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# Convergence of long-memory discrete $k$ -th order Volterra processes

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## Abstract

We obtain limit theorems for a class of nonlinear discrete-time processes  $X(n)$  called the  $k$ -th order Volterra processes of order  $k$ . These are moving average  $k$ -th order polynomial forms:

$$X(n) = \sum_{0 < i_1, \dots, i_k < \infty} a(i_1, \dots, i_k) \epsilon_{n-i_1} \dots \epsilon_{n-i_k},$$

where  $\{\epsilon_i\}$  is i.i.d. with  $\mathbb{E}\epsilon_i = 0$ ,  $\mathbb{E}\epsilon_i^2 = 1$ , where  $a(\cdot)$  is a nonrandom coefficient, and where the diagonals are included in the summation. We specify conditions for  $X(n)$  to be well-defined in  $L^2(\Omega)$ , and focus on central and non-central limit theorems. We show that normalized partial sums of centered  $X(n)$  obey the central limit theorem if  $a(\cdot)$  decays fast enough so that  $X(n)$  has short memory. We prove a non-central limit theorem if, on the other hand,  $a(\cdot)$  is asymptotically some slowly decaying homogeneous function so that  $X(n)$  has long memory. In the non-central case the limit is a linear combination of Hermite-type processes of different orders. This linear combination can be expressed as a centered multiple Wiener-Stratonovich integral.

## 1 Introduction

A common assumption when analyzing a stationary time series  $\{X(n), n \in \mathbb{Z}\}$ , is that  $\{X(n)\}$  is a causal linear process, that is,

$$X_1(n) = \sum_{i=1}^{\infty} a_i \epsilon_{n-i}, \quad (1)$$

where  $\{\epsilon_i\}$  is a sequence of i.i.d. random variables with mean 0 and variance 1. This assumption is based on the Wold's decomposition, which states that if  $\{X(n)\}$  is stationary with mean 0 and finite second moment, and is also purely non-deterministic, then the representation (1) always holds with  $\{\epsilon_i\}$  a sequence of uncorrelated random variables (Brockwell and Davis [5] §5.7). The independence assumption of  $\{\epsilon_i\}$  in (1) obliterates the higher-order dependence structure. In some applications, linear processes provide good approximations, while in others, not, as in the case of the ARCH model for volatility data.

The Volterra process extends linear process by incorporating non-linearity. A (causal) Volterra process with highest order  $K$  is of the form

$$X_K(n) = \sum_{k=1}^K \sum_{0 < i_1, \dots, i_k < \infty} a_k(i_1, \dots, i_k) \epsilon_{n-i_1} \dots \epsilon_{n-i_k}. \quad (2)$$

To understand the importance of (2), suppose that the stationary process is  $X(n) = A(\epsilon_{n-1}, \epsilon_{n-2}, \dots)$  for some regular function  $A$ . Then (2) can be heuristically regarded as its  $K$ -th order Taylor series approximation.

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**Key words** Long memory; Long-range dependence; Volterra process; Wiener chaos; Wiener; Stratonovich; Limit theorems

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The homogeneous polynomial-form expansion in (2) and its continuous-time counterpart where the sums are replaced with integrals, was originally proposed by Vito Volterra (see Volterra [22]) for modeling deterministic nonlinear systems, and later extended by Norbert Wiener (see Wiener [23]) to random systems, which eventually lead to the well-developed theory of Wiener chaos (see, e.g., Cameron and Martin [8], Itô [14], and the recent survey Peccati and Taqqu [19]). In the context of approximation of stationary processes, Nisio [18] shows that any stationary process can be approximated in the sense of finite-dimensional distributions by a Volterra process with  $\epsilon_i$ 's Gaussian. Some nonlinear time series models admit Volterra expansions (2) with  $K = \infty$ . For example, the LARCH( $\infty$ ) model

$$X(n) = a + \sum_{i=1}^{\infty} b_i Y(n-i), \quad Y(n) = X(n)\epsilon_n,$$

under suitable conditions admits the following Volterra expansion (see, e.g., Theorem 2.1 of Giraitis et al. [12]):

$$X(n) = a \left( 1 + \sum_{k=1}^{\infty} \sum_{0 < i_1, \dots, i_k < \infty} b_{i_1} b_{i_2 - i_1} b_{i_3 - i_2} \dots b_{i_k - i_{k-1}} 1_{\{i_1 < i_2 < \dots < i_k\}} \epsilon_{n-i_1} \epsilon_{n-i_2} \dots \epsilon_{n-i_k} \right).$$

We are interested here in stationary processes that have *long memory*, or *long-range dependence*. A common choice is a linear process in (1) with  $a_1(n) \sim cn^{d-1}$  as  $n \rightarrow \infty$ , where  $d \in (0, 1/2)$  is the *memory parameter*, and  $c > 0$  is some constant. This is the case, for instance, when  $X(n)$  is the stationary solution of the fractional difference equation

$$\Delta^d X(n) = \epsilon_{n-1},$$

where  $\Delta = I - B$  is the difference operator with  $I$  being identity operator and  $B$  being the backward shift operator, and  $(I - B)^d$  is understood as a binomial series (see, e.g., Giraitis et al. [11] Chapter 7.2). We note that such long-memory linear processes have an autocovariance decaying like  $n^{2d-1}$  as  $n \rightarrow \infty$ , and a spectral density exploding at the origin as  $|\lambda|^{-2d}$  as  $|\lambda| \rightarrow 0$ .

If one wants to consider a nonlinear long memory model, a natural choice is to have a Volterra process (2) with coefficients  $a_k(i_1, \dots, i_k)$  decaying slowly as  $i_1, \dots, i_k$  tends to infinity, so that the autocovariance has a slow hyperbolic decay. The major goal in this paper is to study the limit of normalized partial sum of some long-memory Volterra processes. When  $X(n)$  is a long-memory linear process, that is, a long-memory Volterra process with  $K = 1$ , then the limit, as is well-known, is fractional Brownian motion (Davydov [9]). When  $X(n)$  is polynomial of a long memory linear processes, that is, when  $a_k(i_1, \dots, i_k) = c_k i_1^{d-1} \dots i_k^{d-1}$  in (2) for some constant  $c_k$ , and  $d$  is large enough, then the limit is a Hermite process of a fixed order (Surgailis [20], Avram and Taqqu [1]). Such limit theorems involving non-Brownian motion limits are often called *non-central limit theorems*.

In this paper, we focus on Volterra processes of a single order  $k \geq 1$ :

$$X(n) = \sum_{0 < i_1, \dots, i_k < \infty} a(i_1, \dots, i_k) \epsilon_{n-i_1} \dots \epsilon_{n-i_k}, \quad (3)$$

which avoids possible cancellations between terms of different orders. Note that the multiple sum (3) *includes* diagonals, that is, it allows  $i_1, \dots, i_k$  to be equal to each other. In the literature, one often considers multiple sums of the type (3) where summation over the diagonals is *excluded*, which greatly simplifies the theory. Although the exclusion of the diagonals is a typical theoretical assumption, it is, from a practical perspective, an artificial one. Expression (3) is the natural one since it includes all the terms.

To obtain a non-central limit theorem for (3), we assume that the coefficient  $a(i_1, \dots, i_k)$  behaves asymptotically as a homogeneous function  $g$  on  $\mathbb{R}_+^k$  which is bounded excluding a neighborhood of the origin. We shall show that in this case, the limit of a normalized sum of centered  $X(n)$  is a linear combination of Hermite-type processes of different orders. These Hermite-type processes that appear in the limit were first introduced in Mori and Oodaira [17], and were called in Bai and Taqqu [2] *generalized Hermite processes*.

They live in Wiener chaos, and extend in a natural way the usual Hermite processes considered in the literature, e.g., Dobrushin and Major [10] and Taqqu [21].

The limit, which is a linear combination involving different orders of multiple Wiener-Itô integrals, can be re-expressed as a single *centered multiple Wiener-Stratonovich integral* with the zeroth-order term excluded. These integrals were introduced by Hu and Meyer [13]. Loosely speaking, in contrast to the usual Wiener-Itô integrals, the multiple Wiener-Stratonovich integrals include diagonals, and intuitively they are the continuous counterpart of the multiple sums in (3) which, as was noted, do include diagonals.

The paper is organized as follows. In Section 2, we introduce the generalized Hermite processes which appear in the formulation of the non-central limit theorem. In Section 3, we provide conditions for the polynomial form (3) to be well-defined in  $L^2(\Omega)$ . In Section 4, we introduce the class of long-memory Volterra processes  $X(n)$  of interest in the non-central limit theorem. In Section 5, we establish central limit theorems when  $a(\cdot)$  in (3) decays fast enough so that  $X(n)$  has short memory. In Section 6, we state a non-central limit theorem for processes  $X(n)$  in (3). Before launching into the article, the reader may want to have a look at this result, formulated as Theorem 6.2, and also at the illustrative Example 6.4. The connection between the limit and multiple Wiener-Stratonovich integrals is indicated in Section 7. Section 8 contains an extended hypercontractivity formula.

## 2 Generalized Hermite processes and kernels

We introduce here the kernels which will be used to define both the coefficient  $a(\cdot)$  in (3), and the processes that will appear in the non-central limit.

First, some notation which will be used throughout the paper. Let  $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ ,  $\mathbf{i} = (i_1, \dots, i_k) \in \mathbb{Z}^k$ ,  $\mathbf{0} = (0, \dots, 0)$ ,  $\mathbf{1} = (1, \dots, 1)$ , and let  $\mathbf{1}_r$  denote the vector made of  $r$  1's. If  $x \in \mathbb{R}$ , then  $[x] = \sup\{n \in \mathbb{Z}, n \leq x\}$ , and  $[\mathbf{x}] = ([x_1], \dots, [x_k])$ . We write  $\mathbf{x} > \mathbf{y}$  if  $x_j > y_j, j = 1, \dots, k$ , and use the following standard notations:  $\|\cdot\|$  denotes a norm in some suitable space,  $1_A(\cdot)$  is the indicator function of a set  $A$ ,  $|A|$  denotes the cardinality of set  $A$ , and if  $g_1$  and  $g_2$  are two functions on  $\mathbb{R}^{k_1}$  and  $\mathbb{R}^{k_2}$  respectively, then  $g_1 \otimes g_2$  defines a scalar function on  $\mathbb{R}^{k_1+k_2}$  as  $(\mathbf{x}_1, \mathbf{x}_2) \rightarrow g_1(\mathbf{x}_1)g_2(\mathbf{x}_2)$ .

The following class of functions was introduced in Bai and Taqqu [2]:

**Definition 2.1.** A *generalized Hermite kernel* (GHK)  $g$  is a nonzero measurable function defined on  $\mathbb{R}_+^k$  satisfying:

1.  $g(\lambda \mathbf{x}) = \lambda^\alpha g(\mathbf{x}), \forall \lambda > 0, \alpha \in (-\frac{k+1}{2}, -\frac{k}{2})$ ;
2.  $\int_{\mathbb{R}_+^k} |g(\mathbf{x})g(\mathbf{1} + \mathbf{x})| d\mathbf{x} < \infty$ .

**Remark 2.2.** As shown in Theorem 3.5 and Remark 3.6 in Bai and Taqqu [2], if  $g(\cdot)$  is a GHK on  $\mathbb{R}^k$ , then for every  $t > 0$ ,

$$\int_0^t |g(s\mathbf{1} - \mathbf{y})| 1_{\{s\mathbf{1} > \mathbf{y}\}} ds < \infty$$

for a.e.  $\mathbf{y} \in \mathbb{R}^k$ . Furthermore,

$$h_t(\mathbf{y}) = \int_0^t g(s\mathbf{1} - \mathbf{y}) 1_{\{s\mathbf{1} > \mathbf{y}\}} ds$$

is a.e. defined, and  $h_t \in L^2(\mathbb{R}^k)$ . In addition, if  $g$  is nonzero, then  $\int_{\mathbb{R}_+^k} g(\mathbf{1} + \mathbf{x})g(\mathbf{x}) d\mathbf{x} > 0$ .

These functions  $g$  were used in Bai and Taqqu [2] as defining kernels for a class of stochastic processes called *generalized Hermite processes*.

