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On group extensions

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ON GROUP EXTENSIONS

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CONTENTS

SECTION PAGE

1. Extension Theory and Factor Systems 1
2. Central Extensions and H-X Extensions 12
3. Extension with Cyclic Factor Group 15
4. Applications 17
5. Double Modules 22
6. Cohomology Groups 23
7. Nonhomogeneous Cochains 26
8. Normalized Cochains 31

BIBLIOGRAPHY.

1.EXTENSION THEORY AND FACTOR SYSTEMS

A group G is called an extension of a group N by a group H, if N is a normal subgroup of G and the factor group G/N is isomorphic to H. For a given N and H, there always exist extensions of N by means of H, for example the direct product of N and H. However the group in general is not uniquely determined by N and H.

A first approach to the extension problem was made by Schreier in 1923. He considered the problem of construct- all groups G such that G will have a normal subgroup N and a given factor group $H \cong G/N$. Group extensions in general have been considered in a broad way by Baer. He reduced the survey of a group N by a group H essentially to the case in which N is abelian. For the abelian case significant progress has been made by the use of the so called cohomology groups. These group-theoretical constructions were built into the framework of the general theory of groups by Eilenberg and Mac Lane.

Let us first examine closely the group G which is the extension of a group N by a group H. Denote the elements of N by 1, a, b, c,.... and the elements of H by e, u, v, w, ... In every coset of N in G choose an element as representative and denote it by $\bar{u}$, where u is the element of H associated with the corresponding coset in the isomorphism between G/N and H. The product of the representatives $\bar{u}$ and $\bar{v}$, by the isomorphism between G/N and H, lies in the coset of N with the representative $\overline{uv}$. In other words there is an element (u,v) in N such that

$$U \quad \bar{u}\cdot\bar{v} = \overline{uv}(u,v) \qquad (u,v) \in N,\ u,v \in H.$$

In particular taking u and v as the element e of H, we have

$$\bar{e}\cdot\bar{e} = \overline{ee}(e,e) = \bar{e}(e,e).$$

Hence $\bar{e} = (e,e)$.

Also the transformation of N by $\bar{u}$ induces an automorphism in N, since N is a normal subgroup. Denote the image of an element a of N under this automorphism by a^u :

$$\bar{u}^{-1} a \bar{u} = a^u$$

Also the element $b^{-1} a b$, $b \in N$ is denoted by a^b.

Then we have

$$\begin{aligned}(a^u)^v &= (\bar{u}^{-1} a \bar{u})^v = \bar{v}^{-1} \bar{u}^{-1} a \bar{u} \bar{v} \\ &= (\overline{uv})^{-1} a \bar{u} \bar{v} \\ &= (\overline{uv}(u,v))^{-1} a(\overline{uv}(u,v)) \\ &= (u,v)^{-1} \overline{uv}^{-1} a \overline{uv} (u,v) \\ &= (u,v)^{-1} a^{uv} (u,v)\end{aligned}$$

So $$(a^u)^v = (u,v)^{-1} a^{uv} (u,v) \qquad (I)$$

Further from the associative law of multiplication in G

$$\begin{aligned}\bar{u}(\bar{v}\bar{w}) &= \bar{u}\cdot\overline{vw}(v,w) \\ &= \overline{uvw}\,(u,vw)(v,w)\end{aligned}$$

Also $(\bar{u}\ \bar{v})\ \bar{w} = \overline{uv}(u,v)\bar{w}$

$$= \overline{uv}\ \bar{w}\ \bar{w}^{-1}\ (u,v)\bar{w}$$
$$= \overline{uvw}(uv,w)\ \bar{w}^{-1}\ (u,v)\bar{w}$$
$$= \overline{uvw}(uv,w)(u,v)^{w}$$

$$(u,vw)(v,w) = (uv,w)(u,v)^{w} \qquad \text{(II)}$$

Finally, if $\bar{u}a$ and $\bar{v}b$ are elements of G we have

$$(\bar{u}a)(\bar{v}b) = \bar{u}\ \bar{v}\ \bar{v}^{-1}\ a\bar{v}b$$
$$= \bar{u}\ \bar{v}\ a^{v}\ b$$
$$= \overline{uv}\ (u,v)\ a^{v}\ b$$

$$(\bar{u}a)(\bar{v}b) = \overline{uv}(u,v)a^{v}b \qquad \text{(III)}$$

So far we have started from a given extension of N by H and have established a correspondence between this extension and a system of elements (u,v), the socalled <u>factor system</u>, and a system of automorphisms $a \longrightarrow a^{u}$.

Conversely, let us assume that in a group G a system of elements (u,v) is chosen where u and v ranges independently over all elementsof a group H, and that every element u of H is associated with some automorphism $a \longrightarrow a^{u}$ of N for which conditions (I)) and (II) are satisfied. We shall now show that there exists an extension G of N by H for which the given elements (u,v) and the given automorphisms correspond to this extension in the above sense.

Here the elements of G will be symbols $\bar{u}a$ where a is an arbitrary element of N and where the symbols $\bar{u}$

correspond one to one to elements u of H. The symbols $\bar{u}a = \bar{v}b$ if and only if $u = v$ and $a = b$. Let us define multiplication in G by the formula (III) and show that G is a group. The associative law of this multiplication can be proved from the definition and the conditions (I) and (II). For

$$
\begin{aligned}
(\bar{u}a\cdot\bar{v}b)\bar{w}c &= (\overline{uv}(u,v)a^{v}\ b)\bar{w}c \\
&= \overline{uvw}(uv,w)(\ (u,v)a^{v}\ b)^{w}\mathcal{C}\ c \\
&= \overline{uvw}(uv,w)(u,v)^{w}\ (a^{v}\ )^{w}\ b^{w}\ c \\
&= \overline{uvw}(uv,w)(u,v)^{w}\ (v,w)^{-1}\ a^{vw}\ (v,w)b^{w}\ c \\
&= \overline{uvw}(u,vw)(v,w)(v,w)^{-1}\ a^{vw}\ (v,w)b^{w}\ c \\
&= \bar{u}a\ \overline{vw}(v,w)b^{w}\ c \\
&= \bar{u}a(\bar{v}b\cdot\bar{w}c)
\end{aligned}
$$

Hence $(\bar{u}a\cdot\bar{v}b)\bar{w}c = \bar{u}a(\bar{v}b\cdot\bar{w}c)$.

