ell^s)\zeta_p(1-s)$, which gives us

$$L_p(\nu^0_{f'}, s) = (1-s)\zeta_p(2-s)(1-\ell^{1-s})\zeta_p(s), \tag{5.125}$$

which was the conjectural p-adic L-function of Φ.

Theorem 5.2.18 tells us that, for each $-k \le 0$, $\nu^{-k}_{f'}$ is an eigenvector for T_q, $q \neq \ell$, and U_p, with eigenvalues

- $\nu^{-k}_{f'}|T_q = q^{k+1}(1 - qq^{-k-2})\nu^{-k}_{f'} = (1+q^{k+1})\nu^{-k}_{f'}$, and
- $\nu^{-k}_{f'}|U_p = p^{k+1}\nu^{-k}_{f'}$.

□

The reader may be troubled by the conspicuous absence of U_ℓ and (rightfully) object that we have not really proven Steven's conjecture. It is not at all clear that $\nu^0_{f'}$ should be an eigenvector for U_ℓ, since f' is *not* and eigenvector for U_ℓ. But if we put

$$W_\ell = \begin{pmatrix} \ell & y \\ \ell p & \ell x \end{pmatrix} \in S_0(p), \tag{5.126}$$

where x, y are integers satisfying $\ell x - py = 1$, then we claim (1) that $\nu^k_{f'}|W_\ell$ is a non-zero eigensymbol for U_ℓ with eigenvalue 1 and (2) $\nu^0_{f'}|W_\ell = -\nu^0_{f'}$. Since W_ℓ commutes with the Hecke operators T_q, $q \neq p, \ell$ and U_p, these claims would imply that $\nu^0_{f'} = -\nu^0_{f'}|W_\ell$ is the conjectural eigensymbol Φ.

We turn now to the first claim.

Proposition 5.5.2. *For all integers $-k \leq 0$, $\nu^{-k}_{f'}|_{-k}W_\ell$ is an eigenvector for U_ℓ with eigenvalue* 1*.*

Proof. First we show $f'|(W_\ell U_\ell W_\ell^{-1})^* - 1 = \ell f'|\langle \ell \rangle^{*-1}$. At ℓ, W_ℓ acts by $\begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}$, and at primes $q \neq p, \ell$, $W_\ell \in \mathrm{SL}_2(\mathbb{Z}_q)$. Therefore,

$$f'|W_\ell^{*-1} = ([\mathbb{Z}_\ell]^2 - \ell[\mathbb{Z}_\ell \times \ell\mathbb{Z}_\ell])\left|\begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}^{*-1}\right. = [\mathbb{Z}_\ell \times \ell\mathbb{Z}_\ell] - \ell[(\ell\mathbb{Z}_\ell)^2]. \tag{5.127}$$

The calculations of Lemma 5.3.2 imply $(f'|W_\ell^{*-1})|U_\ell = \ell(f'|W_\ell^{*-1})\langle \ell \rangle^{*-1}$. Since $\langle \ell \rangle$ is in the center of G, we conclude $f'|(W_\ell U_\ell W_\ell)^{*-1} = \ell f'|\langle \ell \rangle^{*-1}$. Therefore, for all positive integers $k > 0$, (remembering the twist by det), $(\Phi^k_{f'}|_k W_\ell)|_k U_\ell = (\ell^{k-2}\ell)\ell\Phi^k_{f'}|_k W_\ell = \ell^k \Phi_{f'}|W_\ell$. Since the natural numbers are dense in $\mathcal{X}(\mathbb{Z}_p)$, we get $\Phi^{-k}_{f'}|W_\ell|U_\ell = \ell^{-k}\Phi_{f'}|W_\ell$, and thus (remembering the twist by det ^{k}), $(\nu^{-k}_{f'}|W_\ell)|U_\ell = \ell^k \ell^{-k} \nu^{-k}_{f'}|W_\ell = \nu^{-k}_{f'}|W_\ell$. □

The second claim appears to be much harder, but follows, for example, from a deep theorem of Bellaïche. The following is a special case Theorem 1 of [4]:

Theorem 5.5.3. *The $E^{crit}_{2,\ell}$-isotypical subspaces $\mathrm{Symb}^{\pm}_{\Gamma(p\ell)}(\mathcal{D}_0(\mathbb{Z}_p))_{[E^{crit}_{2,\ell}]} \subset \mathrm{Symb}_{\Gamma_0(p\ell)}(\mathcal{D}_0(\mathbb{Z}_p))$ are* 1*-dimensional for both choices of sign.*

One easily checks (by inspecting the moments in equation 5.111) that $\nu^0_{f'}|W_\ell\{\infty, 0\}$ is even, so $\nu^0_{f'}|W_\ell \in \mathrm{Symb}^+_{\Gamma(p\ell)}(\mathcal{D}_0(\mathbb{Z}_p))_{[E_{2,\ell}]}$. A calculation shows that W_ℓ acts on the classical modular symbol $\rho_0(\nu^0_{f'}|W_\ell) \in \mathrm{Symb}_{\Gamma_0(p\ell)}(\mathbb{Q}_p)$ by -1 and since ρ_0 is $S_0(p)$ equivariant, we conclude

$$(\nu^0_{f'}|W_\ell)|W_\ell = -\nu^0_{f'}|W_\ell. \tag{5.128}$$

Since W_ℓ^2 acts by $\langle \ell \rangle$ (and hence trivially at weight 0) we conclude that $\nu_{f'}^0 = -\nu_{f'}^0 | W_\ell$ is in fact the eigensymbol conjecture by Pasol and Stevens.