**Definition 2.3.** The *generalized Hermite processes* are defined through the following multiple Wiener-Itô integrals:

$$Z(t) = I_k(h_t) := \int_0^t \int_{\mathbb{R}^k} g(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds B(dx_1) \dots B(dx_k) \quad (4)$$

where the prime ' indicates that one does not integrate on the diagonals  $x_p = x_q$ ,  $p \neq q \in \{1, \dots, k\}$ ,  $B(\cdot)$  is a Brownian random measure, and  $g$  is a GHK defined in Definition 2.1.

The generalized Hermite processes are self-similar with Hurst exponent

$$H = \alpha + k/2 + 1 \in (1/2, 1), \quad (5)$$

that is,  $\{Z(\lambda t), t > 0\}$  has the same finite-dimensional distributions as  $\{\lambda^H Z(t), t > 0\}$ , and they have also stationary increments.

**Example 2.4.** When  $g$  takes the particular form  $g(\mathbf{x}) = \prod_{j=1}^k x_j^{\alpha/k}$  where  $\alpha/k \in (-\frac{1}{2}(1 + \frac{1}{k}), -\frac{1}{2})$ ,  $Z(t)$  becomes the usual Hermite process obtained through a non-central limit theorem in the context of long memory (e.g., Taqqu [21], Dobrushin and Major [10], Surgailis [20]).

In Bai and Taqqu [2] the following subclass of functions  $g$ , called *generalized Hermite kernel of Class (B)* was considered.

**Definition 2.5.** We say that a nonzero homogeneous function  $g$  on  $\mathbb{R}_+^k$  having homogeneity exponent  $\alpha$  is of Class (B) (abbreviated as ‘‘GHK(B)’’, ‘‘B’’ stands for ‘‘boundedness’’), if

1.  $g$  is a.e. continuous on  $\mathbb{R}_+^k$ ;
2.  $|g(\mathbf{x})| \leq C \|\mathbf{x}\|^\alpha$  for some constant  $C > 0$ , where  $\alpha$  is as in Definition 2.1.

**Remark 2.6.** The norm  $\|\cdot\|$  in Definition 2.5 can be any norm in the finite-dimensional space  $\mathbb{R}^k$  since all the norms are equivalent. For convenience, we choose throughout this paper  $\|\mathbf{x}\| = \sum_{j=1}^k |x_j|$ . The GHK(B) class is a subset of the GHK class, because if  $g$  is a GHK(B), then it is homogeneous and hence satisfies Condition 1 of Definition 2.1. It also satisfies Condition 2 of Definition 2.1. Indeed, we have for some  $C, C' > 0$  that

$$|g(\mathbf{x})| \leq C \|\mathbf{x}\|^\alpha = C \left( \sum_{j=1}^k x_j \right)^\alpha \leq C' \prod_{j=1}^k x_j^{\alpha/k}, \quad \mathbf{x} \in \mathbb{R}_+^k, \quad (6)$$

where the last inequality follows from the arithmetic-geometric mean inequality

$$k^{-1} \sum_{j=1}^k y_j \geq \left( \prod_{j=1}^k y_j \right)^{1/k} \quad \text{and} \quad \alpha < 0.$$

In view of Condition 1 of Definition 2.1, since  $-1 \leq -1/2 - 1/(2k) < \alpha/k < -1/2$ , we hence have

$$\int_{\mathbb{R}_+^k} |g(\mathbf{x})g(\mathbf{1} + \mathbf{x})| d\mathbf{x} \leq C' \left( \int_0^\infty x^{\alpha/k} (1+x)^{\alpha/k} dx \right)^k < \infty.$$

**Example 2.7.** As an example of a GHK(B), we can simply set  $g(\mathbf{x})$  equal to

$$g_1(\mathbf{x}) = \|\mathbf{x}\|^\alpha = |x_1 + \dots + x_k|^\alpha = (x_1 + \dots + x_k)^\alpha, \quad \alpha \in \left(-\frac{k+1}{2}, -\frac{k}{2}\right),$$

since  $\mathbf{x} \in \mathbb{R}_+^k$ .

**Example 2.8.** As another example, consider

$$g_2(\mathbf{x}) = \prod_{j=1}^k x_j^{a_j} / \left( \sum_{j=1}^k x_j^b \right), \quad a_j > 0, \quad b > 0,$$

and

$$\sum_{j=1}^k a_j - b \in \left(-\frac{k+1}{2}, -\frac{k}{2}\right).$$

$g_2$  is continuous and homogeneous with exponent  $\alpha = \sum_{j=1}^k a_j - b$ . It is a GHK(B) because the functions  $\mathbf{x} \rightarrow \prod_{j=1}^k x_j^{a_j}$  and  $\mathbf{x} \rightarrow (\sum_{j=1}^k x_j^b)^{-1}$  are bounded on the  $k$ -dimensional unit sphere restricted to  $\mathbb{R}_+^k$ . For instance,

$$\left(\sum_{j=1}^k x_j^b\right)^{1/b} \leq C\|\mathbf{x}\|$$

by the equivalence of norms on  $\mathbb{R}^k$ . Thus  $g_2(\mathbf{x}) \leq C\|\mathbf{x}\|^\alpha$ .

**Example 2.9.** It is easy to see that the set of GHK(B) functions on  $\mathbb{R}_+^k$  with fixed homogeneity exponent  $\alpha$  (with the zero function added) is closed under linear combinations and taking maximum or minimum. Thus one can consider  $g_1 + g_2$ ,  $g_1 \vee g_2$  and  $g_1 \wedge g_2$  using the  $g_1$  and  $g_2$  in the foregoing examples.

In Bai and Taqqu [2], non-central limit theorems involving GHK(B) are established<sup>1</sup>. These theorems involve sums of a long-memory stationary process called *discrete chaos process* defined as

$$X'(n) = \sum'_{\mathbf{i} \in \mathbb{Z}_+^k} a(i_1, \dots, i_k) \epsilon_{n-i_1} \dots \epsilon_{n-i_k} = \sum'_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i}) \epsilon_{n-i_1} \dots \epsilon_{n-i_k}, \quad (7)$$

where  $a(\mathbf{i}) = g(\mathbf{i})L(\mathbf{i})$ ,  $g$  is a GHK(B),  $L$  is some asymptotically negligible function (see (25) and the lines below), and the prime ' means that we do not sum on the diagonals  $i_p = i_q$ ,  $p \neq q \in \{1, \dots, k\}$ , i.e., the summation in (7) is only over unequal  $i_1, \dots, i_k$ . We note that when  $a(\cdot)$  is symmetric, the autocovariance of  $X'(n)$  in (7) is

$$\gamma(n) = \mathbb{E}X'(n)X'(0) = k! \sum'_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i})a(\mathbf{i} + n\mathbf{1}), \quad n \geq 0.$$

**Remark 2.10.** The difference between the *discrete chaos process*  $X'(n)$  defined in (7) and the Volterra process  $X(n)$  in (3) is the exclusion of the diagonals.

### 3 $L^2(\Omega)$ -definiteness

In this section, we derive conditions under which a  $k$ -th order polynomial form with diagonals is well-defined.

The  $k$ -th order Volterra process in (3) is a polynomial form in i.i.d. random variables  $\{\epsilon_i\}$ . To allow for long memory and obtain non-central limit theorems, the coefficient  $a(\mathbf{i})$  in (3) must be nonzero at an infinite number of  $\mathbf{i} \in \mathbb{Z}_+^k$ . Otherwise  $X(n)$  is an  $m$ -dependent sequence and thus subject to the central limit theorem (Billingsley [4]). So the first problem is to ensure that such a polynomial form with an infinite number of terms is well-defined, that is, to determine when the following random variable is well-defined:

$$X = \sum_{0 < i_1, \dots, i_k < \infty} a(i_1, \dots, i_k) \epsilon_{i_1} \dots \epsilon_{i_k} = \sum_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i}) \epsilon_{i_1} \dots \epsilon_{i_k}, \quad (8)$$

where  $\{\epsilon_i\}$  is an i.i.d. sequence such that

$$\mathbb{E}\epsilon_i = 0, \quad \mathbb{E}\epsilon_i^2 = 1, \quad \mathbb{E}|\epsilon_i|^k < \infty. \quad (9)$$

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<sup>1</sup>In Bai and Taqqu [2], the non-central limit theorem is shown to hold for a larger class of functions which includes functions like  $g(\mathbf{x}) = \prod_{j=1}^k x_j^{\alpha/k}$ , called Class (L). We do not consider this class here, since the main result Theorem 6.2 below does not hold for Class (L) in general.

One can restrict  $a(\mathbf{i})$  to be a symmetric function in  $\mathbf{i}$ , since a permutation of the variables does not affect  $X$ , but we shall not do so unless indicated, because it is easier to write down non-symmetric  $a(\cdot)$ 's.

First, we have the following straightforward criterion for the  $L^1(\Omega)$ -well-definedness of  $X$ :

**Proposition 3.1.** *If  $\sum_{\mathbf{i} \in \mathbb{Z}_+^k} |a(\mathbf{i})| < \infty$ , then  $X$  in (8) is well-defined in the  $L^1(\Omega)$ -sense.*

*Proof.* Let

$$X_m = \sum_{\mathbf{0} < \mathbf{i} \leq m\mathbf{1}} a(i_1, \dots, i_k) \epsilon_{i_1} \dots \epsilon_{i_k}, \quad m > 0.$$

It suffices to check that  $X_m$  is a Cauchy sequence in  $L^1(\Omega)$ . This is true since for any  $n > m > 0$ ,

$$\mathbb{E}|X_m - X_n| \leq \sum_{m\mathbf{1} < \mathbf{i} \leq n\mathbf{1}} |a(\mathbf{i})| \mathbb{E}|\epsilon_{i_1} \dots \epsilon_{i_k}| \leq C \sum_{m\mathbf{1} < \mathbf{i} \leq n\mathbf{1}} |a(\mathbf{i})|,$$

where  $\mathbb{E}|\epsilon_{i_1} \dots \epsilon_{i_k}|$  is bounded above by a constant because of the assumption  $\mathbb{E}|\epsilon_i|^k < \infty$  in (9).  $\square$

The absolute summability assumption in Proposition 3.1 is easy to work with, but it is unfortunately too restrictive for incorporating long memory. We will introduce instead a condition on  $a(\mathbf{i})$  so that  $X$  is well-defined in the  $L^2(\Omega)$ -sense. Beside the obvious assumption  $\mathbb{E}\epsilon_i^{2k} < \infty$ , some delicate assumptions on  $a(\mathbf{i})$  need to be imposed, which are stated in Proposition 3.3 below. We first give an outline of the idea. If  $X$  in (8) is instead defined as an off-diagonal polynomial form:

$$X' = \sum'_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i}) \epsilon_{i_1} \dots \epsilon_{i_k}, \quad (10)$$

then due to the off-diagonality, it is easy to see that the  $L^2(\Omega)$ -well-definedness of  $X'$  is guaranteed by the simple square-summability condition:

$$\sum'_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i})^2 < \infty,$$

which equals  $(k!)^{-1} \mathbb{E}X'^2$  if  $a(\cdot)$  is symmetric. In fact, this  $L^2(\Omega)$ -defineness criterion still holds if one has more generally

$$X' = \sum'_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i}) \epsilon_{i_1}^{(1)} \dots \epsilon_{i_k}^{(k)}, \quad (11)$$

where  $\{\epsilon_i := (\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)}), i \in \mathbb{Z}\}$  forms an i.i.d. sequence of  $k$ -dimensional vector with mean 0 and finite variance in each component. We will need this fact below.

In order to check that the polynomial-form in (8), which includes diagonals, is well-defined, we shall decompose it into a finite number of off-diagonal polynomial forms, and check the well-definedness of each using the simple square-summability condition. In order to do this, we introduce some further notation, which will also be useful in the sequel.

We let  $\mathcal{P}_k$  denote all the partitions of  $\{1, \dots, k\}$ . If  $\pi \in \mathcal{P}_k$ , then  $|\pi|$  denotes the number of sets in the partition. If we have a variable  $\mathbf{i} \in \mathbb{Z}_+^k$ , then  $\mathbf{i}_\pi$  denotes a new variable where its components are identified according to  $\pi$ . For example, if  $k = 3$ ,  $\pi = \{\{1, 2\}, \{3\}\}$  and  $\mathbf{i} = (i_1, i_2, i_3)$ , then  $\mathbf{i}_\pi = (i_1, i_1, i_2)$ . In this case we write  $\pi = \{P_1, P_2\}$  where  $P_1 = \{1, 2\}$  and  $P_2 = \{3\}$ . If  $a(\cdot)$  is a function on  $\mathbb{Z}_+^k$ , then  $a_\pi(i_1, \dots, i_m) := a(\mathbf{i}_\pi)$ , where  $m = |\pi|$ . In the preceding example,  $a_\pi(\mathbf{i}) = a(i_1, i_1, i_2)$  with  $m = 2$ .