If we set $u = v = e$ in (I) we have

$(a^{e})^{e} = (e,e)^{-1}\ a^{e}\ (e,e)$ and since a^{e} ranges over the whole group N as a does we have

$$a^{e} = (e,e)^{-1}\ a(e,e) = a^{(e,e)} \qquad \text{(IV)}$$

Further from (II) when $v = w = e$

$$(u,e)(e,e) = (u,e)(u,e)^{e}$$

$$(e,e) = (u,e)^{e} = (u,e)^{(e,e)} = (e,e)^{-1}\ (u,e)(e,e)$$

$$(u,e) = (e,e) \qquad \text{(V)}$$

Again from (II) when $u = v = e$

$$(e,w)(e,w) = (e,w)(e,e)^{w}$$

$$(e,w) = (e,e)^{w} \qquad \text{(VI)}$$

If $\bar{u}a$ is an arbitrary element of G, then ua

$$\begin{aligned}\bar{u}a\cdot\bar{e}(e,e)^{-1} &= \bar{u}(u,e)a^{e}\,(e,e)^{-1}\\ &= \bar{u}(e,e)a^{(e,e)}\,(e,e)^{-1}\\ &= \bar{u}(e,e)(e,e)^{-1}\,a(e,e)(e,e)^{-1}\\ &= \bar{u}a\end{aligned}$$

So $\epsilon = \bar{e}(e,e)^{-1}$ is a right unit element of G. Let us see whether there is an inverse for $\bar{u}a$. Consider the product

$$\bar{u}a\,\overline{u^{-1}}\,(a^{u^{-1}})^{-1}\,(u,u^{-1})^{-1}(e,e)^{-1}$$

$$= \overline{uu^{-1}}\,(u,u^{-1})a^{u^{-1}}\,(a^{u^{-1}})^{-1}\,(u,u^{-1})^{-1}(e,e)^{-1} \quad \text{from (III)}$$

$$= \bar{e}(e,e)^{-1} = \epsilon$$

This shows that every element $\bar{u}a$ of G has a right inverse $\overline{u^{-1}}(a^{u^{-1}})^{-1}(u,u^{-1})^{-1}(e,e)^{-1}$.

Let us now show that G is the required extension of N by H. If we associate with every element a of N the element

$\hat{a} = \bar{e}(e,e)^{-1}\,a$ of G, then by (IV) and (VI)

$$\begin{aligned}\hat{a}\cdot\hat{b} &= \bar{e}(e,e)^{-1}\,a\,\bar{e}(e,e)^{-1}b\\ &= \overline{ee}(e,e)(\,(e,e)^{-1}a\,)^{e}\,(e,e)^{-1}\,b\\ &= \bar{e}(e,e)(\,(e,e)^{-1}\,a\,)^{(e,e)}\,(e,e)^{-1}b\\ &= \bar{e}(e,e)(e,e)^{-1}(\,(e,e)^{-1}\,a)(e,e)(e,e)^{-1}b\\ &= \bar{e}(e,e)^{-1}\,ab\\ &= \widehat{ab}\end{aligned}$$

Hence $\hat{a}\cdot\hat{b} = \widehat{ab}$.

Also if $\hat{a} = \bar{e}(e,e)^{-1}$ (that is, if $\hat{a}$ is the unit element of G) it follows that a = 1. Thus the elements $\hat{a}$ form a subgroup $\hat{N}$ of G isomorphic to N.

Further if we set

$$\hat{\bar{u}} = \bar{u}I$$

$$\hat{\bar{u}}\cdot\hat{a} = \bar{u}I\cdot\bar{e}(e,e)^{-1}\, a$$

$$= \bar{u}(u,e)I^{e}\ (e,e)^{-1}\, a \qquad \text{from(III)}$$

$$= \bar{u}(e,e)I\ \ (e,e)^{-1}\, a \qquad \text{from (V)}$$

$$= \bar{u}a$$

That is $\hat{\bar{u}}\,\hat{a} = \bar{u}a$ (VII)

It follows that the elements $\hat{\bar{u}}$ lie in distinct cosets of N: for $\hat{\bar{v}} = \hat{\bar{u}}a$, $u \neq v$ would imply

$\bar{v}I = \bar{u}a$ and this is impossible because $u \neq v$.

On the other hand (VII) shows that every left coset of N contains one of the elements of u.

From (VI) and (VII) we deduce that

$$\hat{a}\hat{\bar{u}} = \bar{e}(e,e)^{-1}\ a\,\bar{u}I$$

$$= \bar{u}(e,u)(\ (e,e)^{-1}\ a)^{u}\ I$$

$$= \bar{u}(e,u)(\ (e,e)^{-1}\ )^{u}\ a^{u}$$

$$= \bar{u}a^{u}$$

$$= \hat{\bar{u}}\ \widehat{a^{u}}$$

$$(\hat{\bar{u}})^{-1}\ \hat{a}(\hat{\bar{u}}) = \widehat{a^{u}}\ .$$

Hence $(\hat{\bar{u}})^{-1}\,\hat{a}(\hat{\bar{u}}) = \widehat{a^{u}} \in N$; that is $\hat{N}$ is a normal subgroup of G, and the transformation by $\hat{\bar{u}}$ induces an automorphism in $\hat{N}$ that coincides with the original automorphism $a \longrightarrow a^{u}$ of N.

Finally the equation

$$\begin{aligned}\hat{u}\cdot\hat{v} &= \bar{u}I\cdot\bar{v}I \\ &= \overline{uv}(u,v)I^{v}\cdot I \\ &= \overline{uv}(u,v) \\ &= \widehat{uv}\widehat{(u,v)}\end{aligned}$$

shows that the factor group $G/\hat{N}$ is isomorphic to H, in other words, that G is an extension of N by H and that the factor system $\widehat{(u,v)}$ of this extension corresponds with the given elements (u,v) provided the chosen representatives of the left cosets of N are precisely the elements $\hat{u}$. From these we conclude the theorem:

1.1. To each extension G of a normal subgroup N with a given factor group H there belongs a factor system and a set of automorphisms of N such that conditions (I) and (II) are satisfied. Conversely to a given factor system and a given set of automorphisms of N which fulfill (I) and (II) there belongs an extension of N unique within isomorphism over N. The elements $\bar{u}a$ of this extension $u \in H$, $a \in N$ with the product rule (III) define a group with normal subgroup N and $G/N \cong H$.

Instead of choosing $\bar{u}$ as a representative, we choose

$$\tilde{\bar{u}} = \bar{u}\,\alpha(u)$$ where $\alpha(u) \in N$ then the automorphism $\bar{u}$ of N is replaced by $\tilde{\bar{u}}$.