Let us conclude by remarking that Theorem 5.4.7 will allow us to compute the p-adic L-functions for any the critical slope refinement of any Eisenstein series $E_{k+2,\chi,\tau} \in M_{k+2}(\Gamma_1(N))$, $p \nmid N$. One simply cooks up a test function f' from the characters χ and τ, which will factorize nicely and satisfy the vanishing hypothesis at enough cusps.

Bibliography

[1] Yvette Amice, *Interpolation p-adique*, Bulletin de la Société Mathématique de France **92** (1964), 117–180.

[2] Yvette Amice and Jacques Vélu, *Distributions p-adiques associées aux séries de Hecke*, Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, Bordeaux, 1974), Soc. Math. France, Paris, 1975, pp. 119–131. Astérisque, Nos. 24–25.

[3] Daniel Barsky, *Fonctions zeta p-adiques d'une classe de rayon des corps de nombres totalement réels*, Groupe d'Etude d'Analyse Ultramétrique (5e année: 1977/78), Secrétariat Math., Paris, 1978, pp. Exp. No. 16, 23.

[4] Joël Bellaïche, *Critical p-adic L-functions*, Inventiones Mathematicae **189** (2012), no. 1, 1–60.

[5] Joël Bellaïche and Samit Dasgupta, *The p-adic L-functions of Evil Eisenstein series*, `http://people.ucsc.edu/~sdasgup2/EvilEisenstein.pdf`, March 2012.

[6] Duff Campbell, *The Eisenstein Distribution and p-adic L-functions*, Ph.D. thesis, Boston University, Boston, MA, 1997.

[7] Pierrette Cassou-Noguès, *Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques*, Inventiones Mathematicae **51** (1979), no. 1, 29–59.

[8] Pierre Charollois and Samit Dasgupta, *Integral Eisenstein cocycles on* GL_n*, I: Sczech's cocycle and p-adic L-functions of totally real fields*, `http://people.ucsc.edu/~sdasgup2/Charollois-Dasgupta.pdf`, June 2012.

[9] Pierre Colmez, *Résidu en $s = 1$ des fonctions zêta p-adiques*, Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique **305** (1987), no. 1, 5–8.

[10] Pierre Colmez, *Fontaine's rings and p-adic L-functions*, `http://www.math.jussieu.fr/~colmez/tsinghua.pdf`, 2004.

[11] Samit Dasgupta and Matt Greenberg, *The rank one abelian stark conjecture*, `http://swc.math.arizona.edu/aws/2011/2011DasguptaGreenbergNotes.pdf`, 2011.

[12] Haruzo Hida, *Elementary theory of L-functions and Eisenstein series*, London Mathematical Society Student Texts, vol. 26, Cambridge University Press, Cambridge, 1993.

[13] Richard Hill, *Shintani cocycles on* GL_n, Bulletin of the London Mathematical Society **39** (2007), no. 6, 993–1004.

[14] Shubin Hu and David Solomon, *Properties of higher-dimensional Shintani generating functions and cocycles on* $\mathrm{PGL}_3(\mathbf{Q})$, Proceedings of the London Mathematical Society. Third Series **82** (2001), no. 1, 64–88.

[15] N. Katz, *Another look at p-adic L-functions for totally real fields*, Mathematische Annalen **255** (1981), no. 1, 33–43.

[16] Kalin Kostadinov, *Constructing an explicit modular symbol*, Ph.D. thesis, Boston University, Boston, MA, 2010.

[17] Robert Pollack and Glenn Stevens, *Critical slope p-adic l-functions*, `http://math.bu.edu/people/rpollack/Papers/Critical_slope_padic_Lfunctions.pdf`.

[18] Robert Pollack and Glenn Stevens, *Overconvergent modular symbols and p-adic L-functions*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série **44** (2011), no. 1, 1–42.

[19] Peter Schneider, *Rigid-analytic L-transforms*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Lecture Notes in Mathematics, vol. 1068, Springer, Berlin, 1984, pp. 216–230.

[20] Robert Sczech, *Eisenstein cocycles for* $\mathrm{GL}_2\mathbf{Q}$ *and values of L-functions in real quadratic fields*, Commentarii Mathematici Helvetici **67** (1992), no. 3, 363–382.

[21] Takuro Shintani, *On evaluation of zeta functions of totally real algebraic number fields at non-positive integers*, Journal of the Faculty of Science. University of Tokyo. Section IA. Mathematics **23** (1976), no. 2, 393–417.

[22] David Solomon, *Algebraic properties of Shintani's generating functions: Dedekind sums and cocycles on* $\mathrm{PGL}_2(\mathbf{Q})$, Compositio Mathematica **112** (1998), no. 3, 333–362.

[23] ———, *The Shintani cocycle. II. Partial ζ-functions, cohomologous cocycles and p-adic interpolation*, Journal of Number Theory **75** (1999), no. 1, 53–108.

[24] Michael Spiess, *Shintani cocyles and vanishing order of p-adic Hecke L-series at $s = 0$*, `http://arxiv.org/abs/1203.6689`, March 2012.

[25] Glenn Stevens, *The Eisenstein measure and real quadratic fields*, Théorie des nombres (Quebec, PQ, 1987), de Gruyter, Berlin, 1989, pp. 887–927.

Curriculum Vitae

[redacted]