Suppose that  $\pi = \{P_1, \dots, P_{|\pi|}\}$ , where  $P_i \cap P_j = \emptyset$ ,  $\cup_i P_i = \{1, \dots, k\}$ . We suppose throughout that the  $P_i$ 's are ordered according to their smallest element. In the preceding example,  $P_1 = \{1, 2\}$  and  $P_2 = \{3\}$ . We define the following summation operation on a function  $a(\cdot)$  on  $\mathbb{Z}_+^k$ .

**Definition 3.2.** For any  $T \subset \{1, \dots, |\pi|\}$ , the summation  $S'_T(a_\pi)$  is obtained by summing  $a_\pi$  over its variables indicated by  $T$  off-diagonally, yielding a function with  $|\pi| - |T|$  variables.

For instance, if  $\pi = \{\{1, 5\}, \{2\}, \{3, 4\}\}$ , then  $\mathbf{i}_\pi = (i_1, i_2, i_3, i_3, i_1)$  and if  $T = \{1, 3\}$ , then

$$(S'_T a_\pi)(i) = \sum'_{i_1, i_3} a(i_1, i, i_3, i_3, i_1), \quad (12)$$

provided that it is well-defined. Note that in this off-diagonal sum, we require, in addition to  $i_1 \neq i_3$ , that neither  $i_1$  nor  $i_3$  equals to  $i$ . If  $T = \emptyset$ ,  $S'_T$  is understood to be the identity operator, where no summation is performed.

We need also Appell polynomials which we briefly introduce here. For more details, see, e.g. Avram and Taqqu [1] or Chapter 3.3 of Beran et al. [3]. Given a random variable  $\epsilon$  with  $\mathbb{E}|\epsilon|^K < \infty$ , the  $k$ -th order Appell polynomial with respect to the law of  $\epsilon$ , is defined through the following recursive relation:

$$\frac{d}{dx} A_p(x) = p A_{p-1}(x), \quad \mathbb{E} A_p(\epsilon) = 0, \quad A_0(x) = 1, \quad p = 1, \dots, K.$$

For example, if  $\mu_p = \mathbb{E}\epsilon^p$ , then  $A_1(x) = x - \mu_1$ ,  $A_2(x) = x^2 - 2\mu_1 x + 2\mu_1^2 - \mu_2$ , etc. If in addition  $\mu_1 = 0$ , then  $A_1(x) = x$ , and  $A_2(x) = x^2 - \mu_2$ . For consistency, one sets  $\mu_0 = \mathbb{E}\epsilon^0 = 1$ . We will use an important property of Appell polynomials, namely, for any integer  $p \geq 0$ ,

$$x^p = \sum_{j=0}^p \binom{p}{j} \mu_{p-j} A_j(x). \quad (13)$$

**Proposition 3.3.** *The polynomial form  $X$  in (8) is a random variable defined in the  $L^2(\Omega)$ -sense, if the following three conditions hold:*

1.  $\mathbb{E}\epsilon_i^{2k} < \infty$ ;
2.  $a(\cdot)$  satisfies the following: for any  $\pi = \{P_1, \dots, P_{|\pi|}\} \in \mathcal{P}_k$ , we have

$$\sum'_{0 < i_1, \dots, i_{|\pi|} < \infty} a_\pi(i_1, \dots, i_{|\pi|})^2 < \infty; \quad (14)$$

3. for any  $\pi \in \mathcal{P}_k$  and any nonempty  $T \subset \{1, \dots, |\pi|\}$  satisfying  $|P_t| \geq 2$  for all  $t \in T$ , we have

$$\sum'_{0 < i_1, \dots, i_{|\pi|-|T|} < \infty} [(S'_T |a_\pi|)(i_1, \dots, i_{|\pi|-|T|})]^2 < \infty, \quad (15)$$

where if  $|T| = |\pi|$ , (15) is understood as merely stating that the sum  $S'_T |a_\pi|$  converges.

**Remark 3.4.** To understand the need for (14) and (15), note that, in order to use the  $L^2(\Omega)$ -definiteness of (11), it is necessary to center the powers of  $\epsilon_i$ . For example, consider

$$X = \sum_{i_1, i_2, i_3 > 0} a(i_1, i_2, i_3) \epsilon_{i_1} \epsilon_{i_2} \epsilon_{i_3}.$$

If we focus on the subset  $\{i_1 = i_2 \neq i_3\}$ , then we have

$$\begin{aligned} \sum'_{i_1, i_2 > 0} a(i_1, i_1, i_2) \epsilon_{i_1}^2 \epsilon_{i_2} &= \sum'_{i_1, i_2 > 0} a(i_1, i_1, i_2) (\epsilon_{i_1}^2 - \mu_2) \epsilon_{i_2} + \mu_2 \sum_{i_2 > 0} \sum_{i_1 \neq i_2 > 0} a(i_1, i_1, i_2) \epsilon_{i_2} \\ &= \sum'_{i_1, i_2 > 0} a(i_1, i_1, i_2) A_2(\epsilon_{i_1}) A_1(\epsilon_{i_2}) + \mu_2 \sum_{i_2 > 0} \sum_{i_1 \neq i_2 > 0} a(i_1, i_1, i_2) A_1(\epsilon_{i_2}), \end{aligned}$$



where  $\mu_2 = \mathbb{E}\epsilon_i^2$ . For the preceding two terms to be well-defined in  $L^2(\Omega)$ , we require respectively

$$\sum'_{i_1, i_2 > 0} a(i_1, i_1, i_2)^2 < \infty$$

and

$$\sum_{i_2 > 0} \left( \sum_{i_1 \neq i_2 > 0} a(i_1, i_1, i_2) \right)^2 = \sum_{i_2 > 0} [(S'_T a_\pi)(i_2)]^2 < \infty, \quad \pi = \{\{1, 2\}, \{3\}\}, \quad T = \{1\}.$$

An example of  $a(\cdot)$  satisfying (14) but not (15) is given by:

$$a(i_1, i_2) = (i_1 + i_2)^{-1} (\log i_2)^{-1}.$$

Note that  $a(i_1, i_2)^2 = (i_1 + i_2)^{-2} (\log i_2)^{-2}$  is summable because  $\sum_{i_2=2}^{\infty} i_2^{-1} (\log i_2)^{-2}$  is finite by the integral test, while  $a(i, i) = \frac{1}{2} i^{-1} (\log i)^{-1}$  is not summable.

*Proof of Proposition 3.3.* By collecting various diagonal cases, we express  $X$  as

$$X = \sum_{\pi \in \mathcal{P}_k} \sum'_{0 < i_1, \dots, i_m < \infty} a_\pi(i_1, \dots, i_m) \epsilon_{i_1}^{p_1} \dots \epsilon_{i_m}^{p_m}, \quad (16)$$

where  $\pi = \{P_1, \dots, P_{|\pi|}\} \in \mathcal{P}_k$ ,  $m = |\pi|$ ,  $p_j = |P_j| \geq 1$ ,  $j = 1, \dots, m$ ,  $p_1 + \dots + p_m = k$ . Since  $\mathcal{P}_k$  is finite, one can focus on the  $L^2(\Omega)$ -definedness of each term

$$X_\pi := \sum'_{0 < i_1, \dots, i_m < \infty} a_\pi(i_1, \dots, i_m) \epsilon_{i_1}^{p_1} \dots \epsilon_{i_m}^{p_m}.$$

Let  $A_j(x)$  be the  $j$ -th order Appell polynomial with respect to the law of  $\epsilon_i$ . Let

$$\mu_j = \mathbb{E}\epsilon_i^j.$$

Then by (13),

$$\epsilon_{i_1}^{p_1} \dots \epsilon_{i_m}^{p_m} = \sum_{j_1=0}^{p_1} \dots \sum_{j_m=0}^{p_m} \binom{p_1}{j_1} \dots \binom{p_m}{j_m} \mu_{p_1-j_1} \dots \mu_{p_m-j_m} A_{j_1}(\epsilon_{i_1}) \dots A_{j_m}(\epsilon_{i_m}).$$

Thus to ensure  $X_\pi \in L^2(\Omega)$ , it suffices to show that

$$X_\pi^{\mathbf{j}} := \sum'_{0 < i_1, \dots, i_m < \infty} a_\pi(i_1, \dots, i_m) \binom{p_1}{j_1} \dots \binom{p_m}{j_m} \mu_{p_1-j_1} \dots \mu_{p_m-j_m} A_{j_1}(\epsilon_{i_1}) \dots A_{j_m}(\epsilon_{i_m}) \quad (17)$$

is well-defined in  $L^2(\Omega)$  for any  $(j_1, \dots, j_m) \in \{0, 1, \dots, p_1\} \times \dots \times \{0, 1, \dots, p_m\}$ .

Note now the following crucial fact. Since  $\mu_1 = \mathbb{E}\epsilon_i = 0$  by assumption, we do not need to consider  $p_t - j_t = 1$  in (17). Thus:

$$\text{If } j_t = 0, \text{ then we need to consider only } p_t = |P_t| \geq 2. \quad (18)$$

Suppose first that  $j_1, \dots, j_m \geq 1$ . Since by assumption  $\mathbb{E}A_j(\epsilon_i) = 0$  and  $\mathbb{E}A_j(\epsilon_i)^2 < \infty$  for  $1 \leq j \leq k$ , then in view of the discussion concerning (11), it is sufficient to require (14). Now suppose that some  $j_t = 0$ , and observe that  $A_{j_t}(\epsilon_i)$  is then the constant 1. Thus if  $T$  is the set of  $t$ 's such that  $j_t = 0$ , then

$$X_\pi^{\mathbf{j}} = \sum'_{0 < i_{t_1}, \dots, i_{t_{m-r}} < \infty} (S'_T a_\pi)(i_{t_1}, \dots, i_{t_{m-r}}) c(\mathbf{p}, \mathbf{j}) A_{j_{t_1}}(\epsilon_{i_{t_1}}) \dots A_{j_{t_{m-r}}}(\epsilon_{i_{t_{m-r}}}), \quad (19)$$

where  $\{t_1, \dots, t_{m-r}\} = \{1, \dots, m\} \setminus T$ ,  $r = |T|$  and

$$c(\mathbf{p}, \mathbf{j}) = \binom{p_1}{j_1} \cdots \binom{p_m}{j_m} \mu_{p_1-j_1} \cdots \mu_{p_m-j_m}. \quad (20)$$

So one can bound  $\mathbb{E}(X_\pi^{\mathbf{j}})^2$  by a constant times the sum in (15) since (19) has the form (11).  $\square$

**Remark 3.5.** Since  $\mathbb{E}A_j(\epsilon_i) = 0$  for  $j \geq 1$ , one can see from (17) that  $\mathbb{E}X_\pi^{\mathbf{j}} \neq 0$  only when  $j_1 = \dots = j_m = 0$ , which implies

$$\mathbb{E}X = \sum_{\pi \in \mathcal{P}_k} \sum_{\mathbf{i} \in \mathbb{Z}_+^m} a_\pi(\mathbf{i}) \mu_{p_1} \cdots \mu_{p_m}. \quad (21)$$

Relation (15) with  $|T| = |\pi|$  ensures that  $\mathbb{E}|X| < \infty$ .

We now state here a practical sufficient condition for Proposition 3.3:

**Proposition 3.6.** *Let  $a(\cdot)$  be a function on  $\mathbb{Z}_+^k$  such that*

$$|a(\mathbf{i})| \leq c \prod_{j=1}^k i_j^{\gamma_j},$$

where  $c > 0$  is some constant and  $\gamma_j < -1/2$ ,  $j = 1, \dots, k$ . Then

$$X := \sum_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i}) \epsilon_{i_1} \cdots \epsilon_{i_k}$$

is a well-defined random variable in  $L^2(\Omega)$ , where  $\{\epsilon_i\}$  is i.i.d. with mean 0 and variance 1 and  $\mathbb{E}|\epsilon_i|^{2k} < \infty$ .