We have $$\begin{aligned}a^{\tilde{\bar{u}}} &= (\tilde{\bar{u}})^{-1}\, a\tilde{\bar{u}} \\ &= (\bar{u}\,\alpha(u))^{-1}\, a\bar{u}\,\alpha(u)\end{aligned}$$

$$= \alpha(u)^{-1}\,\bar{u}^{-1}\,a\bar{u}\,\alpha(u)$$
$$= \alpha(u)^{-1}\,a^{u}\,\alpha(u)$$

Also the factor set (u,v) is replaced by

$$\alpha(uv)^{-1}\,(u,v)\,\alpha(u)^{v}\,\alpha(v)$$

for
$$\begin{aligned}\tilde{\bar{u}}\cdot\tilde{\bar{v}} &= \bar{u}\,\alpha(u)\bar{v}\cdot\alpha(v)\\ &= \overline{uv}(u,v)\,\alpha(u)^{v}\,\alpha(v)\\ &= \overline{uv}\,\alpha(uv)\,\alpha(uv)^{-1}\,(u,v)\,\alpha(u)^{v}\alpha(v)\\ &= \tilde{\overline{uv}}\,\alpha(uv)^{-1}\,(u,v)\,\alpha(u)^{v}\alpha(v)\,.\end{aligned}$$

Definition: 1.2. Two factor systems $(a^{u},(u,v))$ and $(a^{\tilde{u}},(\tilde{u},\tilde{v}))$ are said to be equivalent if the automorphisms and factor systems are related by

$$a^{\tilde{u}} = \alpha(u)^{-1}a^{u}\,\alpha(u) \qquad \text{(i)}$$
$$(u,v) = \alpha(uv)^{-1}\,(u,v)\,\alpha(u)^{v}\,\alpha(v) \qquad \text{(ii)}$$

where $\alpha(u)$ is a function of elements $u \in H$ with values in N. Then we write

$$(a^{u},(u,v)) \sim (a^{\tilde{u}},(\tilde{u},\tilde{v})).$$

The equivalence amounts to a change of coset representatives for N in the same group G, and so clearly this is a true equivalence. Hence this relation is symmetric, reflexive and transitive. Two factor systems with sets of automorphisms induce extensions which are isomorphic over N and for which the coset $\bar{u}N$ maps onto the coset $\tilde{\bar{u}}N$ if and only if the factor systems are equivalent.

Suppose we take the direct product $N \times H$. In this case $(u,v) = 1$, and $a^{u} = a$. Hence there exists at least one factor system with $(u,v) \equiv 1$.

Theorem 1.3. A factor system $(a^{u}, (u,v))$ is equivalent to $(a^{\tilde{u}}, I)$ if and only if $I = \alpha(uv)^{-1}(u,v)\,\alpha(u)^{v}\,\alpha(u)$ is solvable.

Proof: If $(a^{u}, (u,v)) \sim (a^{\tilde{u}}, I)$ from (ii) we have $I = \alpha(uv)^{-1}(u,v)\,\alpha(u)^{v}\alpha(v)$.

Suppose $I = \alpha(uv)^{-1}\,\alpha(u,v)\,\alpha(u)^{v}\,\alpha(v)$, that is a function $\alpha(u)$ exists such that the above result holds.

Define $\tilde{a^{u}} = \alpha(u)^{-1}\, a^{u}\, \alpha(u)$, and

$$\tilde{\bar{u}} = \bar{u}\,\alpha(u)$$

$$(\tilde{u},\tilde{v}) = \alpha(uv)^{-1}\,(u,v)\,\alpha(u)^{v}\,\alpha(u) = I$$

Hence $(a^{u}, (u,v)) \sim (a^{\tilde{u}}, (\tilde{u},\tilde{v}))$ Q.E.D.

If $(\tilde{u},\tilde{v}) = 1$, then the coset representatives form a group isomorphic to H, which we may identify with H. In this case we say that <u>G splits over N</u> or that G is the semi direct procuct of N and H.

(Definition: Let N and H be two groups and for every element $u \in H$ and automorphism of N,

$$a \longrightarrow a^{u} \qquad a \in N \quad \text{such that}$$
$$(a^{u})^{v} = a^{uv} \qquad u,v \in H.$$

Then the symbols ua, $u \in H$ and $a \in N$ form a group under the product rule: $ua \cdot vb = uva^{v}b$ called the normal product or the semi direct product of N by H.)

If $(u,v) = I$, the product rule defined in (III) satisfies the condition for the semidirect product. Also equation (I) reduces to $(a^{u})^{v} = a^{uv}$.

Theorem 1.4. The extension G $(a^u, (u,v))$ splits over N if and only if we can find a function $\alpha(u) \in N$ such that

$$\alpha(uv)^{-1}(u,v)\alpha(u)\alpha(v) = 1 \quad \text{for all } u,v \in H.$$

The proof follows from Theorem 1.3.

If a given normal subgroup N of G is abelian, then the automorphism $a \rightarrow a^u$ is isdependent of the choice of the representative. For two representatives $\bar u$ and $\lambda\bar u$ we have $(\lambda\bar u)^{-1} a\lambda\bar u = \bar u^{-1}\lambda^{-1} a\lambda\bar u = \bar u^{-1} a\bar u$. Therefore the necessary and sufficient condition for the extension $(a^u, (u,v))$ will be

$$(a^u)^v = a^{uv} \qquad (I')$$

$$(u,vw)(v,w) = (uv,w)(u,v)^w \qquad (II')$$

The conditions for the equivalence of two factor systems are $(\tilde u,\tilde v) = \alpha(uv)^{-1}(u,v)\,\alpha(u)^v\alpha(v)$ and

$$(\alpha(u)\,\bar u)^{-1} a\,\alpha(u)\,\bar u = \alpha(u)^{-1}a^u\,\alpha(u)$$
$$= a^u.$$

Therefore the automorphisms are invariant under equivalence of factor systems.

Theorem 1.5. An extension G of an abelian group N by an abelian group H is abelian if and only if the factor set associated with it satisfies $(u,v) = (v,u)$ where N is in the center of G.

Proof: If an extension G is abelian, then for any set of representatives

$$\overline{uv}(u,v) = \bar u\,\bar v = \bar v\,\bar u$$
$$= \overline{vu}(v,u)$$

$$= \overline{uv}(v,u)$$

$$\therefore \quad (u,v) = (v,u).$$

Conversely if $(u,v) = (v,u)$ for all u , v H, we have

$$\begin{aligned} \overline{u}a\ \overline{v}b &= \overline{uv}(u,v)a^{u}\ b \\ &= \overline{vu}(v,u)ab \\ &= \overline{vu}(v,u)b^{u}\ a \\ &= \overline{v}b\ \overline{u}a. \end{aligned}$$

Hence G is abelian.

2. CENTRAL EXTENSIONS AND H-$\mathcal{X}$ EXTENSIONS*

Suppose that all factors (u,v) in an extension of a group N by a group H lie in the center Z of N. Then the extension $(a^u,(u,v))$ is called a central extension. Thus for an abelian group N all the extensions are central extensions. For a central extension (I) reduces to $(a^u)^v = a^{uv}$ (Ic)

Therefore the automorphisms $a \rightarrow a^u$ of N form a group homomorphic to H. Denote by $\mathcal{X}$ a particular way of assigning to each element of H an automorphism of N, where the automorphisms that are assigned form a group homomorphic to H. Furthermore, if coset representatives $\overline{u}$ are changed only by factors $\alpha(u)$ lying in Z, the automorphisms are unchanged, for $(\overline{u}\alpha(u))^{-1} a(\overline{u}\alpha(u)) = a^u$. Hence for such extensions, which we call H-$\mathcal{X}$ extensions, the automorphisms are fixed and form a group homomorphic to H.

The condition (II) is

$$(u,vw)(v,w) = (uv,w)(u,v)^w \qquad \text{(IIc)}$$

Here for an equivalent extension,

$$(u,v)^* = \alpha(uv)^{-1}(u,v)\,\alpha(u)^v\alpha(v) \text{ with } \alpha(u) \in Z$$

If the factor sets $(u,v)_1$ and $(u,v)_2$ satisfy (IIc) and if $(u,v)_3 = (u,v)_1(u,v)_2 \quad u,v \in H$, then $(u,v)_3$ also satisfy (IIc). For

$$(u,vw)_1(v,w)_1 = (uv,w)_1(u,v)_1^w$$
$$(u,vw)_2(v,w)_2 = (uv,w)_2(u,v)_2^w$$

Multiplying and since all the elements commute, we have

$$(u,vw)_1(u,vw)_2(v,w)_1(v,w)_2 = (uv,w)_1(uv,w)_2((u,v)_1(u,v)_2)^w$$

---

*Hall, M. Groups Rings and Extensions, I, Annals of of Mathematics, vol.39, 1938, pp.220-234.

i.e. $(u,vw)_3(v,w)_3 \quad (uv,w)_3(u,v)_3^w$.

Hence $(u,v)_3$ are also factor sets determining the H-χ extensions of N. For these factor sets we have the identity $(u,v) = 1$ and an inverse, the set in which (u,v) is replaced by $(u,v)^{-1}$. Moreover for equivalent factor sets if $(u,v)_1^* \sim (u,v)_1$ and $(u,v)_2^* \sim (u,v)_2$, then $(u,v)_1^* (u,v)_2^* \sim (u,v)_1(u,v)_2$. Hence the totality of all H-χ factor sets form an abelian group even if the equivalent sets are identified. The group in which equivalent sets are identified is called the <u>group of extensions</u>.

Theorem 2.1. The order of any element ofthe group of the group of extensions divides the order of H and the least common multiple of orders ofelements of Z.

Proof: Let H be of order m and N of order n.

$\phi(v) = \prod_u (u,v)$ and form the product over u of all equations

$$(u,vw)(v,w) = (uv,w)(u,v)^w.$$ We have

$$\phi(vw)(v,w)^m = \phi(w)\phi(v)^w$$ (since all the elements commute)

$$(v,w)^m = \phi(vw)^{-1}\phi(w)\phi(v)^w$$

Hence $(v,w)^m \sim 1$. But (v,w) is in the group of extensions.

we have $(v,w)^m = 1$.

Also if k is a multiple of every element of Z, we have $(u,v)^k = I$. Hence the result.

Corollary 2.2: If m and k are relatively prime, then all H-$\mathcal{X}$ extensions of N are equivalent to the semi direct product of N by H. In other words if G is the extension of N by H, then G splits over N.

3. EXTENSION WITH CYCLIC FACTOR GROUP

Let the factor group of G over the normal subgroup N be isomorphic to the cyclic group $H = (u)$. Let $\bar{u}$ be a representative of the coset associated with u. If the index $(H:1) = m > 0$, then

$1, \bar{u}, \overline{u^2}, \ldots, \overline{u^{m-1}}$ is a system of representatives of G over N. Also $G = N + N\bar{u} \ldots + N\overline{u^{m-1}}$.

Then $\overline{u^m} = \alpha$ where $\alpha \in N$. Moreover for the automorphisms $a \longrightarrow a^{u^m}$.

3.1 $(a^{u})^{m} = (\overline{u^m})^{-1} a\overline{u^m} = \alpha^{-1} a \alpha$ for all $a \in N$.

Also from the result $\overline{u^{-1}}\,\overline{u^m}\,\bar{u} = \overline{u^m}$ we have

3.2 $\alpha^{u} = \alpha$

The factor sets are given by

$$(u^{i}, u^{k}) = \begin{cases} 1, & i+k > m \\ \alpha, & i+k \leq m \end{cases} \qquad 9\ (0 \leq i,k < m)$$

If $(H:1) = \infty$ then $1, \bar{u}, \overline{u^{-1}}, \bar{u}^{2}, \overline{u^{-2}}, \ldots$ is a system of representatives of G over N and $(u^{k}, u^{j}) = 1$. Conversely if $(H:1) = m > 0$ we must show that 3.1 and 3.2 suffice to determine an extension. The elements of H are $1, u, \ldots, u^{m-1}$.

Let us define the automorphism

3.3 $a^{u^0} = a, \quad a^{u^i} = (a^{u^{i-1}})^{u-1}, \quad i = 1, 2, \cdots, m-2$

and a factor set put

3.4 $$(u^{i}, u^{k}) = \begin{cases} 1, & i+k > m \\ \alpha, & i+k \leq m \end{cases}$$

With these definitions we see that (I) and (II) in section 1 are satisfied. Hence by theorem 1.1 and extension is defined.

If $(H:1)=\infty$ and $a \rightarrow a^u$ is any automorphism of N, then we set $(u^i, u^k) = 1$ for all i,k, then an extension is defined. Hence we have proved the theorem:

Theorem 3.5: Let H be a cyclic group of finite order m. Then an extension G of a group N by H exists, if and only if, we have an automorphism $a \rightarrow a^u$ of N and an element $\alpha \in N$ such that (i) the mth power of automorphism is the inner automorphism of N given by transformation α , and (ii) α is a fixed by the automorphism. If H is of infinite order there is no restriction on the automorphism $a \rightarrow a^u$.

4. APPLICATIONS

Theorem 4.1. Let G be a group of order mn containing a normal subgroup K of order n and having a factor group $H = G/K$ of order m where m and n are relatively prime. Then G splits over K.

Proof: The proof is by induction on n. The theorem is trivial when $n = 1$.

Let $n > 1$ and p be a prime dividng n.

Let S_p be a Sylow subgroup belonging to p in G. Since the normal subgroup K of order n contains at least one S Sylow subgroup S_p, all subgroups conjugate to S_p are in K. Hence all Sylow subgroups in G are in K. So the number of Sylow subgroups in G is the same as the number of Sylow subgroups in K.

$(G:N_G(S_p)) = (K:N_K(S_p))$ where $N_G(S_p)$ and $N_K(S_p)$ are respectively the normalizers of S_p in G and K. Hence $(G:K) = (N_G(S_p):N_K(S_p)) = m$. Also $N_K(S_p) = N_G(S_p) \cap K$. (But if K is normal in G and A is a subgroup, then $A \cap K$ is normal in A). Hence $N_K(S_p)$ is normal in $N_G(S_p)$.

Casei. If $N_G(S_p)$ is a proper subgroup of G, then by induction, it contains a subgroup of order m, hence in G.

Case ii. Let $G = N_G(S_p)$ then $K = N_K(S_p)$.

a) Let S_p be a proper subgroup of K.