*Proof.* We set  $m = |\pi|$ . Relation (14) holds because

$$|a_\pi(\mathbf{i})| \leq c_1 \prod_{j=1}^m i_j^{\beta_j} \quad (22)$$

for some  $c_1 > 0$ , where  $\beta_j \leq \gamma_j < -1/2$ , so

$$\sum_{0 < i_1, \dots, i_m < \infty} a_\pi(i_1, \dots, i_m)^2 \leq c_1^2 \sum_{0 < i_1, \dots, i_m < \infty} i_1^{2\beta_1} \cdots i_m^{2\beta_m} < \infty.$$

To check (15), note that when  $t \in T$ , we have  $|P_t| \geq 2$  by (18), and so we have in addition  $\beta_{j_t} \leq 2\gamma_{j_t} < -1$  in (22). Thus for some finite  $c_2, c_3 > 0$ ,

$$\left[ S'_T |a_\pi(\mathbf{i})| \right]^2 \leq c_2 \left[ \sum_{i_t, t \in T} \left( \prod_{j_t, t \in T} i_{j_t}^{\beta_{j_t}} \right) \right]^2 \left( \prod_{j_s, s \notin T} i_{j_s}^{\beta_{j_s}} \right)^2 = c_3 \left( \prod_{j_s, s \notin T} i_{j_s}^{2\beta_{j_s}} \right),$$

where the summation in the middle is finite, and hence

$$\sum_{0 < i_{j_s} < \infty, s \notin T} \prod_{j_s, s \notin T} i_{j_s}^{2\beta_{j_s}} < \infty.$$

$\square$

## 4 Volterra processes with long memory

We introduce in this section the  $k$ -th order Volterra processes for which we establish non-central limit theorems in Section 6.

### 4.1 The off-diagonal process

We first introduce for convenience the following  $k$ -th order discrete chaos process with different noises:

$$X'(n) := \sum_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i}) \epsilon_{n-i_1}^{(1)} \cdots \epsilon_{n-i_k}^{(k)}, \quad (23)$$

where  $\{\epsilon_i := (\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)}), i \in \mathbb{Z}\}$  is an i.i.d. sequence of vectors, where  $\mathbb{E}\epsilon_i^{(p)} = 0$  and  $\mathbb{E}|\epsilon_i^{(p)}|^2 < \infty$ ,  $p = 1, \dots, k$ . This is just an extension of (7) adapted to (11). For such  $X'(n)$ , it is easy to show that the autocovariance satisfies

$$|\gamma(n)| \leq ck! \sum_{\mathbf{i} \in \mathbb{Z}_+^k} \widetilde{|a|}(\mathbf{i}) \widetilde{|a|}(\mathbf{i} + n\mathbf{1}), \quad n \geq 0, \quad (24)$$

where  $\widetilde{|a|}$  denotes the symmetrization of the absolute value  $|a|(\mathbf{i}) := |a(\mathbf{i})|$ , and  $c > 0$  is a constant which accounts for the covariance between different components of  $\epsilon_i$ . For example, suppose  $X'(n) = \sum_{i_1, i_2 > 0} a(i_1, i_2) \epsilon_{i_1}^{(1)} \epsilon_{i_2}^{(2)}$ , and  $\sigma(p, q) = \mathbb{E}\epsilon_i^{(p)} \epsilon_i^{(q)}$ , then for  $n > 0$ ,

$$\begin{aligned} |\mathbb{E}X'(n)X'(0)| &= \left| \sum_{i_1, i_2 > 0} a(i_1 + n, i_2 + n) a(i_1, i_2) \sigma(1, 1) \sigma(2, 2) + \sum_{i_1, i_2 > 0} a(i_1 + n, i_2 + n) a(i_2, i_1) \sigma(1, 2)^2 \right| \\ &\leq C \left( \sum_{i_1, i_2 > 0} (|a(i_1, i_2)| + |a(i_2, i_1)|) (|a(i_1 + n, i_2 + n)| + |a(i_2 + n, i_1 + n)|) \right) \end{aligned}$$

for some constant  $C > 0$ .

### 4.2 Off-diagonal decomposition of the Volterra process

We will focus on the  $k$ -th order Volterra process  $X(n)$  in (3) with coefficients given as

$$a(\mathbf{i}) = g(\mathbf{i})L(\mathbf{i}), \quad (25)$$

where  $g$  is a GHK(B) on  $\mathbb{R}_+^k$  with homogeneity exponent  $\alpha \in (-\frac{k+1}{2}, -\frac{k}{2})$  (see Definition 2.5), and  $L$  is a bounded real-valued function on  $\mathbb{Z}_+^k$  such that for any  $\mathbf{x} \in \mathbb{R}_+^k$  and any bounded  $\mathbb{Z}_+^k$ -valued function  $\mathbf{B}(n)$ , we have

$$\lim_{n \rightarrow \infty} L([n\mathbf{x}] + \mathbf{B}(n)) = 1 \quad (26)$$

(see Bai and Taqqu [2] equation (25) and Remark 4.5).

**Proposition 4.1.** *The process  $X(n)$  is well-defined in the  $L^2(\Omega)$ -sense.*

*Proof.* Follows from Remark 2.6 and Proposition 3.6.  $\square$

The off-diagonal decomposition (16) of a homogeneous polynomial form obtained in the proof of Proposition 3.3 plays also a crucial role in analyzing the autocovariance of  $X(n)$  and deriving limit theorems. As in (16) and (17), we have

$$X(n) = \sum_{\pi \in \mathcal{P}_k} \sum_{\mathbf{j} \in J_\pi} X_\pi^{\mathbf{j}}(n),$$

where  $\mathcal{P}_k$  is the set of all partitions of  $\{1, \dots, k\}$ ,  $\pi = \{P_1, \dots, P_m\}$ ,  $p_t = |P_t|$ ,  $J_\pi = \{0, 1, \dots, p_1\} \times \dots \times \{0, 1, \dots, p_m\}$ ,

$$X_\pi^{\mathbf{j}}(n) := \sum_{\mathbf{i} \in \mathbb{Z}_+^n} a_\pi(\mathbf{i}) c(\mathbf{p}, \mathbf{j}) A_{j_1}(\epsilon_{n-i_1}) \dots A_{j_m}(\epsilon_{n-i_m}), \quad (27)$$

with  $c(\mathbf{p}, \mathbf{j})$  given as in (20). Note that  $X_\pi^{\mathbf{j}}(n)$  is of the form (23), where  $a_\pi(\mathbf{i}) c(\mathbf{p}, \mathbf{j})$  replaces  $a(\mathbf{i})$  and where  $A_{j_1}(\epsilon_{n-i_1}), \dots, A_{j_m}(\epsilon_{n-i_m})$  are independent random variables replacing  $\epsilon_{n-i_1}^{(1)}, \dots, \epsilon_{n-i_k}^{(k)}$  with  $m$  playing the role of  $k$ . In view of (21), we have

$$X_c(n) := X(n) - \mathbb{E}X(n) = \sum_{\pi \in \mathcal{P}_k} \sum_{\mathbf{j} \in J_\pi^+} X_\pi^{\mathbf{j}}(n), \quad (28)$$

with

$$J_\pi^+ = \{0, 1, \dots, p_1\} \times \dots \times \{0, 1, \dots, p_m\} \setminus \{(0, 0, \dots, 0)\} \quad (29)$$

instead. We recall again that since  $\mu_1 = 0$ , whenever  $j_t = 0$ , we need to consider only  $p_t \geq 2$ ,  $t = 1, \dots, m$ . Thus  $X_\pi^{\mathbf{j}}$  can be further expressed as (19). Note that while  $m$  denotes the number of Appell polynomials in the product (27),  $m - r$  denotes the number of Appell polynomials in the product (19) where each Appell polynomial has a positive order, those of order 0 having been incorporated in  $S'_T$ .

Our first step is to obtain the asymptotic behavior of the *autocovariance* of  $X(n)$  or  $X_c(n)$  when  $g$  is a GHK(B). To this end we need some intermediate results. We will repeatedly use the following elementary asymptotics: if  $\gamma < -1$ , then

$$\sum_{n=1}^{\infty} (l+n)^\gamma = \sum_{n=l+1}^{\infty} n^\gamma \sim -(\gamma+1)^{-1} l^{\gamma+1}, \text{ as } l \rightarrow +\infty. \quad (30)$$

A parallel result but with equality holds for integration:

$$\int_0^\infty (x+y)^\gamma dx = -(\gamma+1)^{-1} y^{\gamma+1}. \quad (31)$$

Relation (30) can be derived using (31) and an integral approximation argument.

**Lemma 4.2.** *Suppose that  $g$  is a GHK(B) of order  $k$  with homogeneity exponent  $\alpha \in (-\frac{k+1}{2}, -\frac{k}{2})$ . Let  $0 \leq r < m = k - r$ , then*

$$g_r(\mathbf{x}) := \int_{\mathbb{R}_+^r} g(y_1, y_1, \dots, y_r, y_r, x_1, \dots, x_{m-r}) dy_1 \dots dy_r \quad (32)$$

is a GHK on  $\mathbb{R}_+^{k_r}$  with  $k_r = k - 2r$  and homogeneity exponent  $\alpha_r = \alpha + r \in (-\frac{k_r+1}{2}, -\frac{k_r}{2})$ .

*Proof.*  $g_r$  is well-defined, since by Definition 2.5 of GHK(B), for some constant  $C > 0$ , we have

$$|g(y_1, y_1, \dots, y_r, y_r, x_1, \dots, x_{m-r})| \leq C(y_1 + \dots + y_r + x_1 + \dots + x_{m-r})^\alpha. \quad (33)$$

Thus by applying (31) iteratively, we need only to note that  $\alpha + r < 0$ , because  $2r < k$  and  $\alpha < -k/2$ . We now check Condition 1 of Definition 2.1, that is, the homogeneity of  $g_r(\cdot)$ . We have for any  $\lambda > 0$  that

$$\begin{aligned} g_r(\lambda \mathbf{x}) &= \int_{\mathbb{R}_+^r} g(y_1, y_1, \dots, y_r, y_r, \lambda x_1, \dots, \lambda x_{m-r}) dy_1 \dots dy_r \\ &= \int_{\mathbb{R}_+^r} g(\lambda y_1, \lambda y_1, \dots, \lambda y_r, \lambda y_r, \lambda x_1, \dots, \lambda x_{m-r}) d(\lambda y_1) \dots d(\lambda y_r) \\ &= \lambda^{\alpha+r} \int_{\mathbb{R}_+^r} g(y_1, y_1, \dots, y_r, y_r, x_1, \dots, x_{m-r}) dy_1 \dots dy_r = \lambda^{\alpha+r} g_r(\mathbf{x}). \end{aligned}$$

We check then Condition 2 of Definition 2.1. Integrating both sides of (33) with respect to  $(y_1, \dots, y_r) \in \mathbb{R}_+^r$  shows  $|g_r(\mathbf{x})| \leq c \|\mathbf{x}\|^{\alpha+r}$  for some  $c > 0$ . So Condition 2 in Definition 2.1 is satisfied in view of Remark 2.6.  $\square$

**Remark 4.3.** The index  $r$  in  $g_r$  refers to the number of pairs of variables in  $g$  that are identified. The number  $m$  denotes the number of different variables in (32), and the number  $k = 2r + (m - r) = m + r$  denotes the total number of variables in  $g$ . Finally,  $k_r = k - 2r = m - r$  indicates the number of  $x$  variables, that is, the size of the argument of  $g_r$ . All the GHK(B)  $g_r$ ,  $r = 0, \dots, [k/2]$ , obtained in (32), have the same  $H = \alpha_r + k_r/2 + 1$  (homogeneity exponent+dimension/2+1). This is because  $\alpha_r = \alpha + r$  and  $k_r = k - 2r$ .

**Remark 4.4.** Lemma 4.2 assumes  $r < m$  or equivalently  $k_r = k - 2r > 0$ . In other words, that there is a positive number of  $x$  variables of  $g_r(\cdot)$  in (32).

### 4.3 Behavior of the autocovariances

We have the following asymptotics for the autocovariance of  $X_\pi^{\mathbf{j}}(n)$ , which are the off-diagonal terms of  $X_c(n)$  in (28). Note that  $\mathbf{j} = (j_1, \dots, j_p) \neq \mathbf{0}$  because of centering, so  $\sum_{t=1}^m 1_{\{j_t=0\}} < m$ . Recall that by assumption we have  $-1 < 2\alpha + k < 0$ .

**Proposition 4.5.** *Let  $m = |\pi|$ , and  $r = \sum_{t=1}^m 1_{\{j_t=0\}} < m$ .*

(i) *If  $m + r = k$ , then the autocovariance  $\gamma(n)$  of  $X_\pi^{\mathbf{j}}(n)$  satisfies*

$$\gamma(n) \sim cn^{2\alpha+k} \quad (34)$$

*as  $n \rightarrow \infty$ , for some constant  $c > 0$ .*

(ii) *If  $m + r < k$ , then*

$$\sum_n |\gamma(n)| < \infty. \quad (35)$$

*Proof.* We claim first that  $m + r = k$  if and only if in the partition  $\pi = \{P_1, \dots, P_m\}$ , every  $|P_t| \leq 2$ , and whenever  $|P_t| = 2$ , one has  $j_t = 0$ . Indeed, as noted in (18), if  $j_t = 0$ , then  $|P_t| \geq 2$ , and thus

$$k = \sum_{t=1}^m 1_{\{j_t=0\}} |P_t| + \sum_{t=1}^m 1_{\{j_t>0\}} |P_t| \geq 2r + (m - r) = m + r.$$

The equality is attained only if when  $j_t = 0$ ,  $|P_t| = 2$ , and when  $j_t > 0$ ,  $|P_t| = 1$ .

Suppose first that  $m + r = k$ . We can assume without loss of generality in (25) that  $a(\cdot) = g(\cdot)$  is symmetric and  $L = 1$  (including a general  $L$  in the following argument is easy).

Using the symmetry of  $g$ ,  $X_\pi^{\mathbf{j}}(n)$  in (27) simplifies. To compute it, note that since  $r$  corresponds to the number of Appell polynomials of order 0 which are all equal to 1, we have

$$j_1 = \dots = j_r = 0, \quad j_{r+1} = \dots = j_m = 1, \quad A_{j_1}(\epsilon) = \dots = A_{j_r}(\epsilon) = 1, \quad A_{j_{r+1}}(\epsilon) = \dots = A_{j_m}(\epsilon) = \epsilon,$$

$$\pi = \left\{ \{1, 2\}, \dots, \{2r - 1, 2r\}, \{2r + 1\}, \dots, \{m + r\} \right\},$$

$$\mathbf{p} = \left( \underbrace{2, \dots, 2}_r, \underbrace{1, \dots, 1}_{m-r} \right), \quad c(\mathbf{p}, \mathbf{j}) = \binom{2}{0} \dots \binom{2}{0} \binom{1}{1} \dots \binom{1}{1} = 1,$$

and  $\mu_0 = \mathbb{E}\epsilon^0 = 1$ ,  $\mu_2 = \mathbb{E}\epsilon^2 = 1$ . We therefore get

$$X_\pi^{\mathbf{j}}(n) = \sum_{(\mathbf{i}, \boldsymbol{\ell}) \in \mathbb{R}_+^k} g(i_1 \mathbf{1}_2, \dots, i_r \mathbf{1}_2, l_1, \dots, l_{m-r}) \epsilon_{n-l_1} \dots \epsilon_{n-l_{m-r}}, \quad (36)$$

where  $\mathbf{1}_2$  denotes the vector made of two 1's. Let

$$\mathbf{i}_t = (i_{1,t}\mathbf{1}_2, \dots, i_{r,t}\mathbf{1}_2), \quad t = 1, 2, \quad \boldsymbol{\ell} = (l_1, \dots, l_{m-r}).$$

Since we are excluding the diagonals, let

$$D(n) = \{(\mathbf{i}_1, \mathbf{i}_2, \boldsymbol{\ell}) > \mathbf{0} : i_{u,t} \neq i_{v,t}, l_u \neq l_v \text{ for } u \neq v; \text{ and } i_{p,1} \neq l_q, i_{p,2} \neq l_q + n \text{ including the case } p = q\}.$$

Then

$$\begin{aligned} \gamma(n) &= (m-r)! \sum_{(\mathbf{i}_1, \mathbf{i}_2, \boldsymbol{\ell}) \in D(n)} g(\mathbf{i}_1, \boldsymbol{\ell}) g(\mathbf{i}_2, \boldsymbol{\ell} + n\mathbf{1}) \\ &= (k-2r)! n^{2\alpha+k} \int_{\mathbb{R}_+^{m-r}} d\mathbf{y} \int_{\mathbb{R}_+^r} d\mathbf{x}_1 \int_{\mathbb{R}_+^r} d\mathbf{x}_2 g\left(\frac{[n\mathbf{x}_1] + \mathbf{1}}{n}, \frac{[n\mathbf{y}] + \mathbf{1}}{n}\right) g\left(\frac{[n\mathbf{x}_2] + \mathbf{1}}{n}, \mathbf{1} + \frac{[n\mathbf{y}] + \mathbf{1}}{n}\right) 1_{E(n)}, \end{aligned} \quad (37)$$

since  $m+r=k$  implies  $m-r=k-2r$  and where  $\mathbf{x}_t = (x_{1,t}\mathbf{1}_2, \dots, x_{r,t}\mathbf{1}_2)$ ,  $t = 1, 2$ ,  $\mathbf{y} = (y_1, \dots, y_{m-r})$ , and  $D(n)$  in the summation is expressed as

$$\begin{aligned} E(n) &= \{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}_+^r, \mathbf{y} \in \mathbb{R}_+^{m-r} : [nx_{u,t}] \neq [nx_{v,t}], [ny_u] \neq [ny_v] \text{ for } u \neq v; \\ &\quad \text{and } [nx_{p,1}] \neq [ny_q], [nx_{p,2}] \neq [ny_q] + n \text{ including the case } p = q\}. \end{aligned}$$

Note first that  $1_{E(n)}$  converges to 1 a.e. as  $n \rightarrow \infty$ . By Definition 2.5,  $|g(\mathbf{z})| \leq c_0 \|\mathbf{z}\|^\alpha$  for some  $c_0 > 0$ . Since  $\frac{[n\mathbf{x}] + \mathbf{1}}{n} > \mathbf{x}$  and  $\alpha < 0$ , we have

$$\begin{aligned} &\int_{\mathbb{R}_+^r} \left| g\left(\frac{[n\mathbf{x}_1] + \mathbf{1}}{n}, \frac{[n\mathbf{y}] + \mathbf{1}}{n}\right) \right| d\mathbf{x}_1 \int_{\mathbb{R}_+^r} \left| g\left(\frac{[n\mathbf{x}_2] + \mathbf{1}}{n}, \mathbf{1} + \frac{[n\mathbf{y}] + \mathbf{1}}{n}\right) \right| d\mathbf{x}_2 \\ &\leq \int_{\mathbb{R}_+^r} g^*\left(\frac{[n\mathbf{x}_1] + \mathbf{1}}{n}, \frac{[n\mathbf{y}] + \mathbf{1}}{n}\right) d\mathbf{x}_1 \int_{\mathbb{R}_+^r} g^*\left(\frac{[n\mathbf{x}_2] + \mathbf{1}}{n}, \mathbf{1} + \frac{[n\mathbf{y}] + \mathbf{1}}{n}\right) d\mathbf{x}_2 \\ &\leq \int_{\mathbb{R}_+^r} g^*(\mathbf{x}_1, \mathbf{y}) d\mathbf{x}_1 \int_{\mathbb{R}_+^r} g^*(\mathbf{x}_2, \mathbf{1} + \mathbf{y}) d\mathbf{x}_2 =: g_r^*(\mathbf{y}) g_r^*(\mathbf{1} + \mathbf{y}), \end{aligned}$$

where  $g^*(\mathbf{z}) = c_1 \|\mathbf{z}\|^\alpha$  is function decreasing in its every variable, and  $g_r^*(\mathbf{y}) = c_2 \|\mathbf{y}\|^{\alpha+r}$ ,  $\mathbf{y} \in \mathbb{R}_+^{m-r}$ . Observe that  $g_r^*$  is a GHK(B) by Definition 2.5 on  $\mathbb{R}_+^{m-r}$ , since  $m-r=k-2r$  and

$$-\frac{k-2r+1}{2} < \alpha+r < -\frac{k-2r}{2} \iff -\frac{k+1}{2} < \alpha < -\frac{k}{2}.$$

So  $g_r^*(\cdot)$  is a GHK by Remark 2.6, and hence

$$\int_{\mathbb{R}_+^{m-r}} d\mathbf{y} \int_{\mathbb{R}_+^r} d\mathbf{x}_1 \int_{\mathbb{R}_+^r} d\mathbf{x}_2 g^*(\mathbf{x}, \mathbf{y}) g^*(\mathbf{x}, \mathbf{1} + \mathbf{y}) = \int_{\mathbb{R}_+^{m-r}} g_r^*(\mathbf{y}) g_r^*(\mathbf{1} + \mathbf{y}) d\mathbf{y} < \infty.$$

One can now let  $n \rightarrow \infty$  in (37) through the Dominated Convergence Theorem to get

$$\gamma(n) \sim (k-2r)! n^{2\alpha+k} C_{g_r}, \text{ as } n \rightarrow \infty,$$

where

$$C_{g_r} = \int_{\mathbb{R}_+^{m-r}} g_r(\mathbf{x}) g_r(\mathbf{1} + \mathbf{x}) d\mathbf{x} > 0$$

with  $g_r$  obtained as in (32). Since we have assumed (without loss of generality) that  $g$  is symmetric, it does not matter which of the  $r$  variables are integrated out. This proves (34)

Consider now the case  $m + r \leq k - 1$ . Again by the assumption of Definition 2.5 and the boundedness of  $L$ ,  $a(\mathbf{i}) \leq c(i_1 + \dots + i_k)^\alpha$  for some  $c > 0$ . Suppose  $T = \{t = 1, \dots, m : j_t = 0\}$ ,  $\ell = (l_1, \dots, l_{m-r})$ . Then by applying (30) iteratively, one has for some  $C > 0$

$$S'_T |a_\pi(\ell)| \leq C(l_1 + \dots + l_{m-r})^{\alpha+r},$$

where by Definition 2.1,

$$\alpha + r < -\frac{k}{2} + r \leq -\frac{m-r}{2} - \frac{1}{2} \leq -1, \quad (38)$$

since  $m - r \geq 1$  by assumption.

In view of (19) and (24), we are left to show that

$$\sum_{n>0} \sum_{\ell \in \mathbb{R}_+^{m-t}} S'_T(\widetilde{|a_\pi|})(\ell) S'_T(\widetilde{|a_\pi|})(\ell + n\mathbf{1}) < \infty.$$

This can be seen as follows:

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{0 < l_1 < \dots < l_{m-r} < \infty} (l_1 + \dots + l_{m-r})^{\alpha+r} (l_1 + \dots + l_{m-r} + n)_+^{\alpha+r} \\ & \leq C \sum_{0 < l_1 < \dots < l_{m-r} < \infty} (l_1 + \dots + l_{m-r})^{2(\alpha+r)+1} < \infty, \end{aligned}$$

for some  $C > 0$ , where we have used (30) and the fact  $2(\alpha + r) + 1 + m - r < 0$  by (38).  $\square$

As a corollary of Proposition 4.5, we have

**Corollary 4.6.** *If  $a(\cdot)$  is as given in (25), then*

$$\text{Var} \left[ \sum_{n=1}^N X(n) \right] \sim CN^{2H}, \text{ as } n \rightarrow \infty \quad (39)$$

for some  $C > 0$ , where  $H = \alpha + k/2 + 1 \in (1/2, 1)$ .