S_p is normal in G and K and hence by induction G contains a subgroup G' of order $(G:S_p)$ isomorphic to G/S_p and K contains a subgroup K' of order $(K:S_p)$

isomorphic to K/S_p and K' is normal in G' and hence by induction G' contains a subgroup of order m and hence G contains a subgroup of order m isomorphic to G/K.

b) Let $K = S_p$.

If S_p is abelian, G is a central extension of S_p, and hence G splits over S_p. (Corollary 2.2)

If S_p is not abelian, then the center Z of S_p is a proper subgroup of S_p and as a characteristic subgroup of S_p, necessarily a normal subgroup of G. Hence K/Z is normal in G/Z. Hence G/Z contains a subgroup U/Z of order m. But Z is normal in U and of index m in the corresponding subgroup U of G. Hence by induction U contains a subgroup of order m. Hence G contains a subgroup of order m. (Q.E.D.)

Theorem 4.2.* If the order m of the finite factor group H is relatively prime to the order n of the finite normal subgroup K of G and if one of the following conditions holds (i) K is abelian

(ii) K is solvable

(iii) H is solvable

then two representative groups of $H \cong G/K$ are conjugate.

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* For proof see Zassenhaus, <u>Theory of Groups</u>, pp. 182 (It has been conjectured that the above theorem is true in general)

Theorem 4.3[*] In a finite solvable group G of order N

(i) For every decomposition $N = nm$ of the group order into a product of relatively prime factors, there is a subgroup of order m and index n.

(ii) All subgroups whose order is a divisor of m lie in a subgroup of order m.

(iii) All subgroups of order m are conjugate.

(iv) The normalizer of a subgroup of order m is its own normalizer.

Proof: The proof is by induction on the length of the principal series. The least normal subgroup in a principal series is of order p^α. Let this group be H. There are four possibilities $p^\alpha = m_1$, m, m_1 or m where $m = m_1m_2$, $n = n_1n_2$. If $p^\alpha =$ m the theorem follows. Therefore let us consider the other three cases.

(i) Case 1. $p^\alpha = m_1$

Since G is solvable, G/H is solvable, hence by induction G/H contains a subgroup of order m_2 - So G Contains a subgroup corresponding to this of order $m_1m_2 = m$.

Case 2. $p^\alpha = n$ follows from theorem 4.1.

Case 3. $p^\alpha = n_1 < n$. Then G/H contains a subgroup order m which corresponds to a subgroup G' of order mn_1 in G. Hence by the theorem 4.1, G' contains a subgroup of order m. Therefore G contains a subgroup of order m.

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[*] For a different proof of this theorem see Hall, M. The Theory of Groups. pp. 141

(ii) Case 1. $p^\alpha = m_1$. Let M' be a subgroup of order m', divisor of m. The order of $(M' \cup H)/H$ is a divisor of m_2 and hence it is in a subgroup of G/H of order m_2. Thus M' belongs to the corresponding subgroup of order $m_2 m_1 = m$.

Case 2. $p^\alpha = n$. If M is a subgroup of order m in G, then $M \cap (M' \cup H) = M^*$ is of order m'. By theorem 4.2 the subgroups of order m' in $M' \cup H$ are conjugate. Hence M^* and M' are conjugate. Hence M' is contained in a conjugate of M.

Case 3. $p^\alpha = n_1$. $(M' \cup H)/H$ is of order m' and $(M' \cup H)/H$ is a subgroup of G/H. Hence by induction this is contained in a subgroup of order m in G/H. Hence M' is contained in the corresponding subgroup G' of G of order mn_1. From case 2 M' is in a subgroup of order m in G', which proves that M' is in a subgroup of order m of G.

(iii) Let M_1 and M_2 be subgroups of order m.

Case1. $p^\alpha = m_1$. Then $M_1 = M_1 \cup H$ and $M_2 = M_2 \cup H$ are subgroups of order m. $(M_1 \cup H)/H$ and $(M_2 \cup H)/H$ are subgroups of G/H of order m_2, hence by induction conjugate. If g^* transforms $(M_2 \cup H)/H$ into $(M_1 \cup H)/H$ and g in G is mapped into g^* by homomorphism $G \to G/H$, then $g^{-1}(M_2 \cup H)g$ is mapped into $M_1 \cup H$. In other words $g^{-1}(M_2 \cup H)g = M_1 \cup H$. Hence M_1 and M_2 are conjugate.

Case 2. $p^\alpha = n$ follows from theorem 4.2.

Case 3. $p^\alpha = n_1$. $(M_1 \cup H)/H$ and $(M_2 \cup H)/H$ are subgroups of G/H or order m. Hence by induction they are conjugate. Rest as in case 1.

(iv) Let $N_G(M)$ be the normalizer of M in G where M is a subgroup of order m.

the order of $N_G(M)$ must be of the form mn' where n' is a factor of n. Then M is normal in $N_G(M)$ and the normalizer contains a subgroup of order n'. Let g be an element of G where $g \notin N_G(M)$ such that

$$g^{-1}N_G(M)\ g = N_G(M).$$

Since M is a normal subgroup of the normalizer there is no conjugate subgroup of order m in the normalizer. But the subgroups of order m in the normalizer are conjugate. Hence no other element outside the normalizer can transform M into itself. Hence the result. Q.E.D.

5. DOUBLE MODULES

Let π be a discrete group written multiplicatively, N an abelian discrete group written additively and the elements of π operate on the left of N. This third assumption means that for each $x \in \pi$ and $a \in N$ there is an element $xa \in N$, subject to the following conditions:

$$\text{(i)} \quad x(a+b) = xa + xb \qquad a,b \in N$$

$$\text{(ii)} \quad x_2(x_1 g) = (x_2 x_1)g \qquad x_1, x_2 \in \pi$$

$$\text{(iii)} \quad 1a = a$$

From (i) we have $-\xi a = \xi(-a)$ and $\xi 0 = 0$. If for every $x \in \pi$ and $a \in N$ we have $xa = a$, then we say that operates on N simply. If N admits as a group of operators both on the right and on the left, then we say that N is a <u>double π module</u>. Let us first consider the one sided case. Let N be a normal abelian subgroup of some group G, and let $\pi = G/N$. If $x = Nu_x$, then $u_x^{-1} a\, u_x$ depends only on a and x, but not on the choice of the representative u_x in the coset. Hence we can write $u_x^{-1} a u_x = a^x$. This is an example of the one-sided case, taking N multiplicatively.

6. COHOMOLOGY GROUPS

Let π and N be two groups as defined in the preceding section. Let us consider the functions F of $(n + 1)$ variables defined on π and the values in N. If F satisfies the homogeneity condition

$$F(xx_0,xx_1,\dots,xx_n) = xF(x_0,x_1,\dots,x_n) \quad \text{then } F$$

is called an n dimensional cochain $(n = 0,1,\dots)$ of π over N. Let F_1 and F_2 be two n dimensioanl cochains such that their sum $F_1 + F_2$ is defined as

$$(F_1 + F_2)(x_0,\dots,x_n) = F_1(x_0,\dots,x_n) + F_2(x_0,\dots,x_n)$$

isa also a cochain. With this operation of addition, the n dimensional cochains form an additive abelian group and denoted by $C^n(\pi,N)$. $C^o(\pi,N) \cong N$ by definition and so a zero dinensional cochain is simply an element of N.