*Proof.* If  $\gamma(n)$  is the autocovariance of a stationary process  $Y(n)$ , then

$$\text{Var} \left[ \sum_{n=1}^N Y(n) \right] = N \sum_{|n| < N} \gamma(n) - \sum_{|n| < N} |n| \gamma(n).$$

It is well-known that if  $\sum_n |\gamma(n)| < \infty$ , then for some constant  $c_1 > 0$ ,  $\text{Var} \left[ \sum_{n=1}^N Y(n) \right] \leq c_1 N$ ; if  $\gamma(n) \sim c_2 n^{2H-2}$  for  $H \in (1/2, 1)$  and some constant  $c_2 > 0$ , then  $\text{Var} \left[ \sum_{n=1}^N Y(n) \right] \sim c_3 N^{2H}$  for some constant  $c_3 > 0$ . Now apply these to  $X_\pi^j(n)$  in the decomposition (28) to the two cases  $m + r < k$  and  $m + r = k$  in Proposition 4.5 respectively. The variance of the sum of  $X_\pi^j(n)$  with  $m + r = k$  dominates those with  $m + r < k$ . Note that the off-diagonal polynomial forms  $X_\pi^j(n)$ 's are uncorrelated if they have different values of  $r$ 's because then they have different orders. In addition, the exponent in (34) is  $2\alpha + k = 2H - 2$ , by (5) and by the definition of  $\alpha$ , one has  $H \in (1/2, 1)$ . Therefore (39) holds.  $\square$

**Remark 4.7.** In view of the preceding proof, when  $m + r < k$ ,  $X_\pi^j(n)$  has a summable autocovariance, which is the typical definition of *short memory* or *short-range dependence*, while if  $m + r = k$ , the autocovariance of  $X_\pi^j(n)$  has a hyperbolic decay with a power in  $(-1, 0)$ , which is the typical definition of *long memory* or *long-range dependence*, with a Hurst exponent  $H = \alpha + k/2 + 1$ .

## 5 Central limit theorems for $k$ -th order Volterra processes

We establish in this section a central limit theorem for  $X(n)$  in (3) using the off-diagonal decomposition (28).

We state first a lemma concerning a comparison of moments of the off-diagonal discrete chaos in (11), which will be used later to establish tightness in the space  $D[0, 1]$  with uniform topology in the central limit theorem.

**Lemma 5.1.** *Suppose that  $\{\epsilon_i := (\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)}), i \in \mathbb{Z}\}$  forms an i.i.d. sequence of  $k$ -dimensional vector with mean  $\mathbf{0}$ , and  $\mathbb{E}|\epsilon_i^{(j)}|^r < \infty$  for some  $r > 2$ ,  $j = 1, \dots, k$ . Suppose  $a(\cdot)$  is a function defined on  $\mathbb{Z}_+^k$  satisfying  $\sum_{\mathbf{i} \in \mathbb{Z}_+^k} a(\mathbf{i})^2 < \infty$ , so that*

$$Y = \sum_{0 < i_1 < \dots < i_k < \infty} a(i_1, \dots, i_k) \epsilon_{i_1}^{(1)} \dots \epsilon_{i_k}^{(k)} \quad (40)$$

is well defined. Then for any  $p \in (2, r)$ , there exists a constant  $C$  which doesn't depend on  $a(\cdot)$ , such that

$$[\mathbb{E}|Y|^p]^{1/p} \leq C[\mathbb{E}|Y|^2]^{1/2}. \quad (41)$$

Lemma 4.3 of Krakowiak and Szulga [16] yields (41) when  $\epsilon_i^{(1)} = \dots = \epsilon_i^{(k)} = \epsilon_i$  and  $a(\mathbf{i}) = a(\mathbf{i})1_{\{\mathbf{i} \leq n\mathbf{1}\}}$  for some  $n \geq k$ , and it is extended straightforwardly to the case  $\sum_{0 < i_1 < \dots < i_k < \infty} a(i_1, \dots, i_k)^2 < \infty$  in Bai and Taquq [2]. The proof, which develops a martingale structure for

$$\left\{ X_n := \sum_{0 < i_1 < \dots < i_k \leq n} a(i_1, \dots, i_k) \epsilon_{i_1} \dots \epsilon_{i_k}, n \geq k \right\}$$

and uses the square function inequality (Theorem 3.2 of Burkholder [7]), needs to be modified to allow non-identical components in  $\epsilon_i$  as in the preceding lemma. We include a proof in Section 8 for completeness.

**Theorem 5.2.** *Suppose that the coefficient  $a(\cdot)$  defining the Volterra process  $X(n)$  in (3) satisfies the assumptions in Proposition 3.3. Suppose also that for any  $\pi = \{P_1, \dots, P_{|\pi|}\} \in \mathcal{P}_k$ ,*

$$\sum_{n=1}^{\infty} \sum_{\mathbf{i} \in \mathbb{Z}_+^{|\pi|}} \widetilde{|a_\pi|}(\mathbf{i}) \widetilde{|a_\pi|}(\mathbf{i} + n\mathbf{1}) < \infty, \quad (42)$$

where  $\sim$  stands for symmetrization, and that for every  $T \subset \{1, \dots, |\pi|\}$ ,  $|T| < |\pi|$ , satisfying  $|P_t| \geq 2$  for all  $t \in T$ , we have

$$\sum_{n=1}^{\infty} \sum_{\mathbf{i} \in \mathbb{Z}_+^{|\pi| - |T|}} (\widetilde{S'_T |a_\pi|})(\mathbf{i}) (\widetilde{S'_T |a_\pi|})(\mathbf{i} + n\mathbf{1}) < \infty. \quad (43)$$

Then if in addition  $\sigma^2 := \sum_n \gamma(n) := \sum_n \text{Cov}(X(n), X(0)) > 0$ , we have  $\sigma^2 < \infty$  and

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor Nt \rfloor} [X(n) - \mathbb{E}X(n)] \xrightarrow{f.d.d.} \sigma B(t), \quad (44)$$

where  $B(t)$  is a standard Brownian motion.

If in addition, the noise  $\{\epsilon_i\}$  defining  $X(n)$  satisfies  $\mathbb{E}\epsilon_i^{2k+\delta} < \infty$  for some  $\delta > 0$ , then  $\xrightarrow{f.d.d.}$  in (44) can be replaced by the weak convergence in  $D[0, 1]$  with uniform topology.

*Proof.* In (28),  $X(n) - \mathbb{E}X(n)$  is expressed as a finite sum of off-diagonal terms  $X_\pi^j(n)$  given in (27). This is, however, similar to Theorem 6.14 of Bai and Taquq [2] by noting that (42) and (43) are essentially the same as the SRD condition in Definition 6.1 of Bai and Taquq [2]. The only difference is the presence



of non-identically distributed noises since here, Appell polynomials  $A_j(\epsilon_i)$  of different orders are involved. This extension is easy to include. We thus omit the details but mention just the following two points: the relations (42) and (43) imply that the auto(cross-)covariances of  $X_\pi^{\mathbf{j}}(n)$ 's are absolutely summable, in particular,  $\sigma^2 < \infty$ . The proof of the convergence in finite-dimensional distributions uses a truncation argument to reduce the  $X_\pi^{\mathbf{j}}(n)$ 's to  $m$ -dependent sequences. The tightness in  $D[0, 1]$  can be established with the help of (41).  $\square$

We will now state a more practical condition than (42) and (43):

**Proposition 5.3.** *Relation (42) and (43) hold, if*

$$|a(i_1, \dots, i_k)| \leq C i_1^{\gamma_1} \dots i_k^{\gamma_k} \quad (45)$$

where  $C > 0$  is some constant and each  $\gamma_j < -1$ .

*Proof.* It suffices to show (42) and (43) for  $a(\mathbf{i}) = i_1^{\gamma_1} \dots i_k^{\gamma_k}$ , which is easily checked by the separability of the product and that  $\sum_{i>0} \sum_{n>0} i^a (i+n)^b < \infty$  for any  $a, b < -1$ .  $\square$

In contrast, if  $X'(n)$  is the discrete chaos process as defined in (23), the central limit theorem holds for this process under weaker assumptions, namely,  $\gamma_j < -1/2$  and  $\sum_{j=1}^k \gamma_j < -k/2 - 1/2$  instead of  $\gamma_j < -1$ . Indeed:

**Proposition 5.4.** *Let  $X'(n)$  be given as in (23), with  $a(\cdot)$  satisfying the following:*

$$|a(i_1, \dots, i_k)| \leq C i_1^{\gamma_1} \dots i_k^{\gamma_k}, \quad (46)$$

where  $C$  is a positive constant and each  $\gamma_j < -1/2$ , and  $\sum_{j=1}^k \gamma_j < -k/2 - 1/2$ . Then the autocovariance  $\gamma(n)$  of  $X'(n)$  is absolutely summable. If in addition  $\sigma^2 := \sum_{n=-\infty}^{\infty} \text{Cov}(X'(n), X'(0)) > 0$ , then  $X'(n)$  satisfies the central limit theorem (44). If a moment higher than 2 of each  $\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)}$  exists, then (44) holds with  $\xrightarrow{\text{f.d.d.}}$  replaced by weak convergence  $\Rightarrow$  in  $D[0, 1]$ .

The above  $\xrightarrow{\text{f.d.d.}}$  or  $\Rightarrow$  convergence also holds for a linear combination of different  $X'(n)$ 's defined using a common i.i.d. noise vector  $\epsilon_i$ , where the different  $X'(n)$ 's in the linear combination can have different orders and involve subvectors of  $\epsilon_i$ , provided that each  $X'(n)$  satisfies the conditions mentioned above.

*Proof.* In view of the relation (24) and the extension of Theorem 6.14 in Bai and Taqqu [2] mentioned in the proof of Theorem 5.2, we only need to show that

$$\sum_{n=1}^{\infty} \sum_{\mathbf{i}>\mathbf{0}} |a(\mathbf{i} + n\mathbf{1})| |a(\mathbf{i})| < \infty.$$

In view of the bound (46), this holds if

$$\sum_{n=1}^{\infty} \sum_{\mathbf{i}>\mathbf{0}} (i_1 + n)^{\gamma_1} \dots (i_k + n)^{\gamma_k} i_1^{\gamma_{\sigma(1)}} \dots i_k^{\gamma_{\sigma(k)}} = \sum_{n=1}^{\infty} r_1(n) \dots r_k(n) < \infty, \quad (47)$$

where  $\sigma$  is any permutation of  $\{1, \dots, k\}$ ,  $r_j(n) = \sum_{i=1}^{\infty} (i+n)^{\gamma_j} i^{\gamma_{\sigma(j)}}$ . Without loss of generality, we may assume that  $-1 < \gamma_j < -1/2$ . In this case, using the fact  $\int_0^{\infty} x^a (1+x)^b dx = B(a+1, -1-a-b)$  for  $a, b \in (-1, -1/2)$ , where  $B(\cdot, \cdot)$  is the beta function, and an integral approximation, one gets

$$r_j(n) \sim B(\gamma_{\sigma(j)} + 1, -1 - \gamma_j - \gamma_{\sigma(j)}) n^{\gamma_j + \gamma_{\sigma(j)} + 1}$$

as  $n \rightarrow \infty$ . But  $\sum_{j=1}^k (2\gamma_j + 1) < -k - 1 + k = -1$  by assumption. So (47) holds.  $\square$

**Example 5.5.** To illustrate the statement about linear combinations, let  $\{\epsilon_i := (\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)}), i \in \mathbb{Z}\}$  be an i.i.d. sequence as in Lemma 5.1. Suppose that  $\{j_1, \dots, j_{k_1}\}$  and  $\{l_1, \dots, l_{k_2}\}$  are two subsets of  $\{1, \dots, k\}$ . Then Proposition 5.4 applies to  $X'_1(n) + X'_2(n)$ , where

$$\begin{aligned} X'_1(n) &= \sum'_{0 < i_1, \dots, i_{k_1} < \infty} a^{(1)}(i_1, \dots, i_{k_1}) \epsilon_{n-i_1}^{(j_1)} \dots \epsilon_{n-i_{k_1}}^{(j_{k_1})} \\ X'_2(n) &= \sum'_{0 < i_1, \dots, i_{k_2} < \infty} a^{(2)}(i_1, \dots, i_{k_2}) \epsilon_{n-i_1}^{(l_1)} \dots \epsilon_{n-i_{k_2}}^{(l_{k_2})}, \end{aligned}$$

where  $a^{(1)}$  and  $a^{(2)}$  satisfy the conditions of Proposition 5.4 with  $k$  replaced by  $k_1$  and  $k_2$  respectively.

## 6 Non-central limit theorem for $k$ -th order Volterra processes

The non-central limit theorem (NCLT) builds on a result concerning convergence of a discrete chaos to a Wiener chaos. Let  $h$  be a function defined in  $\mathbb{Z}^k$  such that  $\sum'_{\mathbf{i} \in \mathbb{Z}^k} h(\mathbf{i})^2 < \infty$ , where  $'$  indicates the exclusion of the diagonals  $i_p = i_q, p \neq q$ . Let  $Q_k(h)$  be defined as follows:

$$Q_k(h) = Q_k(h, \epsilon) = \sum'_{(i_1, \dots, i_k) \in \mathbb{Z}^k} h(i_1, \dots, i_k) \epsilon_{i_1} \dots \epsilon_{i_k} = \sum'_{\mathbf{i} \in \mathbb{Z}^k} h(\mathbf{i}) \prod_{p=1}^k \epsilon_{i_p}, \quad (48)$$

where  $\epsilon_i$ 's are i.i.d. noise. It is easy to see that switching the arguments of  $h(i_1, \dots, i_k)$ , does not change  $Q_k(h)$ . So if  $\tilde{h}$  is the symmetrization of  $h$ , then  $Q_k(h) = Q_k(\tilde{h})$ .

Suppose now that we have a sequence of function vectors  $\mathbf{h}_n = (h_{1,n}, \dots, h_{j,n})$  where each  $h_{j,n} \in L^2(\mathbb{Z}^{k_j})$ ,  $j = 1, \dots, J$ .

**Proposition 6.1.** (Proposition 4.1 of Bai and Taqqu [2]) Let

$$\tilde{h}_{j,n}(\mathbf{x}) = n^{k_j/2} h_{j,n}([n\mathbf{x}] + \mathbf{c}_j), \quad j = 1, \dots, J,$$

where  $\mathbf{c}_j \in \mathbb{Z}^{k_j}$ . Suppose that there exists  $h_j \in L^2(\mathbb{R}^{k_j})$ , such that

$$\|\tilde{h}_{j,n} - h_j\|_{L^2(\mathbb{R}^{k_j})} \rightarrow 0 \quad (49)$$

as  $n \rightarrow \infty$ . Then, as  $n \rightarrow \infty$ ,

$$\mathbf{Q} := (Q_{k_1}(h_{1,n}), \dots, Q_{k_J}(h_{J,n})) \xrightarrow{d} \mathbf{I} := (I_{k_1}(h_1), \dots, I_{k_J}(h_J)),$$

where the multiple Wiener-Itô integrals  $I_{k_j}(\cdot)$ 's are defined using the same Brownian random measure.

We are now ready to state the non-central limit theorem. We always assume in the sequel that the coefficient  $a(\cdot)$  is of the form (25) and symmetric, with  $g$  a symmetric GHK(B). Proposition 4.5 and Corollary 4.6 show that the basic terms  $X_{\pi}^j(n)$  in the decomposition (28) will either be long-range dependent or short-range dependent, and the short-range dependent ones will vanish if the normalization  $N^{-H}$  used for long-range dependent terms is applied.

**Theorem 6.2.** Let  $X(n)$  be a  $k$ -th order Volterra process given in (3), with the coefficient  $a(\mathbf{i}) = g(\mathbf{i})L(\mathbf{i})$  given as in (25), where  $g$  is a symmetric GHK(B) on  $\mathbb{R}_+^k$  with homogeneity exponent  $\alpha \in (-k/2 - 1/2, -k/2)$ . Then one has the following weak convergence in  $D[0, 1]$ :

$$\frac{1}{N^H} \sum_{n=1}^{[Nt]} (X(n) - \mathbb{E}X(n)) \Rightarrow Z(t) := \sum_{r=0}^{[k/2]} d_{k,r} Z_{k-2r}(t), \quad (50)$$

where

$$H = \alpha + k/2 + 1, \quad (51)$$

$$d_{k,r} = \frac{k!}{2^r (k-2r)! r!}, \quad (52)$$

$Z_0(t) := 0$ , and if  $k - 2r > 0$ ,

$$Z_{k-2r}(t) := \int_{\mathbb{R}^{k-2r}}' \int_0^t g_r(\mathbf{s}\mathbf{1} - \mathbf{x}) \mathbf{1}_{\{\mathbf{s}\mathbf{1} > \mathbf{x}\}} ds B(dx_1) \dots B(dx_{k-2r})$$

is a  $(k - 2r)$ -th order generalized Hermite process and

$$g_r(\mathbf{s}\mathbf{1} - \mathbf{x}) := \int_{\mathbb{R}_+^r} g(y_1, y_1, \dots, y_r, y_r, s - x_1, \dots, s - x_{k-2r}) dy_1 \dots dy_r. \quad (53)$$

*Proof.* The process  $X(n)$  is well-defined in the  $L^2(\Omega)$ -sense by Proposition 4.1. We now use the notation in Proposition 4.5. If the basic off-diagonal term  $X_\pi^{\mathbf{j}}(n)$  in (28) satisfies  $m + r < k$ , in view of that proposition and the proof of Corollary 4.6, one has

$$N^{-H} \text{Var} \left[ \sum_{n=1}^{\lfloor Nt \rfloor} X_\pi^{\mathbf{j}}(n) \right] \rightarrow 0,$$

as  $N \rightarrow \infty$ . So these terms converge in probability to zero in  $D[0, 1]$ .

Suppose now that  $m + r = k$  or equivalently  $k - 2r = m - r$ . The goal is show the weak convergence in  $D[0, 1]$  of  $N^{-H} \sum_{n=1}^{\lfloor Nt \rfloor} X_\pi^{\mathbf{j}}(n)$  to  $Z_{k-2r}(t)$ . The tightness is standard since  $H > 1/2$  (see, e.g., Proposition 4.4.2 of Giraitis et al. [11]). It remains to show convergence of the finite-dimensional distributions. To do so, we will use Proposition 6.1, which only requires to show that the convergence in (49) holds separately for each order  $k - 2r = m - r$  with  $r = 1, \dots, \lfloor k/2 \rfloor$  and for a single  $t > 0$ .

For simplicity, we assume  $a(\cdot) = g(\cdot)$  (including a general  $L$  satisfying (26) is easy), and further one can assume without loss of generality that

$$a_\pi(i_1, \dots, i_m) = g(i_1, i_1, \dots, i_r, i_r, i_{r+1}, \dots, i_m),$$

and thus  $X_\pi^{\mathbf{j}}(n)$  is as given in (36). Let

$$(\mathbf{i}, \boldsymbol{\ell}) = (i_1, i_1, \dots, i_r, i_r, l_1, \dots, l_{m-r}),$$

and since  $X_\pi^{\mathbf{j}}(n)$  has no diagonals, we let

$$F(\boldsymbol{\ell}, n) = \{\mathbf{i} \in \mathbb{R}_+^r : i_u \neq i_v \text{ for } u \neq v; \text{ and } i_p \neq n - l_q \text{ including the case } p = q\},$$

so that we can write

$$N^{-H} \sum_{n=1}^{\lfloor Nt \rfloor} X_\pi^{\mathbf{j}}(n) = \sum_{\boldsymbol{\ell} \in \mathbb{R}_+^{m-r}} \frac{1}{N^{\alpha+k/2+1}} \sum_{n=1}^{\lfloor Nt \rfloor} \sum_{\mathbf{i} \in F(\boldsymbol{\ell}, n)} g(\mathbf{i}, n\mathbf{1} - \boldsymbol{\ell}) \mathbf{1}_{\{n\mathbf{1} > \boldsymbol{\ell}\}} \epsilon_{l_1} \dots \epsilon_{l_{m-r}} =: Q_{m-r}(h_{t,N}).$$

By associating  $\mathbf{i}$  to  $\lfloor N\mathbf{x} \rfloor + \mathbf{1}$  and  $n$  to  $\lfloor Ns \rfloor + 1$ , we write the inner sums into integrals, namely,

$$h_{t,N}(\boldsymbol{\ell}) = \frac{1}{N^{\alpha+k/2+1}} \sum_{n=1}^{\lfloor Nt \rfloor} \sum_{\mathbf{i} \in F(\boldsymbol{\ell}, n)} g(\mathbf{i}, n\mathbf{1} - \boldsymbol{\ell}) \mathbf{1}_{\{n\mathbf{1} > \boldsymbol{\ell}\}} \quad (54)$$

$$= N^{-(m-r)/2} \int_0^t ds \int_{\mathbb{R}_+^r} d\mathbf{x} g \left( \frac{\lfloor N\mathbf{x} \rfloor + \mathbf{1}}{N}, \frac{\lfloor Ns \rfloor \mathbf{1} + \mathbf{1} - \boldsymbol{\ell}}{N} \right) \mathbf{1}_{\{\lfloor Ns \rfloor + \mathbf{1} > \boldsymbol{\ell}\}} \mathbf{1}_{G(\boldsymbol{\ell}, N)} - R_{t,N}(\boldsymbol{\ell}), \quad (55)$$

where

$$G(\boldsymbol{\ell}, N) = \{[Nx_u] \neq [Nx_v] \text{ for } u \neq v; \text{ and } [Nx_p] \neq [Ns] - l_q \text{ including the case } p = q\},$$

and where

$$R_{t,N}(\boldsymbol{\ell}) = N^{-(m-r)/2} \frac{Nt - [Nt]}{N} \int_{\mathbb{R}_+^r} d\mathbf{x} g \left( \frac{[N\mathbf{x}] + \mathbf{1}}{N}, \frac{[Nt]\mathbf{1} + \mathbf{1} - \boldsymbol{\ell}}{N} \right) \mathbf{1}_{\{[Nt] + \mathbf{1} > \boldsymbol{\ell}\}} \mathbf{1}_{G(\boldsymbol{\ell}, N)}$$

is a residual term which will be asymptotically negligible<sup>2</sup>.

In view of Proposition 6.1, it is sufficient to show that

$$\lim_{N \rightarrow \infty} \|\tilde{h}_{t,N} - h_t\|_{L^2(\mathbb{R}^{m-r})} = 0, \quad (56)$$

where

$$h_t(\mathbf{y}) = \int_0^t g_r(s\mathbf{1} - \mathbf{y}) \mathbf{1}_{\{s\mathbf{1} > \mathbf{y}\}} ds,$$

and

$$\tilde{h}_{t,N}(\mathbf{y}) = N^{(m-r)/2} h_{t,N}([N\mathbf{y}] + \mathbf{1}) = \int_0^t ds \int_{\mathbb{R}_+^r} d\mathbf{x} g \left( \frac{[N\mathbf{x}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{y}]}{N} \right) \mathbf{1}_{\{[Ns]\mathbf{1} > [N\mathbf{y}]\}} \mathbf{1}_{H(N)} - \tilde{R}_{t,N}(\mathbf{y}), \quad (57)$$

where

$$\tilde{R}_{t,N}(\mathbf{y}) = \frac{Nt - [Nt]}{N} \int_{\mathbb{R}_+^r} d\mathbf{x} g \left( \frac{[N\mathbf{x}] + \mathbf{1}}{N}, \frac{[Nt]\mathbf{1} - [N\mathbf{y}]}{N} \right) \mathbf{1}_{\{[Nt]\mathbf{1} > [N\mathbf{y}]\}} \mathbf{1}_{H(N)},$$

with

$$H(N) = \{[Nx_u] \neq [Nx_v] \text{ for } u \neq v; \text{ and } [Nx_p] \neq [Ns] - [Ny_q] - 1 \text{ including the case } p = q\}.$$

The term  $H(N)$  comes from  $G(\boldsymbol{\ell}, N)$ .

We first deal with the term involving  $g(\cdot)$  in (57), and then with  $\tilde{R}_{t,N}$ . First, the a.e. convergence of

$$g \left( \frac{[N\mathbf{x}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{y}]}{N} \right) \mathbf{1}_{\{[Ns]\mathbf{1} > [N\mathbf{y}]\}} \mathbf{1}_{H(N)}$$

to

$$g(\mathbf{x}, s\mathbf{1} - \mathbf{y}) \mathbf{1}_{\{s\mathbf{1} > \mathbf{y}\}} \quad \text{as } N \rightarrow \infty$$

follows from the a.e. continuity of  $g$ , and the a.e. convergence of  $\mathbf{1}_{H(N)}$  to 1<sup>3</sup>. We are thus left to establish suitable bounds in order to apply the Dominated Convergence Theorem.