For each cochain F, the function δF of $(n+2)$ variables is defined as

5.1. $$\delta F(x_0,x_1,\dots,x_{n+1}) = \sum_{i=0}^{n+1}(-1)^i F(x_0,\dots,\hat{x}_i,\dots,x_{n+1}),$$

where $\hat{x}_i$ means that the symbol x_i has been ommitted.

F has the following properties:

(i) δF is a $(n+1)$ cochain

(ii) $\delta(F_1 + F_2) = \delta F_1 + \delta F_2$ for

$$\delta(F_1 + F_2)(x_1,\dots,x_{n+1}) = \sum_{i=0}^{n+1}(-1)^i (F_1 + F_2)(x_0,x_1,\dots,\hat{x}_i,\dots x_{n+1})$$

$$= \sum_{i=0}^{n+1} (-1)^i (F_1(x_0,\dots,\hat{x}_i,.x_{n+1}) + (F_2(x_0,\dots,\hat{x}_i,\dots x_{n+1}))$$

$$= \sum_{i=0}^{n+1} (-1)^{i} F_1(x_0,..,\hat{x}_i,..,x_{n+1}) + \sum_{i=0}^{n+1} (-1)^{i} F (x_0,..,\hat{x}_i,..,x_{n+1})$$

$$= \delta F_1 + \delta F_2.$$

(iii) $\delta(\delta F) = 0$, in other words the coboundary of a coboundary is zero.

$$\begin{aligned}\delta(\delta F) &= \sum_{i=0}^{n+1} (-1)^{i} \delta F(x_0,...,\hat{x}_i, ..,x_{n+1}) \\ &= \sum_{i=0}^{n+1} \sum_{j<i} (-1)^{j+i} F(x_0,...,\hat{x}_j,..,\hat{x}_i, ..x_{n+1}) \\ &+ \sum_{i=0}^{n+1} \sum_{j>i} (-1)^{j+i-1} F(x_0,..,\hat{x}_i,..,\hat{x}_j,.., x_{n+1})\end{aligned}$$

But $F(x_0,..,\hat{x}_i,..\hat{x}_j,..x_{n+1}) = F(x_0,...\hat{x}_i,..\hat{x}_j,..,x_{n+1})$ for all i,j. Hence $\delta(\delta F) = 0$.

The operator is called the *coboundary operator*. The coboundary formula coincides with the usual coboundary formula of the combinatorial topology. The cochain δF will be called the coboundary of F.

If $F = C^n(\pi,N)$ and $\delta F = 0$, then F is called an n-cocycle. These n-cocycles form a subgroup $Z^n(\pi,N)$ of $C^n(\pi,N)$. Also these cocycles form the kernel of the homomorphism of $C^n(\pi,N)$ into $C^{n+1}(\pi,N)$ induced by δ.

If $n > 0$, the n-cochains F such that $F = \delta F'$ for some $F' \in C^{n-1}(\pi,N)$ are called an n dimensional coboundary. They form a subgroup $B^n(\pi,N)$ of $C^n(\pi,N)$. If $n = 0$, we define $B^0(\pi,N) = 0$. Since $\delta(\delta F') = 0$, we have $\delta F = 0$. Hence $B^n(\pi, N)$ is a subgroup of $Z^n(\pi, N)$. The nth cohomology group $H^n(\pi, N)$ of π over N is defined on the factor group

$$H^n(\pi, N) = Z^n(\pi, N) / B^n(\pi, N) .$$

The elements of $H^n(\pi, N)$ are called cohomology classes. Two cochains are cohomologous if their difference is a coboundary. Thus two cocycles are cohomologous if they belong to the same cohomology class. Note that the cohomology group $H^n(\pi, N)$ not only depends on the variables n, and N but also upon the way in which π operates on N.

7. NON HOMOGENEOUS COCHAINS

We have considered the homogenious cochain defined in the last sections. They can be replaced by certain equivalent non-homogeneous cochains. An n-dimensional non-homogeneous cochain is a function of n variables, defined on π and with values in N where π and N are two groups defined as in section 5 and satisfying the three conditions stated there. In particular, by a function of zero variables on π to N we mean any element $a \in N$.

The formulae

$$F(x_o,x_1,\dots,x_n) = x_of(x_o^{-1}x_1,\ x_1^{-1}x_2,\dots,\ x_{n-1}^{-1}\ x_n)$$

and

$$f(x_1,x_2,\dots,x_n) = F(1,x_1,x_1x_2,\dots,x_1x_2\dots x_n)$$

provide a one-one correspondence $F \Leftrightarrow f$ between homogeneous and non-homogeneous n-cochains. These correspondences are additive in both directions. So we shall use the same symbol $C^n(\pi,N)$ to denote both the group ,N) of the homogeneous cochains and the group of the non-homogeneous cochains.

The coboundary δf of a non-homogeneous n-cochain is a function of $(n+1)$ variables with the condition $f \Leftrightarrow F$ implying $\delta f \Leftrightarrow \delta F$. Hence

$$(\delta f)(x_1,\dots,x_{n+1}) = (\delta F)(1,x_1,\ x_1x_2,\dots,x_1x_2\dots x_{n+1})$$

$$= F(x_1,x_1x_2,\dots,x_1x_2\dots x_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i\ F(1,x_1,\dots,x_1x_2\dots x_i,\dots,x_1\dots x_{n+1})$$
$$+ (-1)^{n+1}\, F(1,x_1,\ x_1x_2,\dots,x_1\dots x_n)$$

But $F(x_1,x_1x_2,\dots,x_1x_2x_3\dots x_{n+1}) = x_1F(1,x_2,\dots,x_2\dots x_{n+1})$

$$= x_1 f(x_2,x_3,\dots,x_{n+1})$$

for $0 < i < n+1$.

$$F(1,x_1,\dots,x_1\widehat{x_2}\cdots x_i,\dots,x_1x_2\,x_{n+1})$$
$$= f(x_1,x_2, \quad ,x_{i-1},x_ix_{i+1}, \quad ,x_{n+1})$$

The last term

$$F(1,x_1,x_1x_2,\dots,x_1x_2\cdots x_n) = f(x_1,x_2,\dots,x_n)$$

Thus we have

7.1 $$(\delta f)(x_1,\dots,x_{n+1}) = x_1 f(x_1,\dots,x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1,\dots,x_ix_{i+1},\dots,x_{n+1}) + (-1)^{n+1} f(x_1x_2,\dots,x_n)$$

The result $\delta\delta f = 0$ can be verified for the non-homogeneous case also. Consequently the same symbols $Z^n(\pi,N)$ and $B^n(\pi,N)$ can be used for the groups of cocycles and the coboundaries in non-homogeneous case.

Let us consider the four cases $n = 0,1,2,3$.

Case1. $n = 0$.

A 0-cochain by definition is an element $a \in N$. We have $(\delta f)(x_1) = x_1a_1 - a$. Hence f is a cocycle if and only if $xa = a$ for all $x \in \pi$; that is if and only if π operates simply on N. Since $B^0(\pi,N) = 0$ we have the theorem

The group $H^0(\pi,N)$ is the subgroup of those elements

a of N on which π operates simply; i.e. those a for which $xa = a$ for all $x \in \pi$. Notice that if π operates on N simply, then $H^0(\pi, N) = N$.

case 2. $n = 1$.

A 1-cochain $f \in C^1(\pi, N)$ is a function of one variable on to N and $(\delta f)(x_1 x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1)$. Therefore f is a 1-cocycle if and only if

$$f(x_1 x_2) = f(x_1) + x_1 f(x_2).$$

In order that $f \in B^1(\pi, N)$ we must have

$$f(x) = xa - a \quad \text{for some } a \in N.$$

Case 3. $n = 2$.

A 2-cochain $f \in C^2(\pi, N)$ is a function of two variables on π to N. We have

$$(\delta f)(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2)$$

Hence f is a cocycle if and only if

7.2. $$x_1 f(x_2, x_3) + f(x_1, x_2 x_3) = f(x_1, x_2) + f(x_1 x_2, x_3)$$

Before we proceed further let us compare this result with the factor sets (u,v) considered in section I. Let N be a normal abelian subgroup of some group G and $\pi = G/N$ be the factor group. If the coset $Nu_x = x$ is an element of π, then for a N, $u_x^{-1} a u_x$ depends only on x and a and not on the choice u_x in the coset. Hence we can write $u_x^{-1} a \, u_x = xa$. So is a group of operators on the left. Since N is an abelian additive group, the equation (11) in section 1 becomes

7.3 $(u, vw) + (v, w) = (uv, w) + w(u, v)$.

If we change the function $(u,v) = f(u,v)$ the above result reduces to the condition for a function to be a cocycle of dimension two.

If $f^{*}(u,v)$ and $f(u,v)$ are two equivalent factor systems then by

$$f^{*}(u,v) = -\alpha(uv) + f(u,v) + v\,\alpha(u) + \alpha(v)$$

i.e., $f^{*}(u,v) - f(u,v) = v\,\alpha(u) - \alpha(uv) + \alpha(v)$

i.e., $f^{*} - f = \delta\alpha$. Thus both factor sets f^{*} and f belong to the same cohomology class. Hence the group of extensions is the second cohomology group $H^2(\pi,N)$. We have the theorem:

Theorem 7.3.

The group of extensions of an abelian group N by a group π is the second cohomology group $H^2(\pi,N)$ where

- (i) π operates on the left to induce automorphisms in N
- (ii) Factor sets $f(u,v)$ are the cocycles in $Z^2(\pi,N)$.
- (iii) Equivalent factor sets differ by coboundaries of $B^2(\pi,N)$.

Case 4. $n = 3$

A 3-cochain f $C^3(\pi,N)$ is a function of three variables on π to N, where

$$(\delta f)(x_1,x_2,x_3,x_4) = x_1 f(x_2,x_3,x_4) - f(x_1x_2,x_3,x_4) + f(x_1,x_2x_3,x_4) - f(x_1,x_2,x_3x_4) + f(x_1,x_2,x_3)$$

Hence f is a cocycle if and only if

$$x_1 f(x_2,x_3,x_4) + f(x_1,x_2x_3,x_4) + f(x_1,x_2,x_3) = f(x_1x_2,x_3x_4) + f(x_1,x_2,x_3x_4)$$

Some interpretation of cohomology group $H^3(\pi,N)$ can be found in papers by Eilenberg and MacLane: Cohomology Theory in Abstract Groups II. Annals of Math. 48 (1947) pp 325 - 341.

8. NORMALIZED COCHAINS

An n-cochain $f \in C^n(\pi, N)$ (consider here only the non-homogeneous cochains) is said to be normalized if $f(x_1, x_2, \ldots x_n) = 0$ in N whenever anyone of the variables x is the unit element (say I) of π. Clearly the set of all normalized cochains form a subgroup $C'^n(\pi, N)$ of $C^n(\pi, N)$. The cocycles which are normalized form a subgroup $Z'^n(\pi, N)$ of $Z^n(\pi, N)$. Also the group $B'^n(\pi, N)$ of all coboundaries of normalized cochains form a subgroup of B^n.

Note: when $n = 2$, if we choose the identity as the representative of N in writing G as a sum of cosets of N leads to the normalization $f(1,1) = 0$. Hence putting $u = v = 1$ in 7.3

$$f(1,w) + f(1,w) = f(1,w) + wf(1,1) \quad \text{whence}$$

$$8.1 \qquad f(1,w) = 0$$

Similarly putting $v = w = 1$ we have

$$f(u,1) + f(1,1) = f(u,1) + f(u,1)$$

so $f(u,1) = 0$ showing that we are dealing with normalized cocycles.

Definition: Any cochain f will be said to be i-normalized for $i = 0, 1, \ldots, n$ if $f(x_1, x_2, \ldots x_n) = 0$ whenever one of the first i variables $x_1, x_2, \ldots, x_i$ is 1.

Clearly every cochain is zero-normalized and the normalized cochain defined originally is n-normalized cochain. Also the coboundary of an i-normalized cochain is i-normalized. For from formula 7.1 for the coboundary we have

$$
\begin{aligned}
(\delta f)(x_1,x_2,\ldots,x_{n+1}) &= x_1 f(x_2,\ \ldots,x_{n+1}) \\
&\quad \sum_{i=1}^{n}(-1)^{i}\ f(x_1,\ldots,x_i x_{i+1},\ldots,x_{n+1}) \\
&\quad +(-1)^{n+1} f(x_1,x_2,\ \ldots x_n) \\
&= x_1 f(x_2,\ldots,x_{n+1}) \\
&\quad - f(x_1 x_2,x_3,\ldots,x_{n+1}) \\
&\quad + f(x_1,x_2 x_3,\ldots,x_{n+1}) \\
&\quad \ldots \\
&\quad (-1)^{n+1} f(x_1,x_2,\ldots,x_{n+1})
\end{aligned}
$$

If $x_1 = 1$, all the terms except the first and the second become zero, since f is i-normalized. But in this case the first two terms are equal but opposite in sign and hence they cancel off. So $(\delta f)(x_1,\ldots,x_{n+1}) = 0$. If $i > 1$ we see that all the terms except two becoming zero and those two terms remaining will cancel.

Starting with any cochain $f \in C^n(\pi,N)$ let us construct cochains $f_o, f_1,\ldots,f_n \in C^n(\pi,N)$ and $g_1,g_2,\ldots,g_n \in C^{n-1}(\pi,N)$ by induction as

8.2 $\quad f_o = f$ and $f_i = f_{i-1} - \delta g_i$

8.3 $\quad g_i(x_1,x_2,\ldots,x_{n-1}) = (-1)^{i-1} f_{i-1}(x_1,x_2,\ldots,x_{i-1},1,x_i\ldots x_{n-1})$

$i = 1,2,\ \ ,n$

From 8.2 we have

$$
\begin{aligned}
\delta f_i &= \delta f_{i-1} - \delta^2 g_i \\
&= \delta f_{i-1} \quad i=1,\ldots,n
\end{aligned}
$$

$\delta f_i = \delta f_o = \delta f$ for all i.

Lemma 8.4. If δf is i-normalized then f_i is i-normalized.

Proof. When $i = 0$ the statement is trivial. Suppose the lemma to be true and consider it for $i+1$, it being necessary to that $f_{i+1}(x_1,x_2,\ldots x_i,1,x_{i+2},\ldots x_n) = 0)$

From the definition of f_{i+1} in 8.2

$$
\begin{aligned}
&f_{i+1}(x_1,x_2,\ldots x_i,1,x_{i+2},\ldots,x_n)\\
&\quad = f_i(x_1,\ldots,x_i,1,x_{i+2},\ldots,x_n) - (\delta g_{i+1})(x_1,\ldots x_i,1,x_{i+2},\ldots x_n)\\
&\quad = f_i(x_1,\ldots\ldots x_i,1,x_{i+2},\ldots x_n) - x_1 g_{i+1}(x_2,\ldots,x_i,1,x_{i+2},\ldots,x_n)\\
&\qquad - \sum_{j=1}^{i-1}(-1)^j g_{i+1}(x_1,\ldots,x_j x_{j+1},\ldots x_i,1,x_{i+2},\ldots x_n)\\
&\qquad -(-1)^i g_{i+1}(x_1,\ldots,x_{i-1},x_i 1,x_{i+2},\ldots,x_n)\\
&\qquad -(-1)^{i+1} g_{i+1}(x_1,\ldots,x_i,1 x_{i+2},\ldots,x_n)\\
&\qquad - \sum_{j=i+2}^{n-1}(-1)^j g_{i+1}(x_1,\ldots x_i,1,x_{i+2},\ldots,x_j x_{j+1},\ldots x_n)\\
&\qquad -(-1)^n g_{i+1}(x_1,x_2,\ldots,x_i,1,x_{i+2},\ldots,x_{n-1})
\end{aligned}
$$

But by induction f_i is i-normalized, so is g_{i+1} (by 8.3). So the first term in g_{i+1} and the first summation vanish by the i-normalization of g_{i+1}; the next two terms cancel. Now replacing g_{i+1} in terms of f_i as defined in 8.3

$$
\begin{aligned}
f_{i+1}(x_1,x_2,\ldots,x_i,1,x_{i+2},\ldots,x_n) &= f_i(x_1,\ldots,x_i,1,x_{i+2},\ldots,x_n)\\
&\quad (-1)^i \sum_{j=i+2}^{n-1}(-1)^{j+1} f_i(x_1,\ldots,x_i,1,1,x_{i+2},\ldots,x_j x_{j+1},\ldots,x_n)\\
&\quad (-1)^{i+n+1} f_i(x_1,\ldots,x_i,1,1,x_{i+2},\ldots,x_{n-1})
\end{aligned}
$$

$$= (A)\begin{cases} +(-1)^i x_1 f_i(x_2,\ldots,x_i,1,1,x_{i+2},\ldots,x_n) \\ +(-1)^i \sum_{j=1}^{i-1} (-1)^j f_i(x_1,\ldots,x_jx_j \; ,xxx,x_i,1,1,x_{i+2},\ldots,x_n) \end{cases}$$

$$+(-1)^i(-1)^i f_i(x_1,\ldots,x_i 1,1,x_{i+2},\ldots,x_{n-1})$$

$$(B)\begin{cases} +(-1)^i(-1)^{i\ 1} f_i(x_1,\ldots,x_i,1,x_{i+2},\ldots,x_n) \\ +(-1)^i(-1)^{i+2} f_i(x_1,\ldots,x_i,1,1x_{i+2},\ldots,x_n) \end{cases}$$

$$\sum_{j=i+2}^{n-1} (-1)^{j\ 1} f_i(x_1,\ldots,x_i,1,x_{i+2},\ldots,x_jx_{j+1},\ldots x_n)$$

$$+(-1)^i(-1)^{n+1} f_i(x_1,x_2, \ldots x_i,1,1,x_{i+2},\ldots,x_{n-1})$$

$$\pm(-1)^i \delta f_i(x_1,\ldots,x_i,1,1,x_{i+2},x\ldots x_n)$$

The terms added (A) will vanish by the assumed normalization of f_i and (B) will cancel. But $\delta f_i = \delta f$ is $(i+1)$ normalized, whence $(-1)^i \delta f_i \; (x_1,xx,x_i,1,1,x_{i+2},\ldots x_n) = 0$. So $f_{i+1}(x_1,x_2,\ldots,x_i,1,x_{i+2},\ldots,x_n) = 0$. The lemma is proved.

Lemma 8.5. Every (unnormalized) cocycle is cohomologous to a normalized cocycle.

Lemma 8.6 If the coboundary of some chain is normalized, then it is the coboundary of a normalized cochain.

The proof of these two lemmas follows from lemma 8.4. For lemma 8.5. If f is a cocycle then $\delta f = 0$, which is trivially normalized. Hence from lemma 8.4 f_n is normalized. But from 8.2 f_n is cohomologous to $f_o = f$. Hence the result. For lemma 8.6. Suppose f

is any n-dimensional cochain with a normalized coboundary δf. Since $f = f_0$ is zero normalized, it follows from lemma 8.4 that f_n is normalized. But $\delta f_n = \delta f = g \in C^{n+1}$(Hence the result.

Finally we shall prove the theorem.

Theorem 8.7. The cohomology groups $H^n(\Pi, N)$ of every dimension n for unnormalized cochains are isomorphic to those for normalized cochains.

Proof: For n = 0, we have $B'^0 = B^0 = 0)$ and $Z'^0 = Z^0$
For $n = 1$, if $f(u) \in Z'$ then $uf(v) - f(uv) + f(u)v = 0$ hence putting $u = v = 1$, we have $f(1) = 0$, so $f(u)$ is normalized which means $Z^1 = Z'^1$. Also $B^1 = B'^1$. Therefore $H^0(\Pi, N)$ and $H^1(\Pi, N)$ are the same in both cases. Suppose $n > 1$. Clearly $B'^n \subseteq B^n$ and $Z'^n \subseteq Z^n$. Hence a cohomology class for C'^n, i.e. a coset of B'^n in C'^n corresponds to a unique cohomology class for C^n, namely the coset of B^n in C^n. This correspondence is, of course, a homomorphism of $H'^n(\Pi, N)$ into $H^n(\Pi, N)$. The isomorphism follows from lemma 8.5 and lemma 8.6.

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