By the definition of a GHK(B),

$$|g(\mathbf{x})| \leq c \|\mathbf{x}\|^\alpha = c(x_1 + \dots + x_k)^\alpha$$

for some constant  $c > 0$ . Recall that  $\alpha < -k/2$ . We hence claim that for any  $\mathbf{x} > \mathbf{0}$ ,

$$\left| g \left( \frac{[N\mathbf{x}] + \mathbf{1}}{N}, \frac{[Ns]\mathbf{1} - [N\mathbf{y}]}{N} \right) \right| \mathbf{1}_{\{[Ns]\mathbf{1} > [N\mathbf{y}]\}} \leq g^*(\mathbf{x}, s\mathbf{1} - \mathbf{y}) \mathbf{1}_{\{s\mathbf{1} > \mathbf{y}\}}, \quad (58)$$

<sup>2</sup>The presence of  $R_{t,N}$  is due to the fact that the sum over  $n$  in (54) goes up to  $[Nt]$ , whereas the corresponding integral in (55), before the change of variable, goes up to  $Nt$ . Such a remainder  $R_{t,N}$  which turns out to be asymptotically negligible in  $L^2$ , should also have been included in Bai and Taquq [2], following Relation (38) and also in the proof of Theorem 6.10 of that paper.

<sup>3</sup>If  $L$  had not been taken to be 1, we would have an additional term  $L([N\mathbf{x}] + \mathbf{1}, [Ns]\mathbf{1} - [N\mathbf{y}])$ , which tends to 1 as  $N \rightarrow \infty$  by (26). Indeed, we can write  $([N\mathbf{x}] + \mathbf{1}, [Ns]\mathbf{1} - [N\mathbf{y}]) = ([N\mathbf{x}], [N(s\mathbf{1} - \mathbf{y})]) + \mathbf{B}_N(s, \mathbf{y})$ , where for each fixed  $(s, \mathbf{y})$ , the remainder  $\mathbf{B}_N(s, \mathbf{y}) = (\mathbf{1}, [Ns]\mathbf{1} - [N\mathbf{y}] - [N(s\mathbf{1} - \mathbf{y})])$  is bounded with respect to  $N$ , thus we apply (26)

where

$$g^*(\mathbf{z}) = C\|\mathbf{z}\|^\alpha = C(|z_1| + \dots + |z_m|)^\alpha, \quad \mathbf{z} \in \mathbb{R}^m,$$

for some constant  $C > 0$ . This is because  $\{[Ns]\mathbf{1} > [Ny]\} \subset \{s\mathbf{1} > \mathbf{y}\}$ , and on the set  $\{\mathbf{x} > \mathbf{0}, [Ns]\mathbf{1} > [Ny]\}$ , we have  $([N\mathbf{x}] + 1)/N > \mathbf{x}$ , as well as  $([Ns] - [Ny_j])/N \geq \frac{1}{2}(s - y_j)$  (see Relation (40) in the proof of Theorem 6.5 of Bai and Taqqu [2]),  $j = 1, \dots, m - r$ . But by Remark 2.2, for any  $t > 0$  and a.e.  $\mathbf{y} \in \mathbb{R}^{m-r}$ ,

$$\int_0^t ds \int_{\mathbb{R}_+^r} d\mathbf{x} g^*(\mathbf{x}, s\mathbf{1} - \mathbf{y}) 1_{\{s\mathbf{1} > \mathbf{y}\}} = \int_0^t g_r^*(s\mathbf{1} - \mathbf{y}) 1_{\{s\mathbf{1} > \mathbf{y}\}} ds < \infty, \quad (59)$$

where  $g_r^*(\mathbf{y}) = C'\|\mathbf{y}\|^{\alpha+r}$  for some  $C' > 0$  is a GHK on  $\mathbb{R}_+^{m-r}$  (see Lemma 4.2). One hence obtains by (58), (59) and the Dominated Convergence Theorem that  $\tilde{h}_{t,N}(\mathbf{y})$  converges to  $h_t(\mathbf{y})$  for a.e.  $\mathbf{y} \in \mathbb{R}^{m-r}$ . To conclude the  $L^2$ -convergence of  $\tilde{h}_{t,N}$  to  $h_t$ , note that

$$|\tilde{h}_{t,N}(\mathbf{y})| \leq \tilde{h}_t^*(\mathbf{y}) := \int_0^t ds g_r^*(s\mathbf{1} - \mathbf{y}) 1_{\{s\mathbf{1} > \mathbf{y}\}}, \quad (60)$$

where  $\tilde{h}_t^* \in L^2(\mathbb{R}^{m-r})$  by Remark 2.2. Since  $h_t(\mathbf{y}) \in L^2(\mathbb{R}^{m-r})$  as well, we can apply the  $L^2$ -version Dominated Convergence Theorem to conclude (56), because the remainder term  $\tilde{R}_{t,N}$  in (57) satisfies

$$\begin{aligned} \|\tilde{R}_{t,N}(\mathbf{y})\|_{L^2(\mathbb{R}^{m-r})}^2 &\leq \left(\frac{Nt - [Nt]}{N}\right)^2 N^{-(m-r)} \sum_{\ell > \mathbf{0}} \left(\int_{\mathbb{R}_+^r} d\mathbf{x} g^*\left(\mathbf{x}, \frac{\ell}{N}\right)\right)^2 \\ &= N^{-2H} (Nt - [Nt])^2 \sum_{\ell > \mathbf{0}} g_r^*(\ell)^2 \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$  since  $\sum_{\ell > \mathbf{0}} g_r^*(\ell)^2 < \infty$ . We also used the fact that  $\int_{\mathbb{R}_+^r} d\mathbf{x} g^*\left(\mathbf{x}, \frac{\ell}{N}\right) = N^{-\alpha-r} g_r^*(\ell)$  and  $H = \alpha + (m+r)/2 + 1$ .

Finally, the combinatorial coefficients  $d_{k,r}$  in (50) are obtained by counting the ways of choosing  $r$  subsets out of the  $k$  variables, where each subset contains 2 variables, and where the order of the  $r$  subsets does not matter. One can apply the multinomial formula involving  $k$  variables to be divided into one group of  $k - 2r$  variables and  $r$  groups of 2 variables, but since the order of these  $r$  groups is irrelevant, there is an additional division by  $r!$ . Hence

$$d_{k,r} = \frac{k!}{(k-2r)!(2!)^r r!}.$$

□

**Remark 6.3.** We have considered only causal forms because for the coefficient  $a(\mathbf{i})$  for non-causal forms,  $\mathbf{i} \in \mathbb{Z}^k$ , one can specify different homogeneity exponents  $\alpha$  in different orthotopes of  $\mathbf{i}$  for  $a(\mathbf{i})$ , and only the orthotope with the highest  $\alpha$  will contribute in the limit.

**Example 6.4.** Set in Theorem 6.2  $a(\mathbf{i}) = g(\mathbf{i}) = (i_1 + i_2 + i_3 + i_4 + i_5)^\alpha$ , where  $-3 < \alpha < -5/2$ . Hence by (51),

$$H = \alpha + 5/2 + 1 = \alpha + 7/2 \in (1/2, 1).$$

That is, we consider

$$X(n) = \sum_{0 < i_1, \dots, i_5 < \infty} (i_1 + i_2 + i_3 + i_4 + i_5)^\alpha \epsilon_{n-i_1} \epsilon_{n-i_2} \epsilon_{n-i_3} \epsilon_{n-i_4} \epsilon_{n-i_5}.$$

Here  $k = 5$ , and hence  $r$ , which denotes the possible number of pairings of variables, can be 0, 1, or 2. The corresponding functions  $g_r$ 's in (32), are  $g_0 = g$ , where no pairing takes place,

$$g_0(x_1, x_2, x_3) = \int_0^\infty (x_1 + x_2 + x_3 + 2y)^\alpha dy = \frac{1}{2(\alpha+1)} (x_1 + x_2 + x_3)^{\alpha+1},$$

where there is one pairing, and

$$g_2(x_1) = \int_0^\infty \int_0^\infty (x_1 + 2y_1 + 2y_2)^\alpha dy_1 dy_2 = \frac{1}{4(\alpha+1)(\alpha+2)} x_1^{\alpha+2},$$

where there are two pairings. Moreover, by (52),

$$d_{5,0} = 1, \quad d_{5,1} = \frac{5!}{2 \times 3! \times 1} = 10, \quad \text{and} \quad d_{5,2} = \frac{5!}{2^2 \times 1 \times 2} = 15.$$

We have then the following convergence in  $D[0, 1]$ :

$$\frac{1}{N^H} \sum_{n=1}^{[Nt]} [X(n) - \mathbb{E}X(n)] \Rightarrow Z_5(t) + 10Z_3(t) + 15Z_1(t),$$

where

$$Z_5(t) := \int_{\mathbb{R}^5} \int_0^t (5s - x_1 - x_2 - x_3 - x_4 - x_5)^{\alpha+1} 1_{\{s_1 > \mathbf{x}\}} ds B(dx_1)B(dx_2)B(dx_3)B(dx_4)B(dx_5),$$

$$Z_3(t) := \frac{1}{2(\alpha+1)} \int_{\mathbb{R}^3} \int_0^t (3s - x_1 - x_2 - x_3)^{\alpha+1} 1_{\{s_1 > \mathbf{x}\}} ds B(dx_1)B(dx_2)B(dx_3)$$

and

$$Z_1(t) := \frac{1}{4(\alpha+1)(\alpha+2)} \int_{\mathbb{R}} \int_0^t (s - x_1)_+^{\alpha+2} ds B(dx_1).$$

Observe that  $Z_1(t)$  is fractional Brownian motion with  $H = \alpha + 7/2$ , and can be expressed as

$$Z_1(t) = \frac{1}{4(\alpha+1)(\alpha+2)(\alpha+3)} \int_{\mathbb{R}} [(t - x_1)_+^{\alpha+3} - (-x_1)_+^{\alpha+3}] B(dx_1).$$

## 7 Expressing the NCLT limit as a centered multiple Wiener-Stratonovich integral

When Norbert Wiener (see, e.g., Wiener [23]) first introduced the multiple integral with respect to a Brownian motion, he did not exclude the diagonals to render integrals of different orders orthogonal to each other, although the idea of orthogonalization was in fact informally developed (see Lecture 4 in Wiener [23]). Itô [14] modified Wiener's definition by excluding the diagonals, and made the  $k$ -tuple integral  $I_k(f)$  well-defined for all  $f \in L^2(\mathbb{R}^k)$ . Since then, the literature had focused on Itô's off-diagonal integrals. Hu and Meyer [13], however, considered integrals with diagonals and related them to the iterated Stratonovich integrals. Formal theories were later developed in Johnson and Kallianpur [15] and Budhiraja and Kallianpur [6].

We denote the  $k$ -tuple Wiener-Stratonovich integral as  $\mathring{I}_k(\cdot)$ . The integral  $\mathring{I}_k(\cdot)$  and the Wiener-Itô integral  $I_k(\cdot)$  are related through the following Hu-Meyer formula: for a symmetric function  $h \in L^2(\mathbb{R}^k)$ ,

$$\mathring{I}_k(h) = \sum_{r=0}^{[k/2]} d_{k,r} I_{k-2r}(\tau^r h), \quad (61)$$

where  $d_{k,r}$  is as in (52), and  $\tau^r$  is the so-called  $r$ -th  $\tau$ -trace defined as

$$(\tau^r h)(\mathbf{x}) = \int_{\mathbb{R}^r} h(y_1, y_1, y_2, y_2, \dots, y_r, y_r, x_1, \dots, x_{k-2r}) d\mathbf{y},$$

provided that  $\tau^r(h) \in L^2(\mathbb{R}^{k-2r})$  (see Definition 2.7 of Budhiraja and Kallianpur [6]). In the integral defining  $\tau^r(h)$ , we have  $r$  pairs of  $y$ 's. We note that the formula (61) was in fact known to Wiener (see (5.14) of

Wiener [23]). There is also a more general notion of trace than  $\tau^r$ , called the limiting trace and denoted by  $\overrightarrow{\text{Tr}}^r$  (see Definition 2.3 of Budhiraja and Kallianpur [6]), involving tensor products of Hilbert space. It is sufficient for our purpose to focus on the  $\tau$ -trace. Note that if  $k$  is even and  $r = k/2$ , then  $r - k/2 = 0$  in (61). In addition, the following convention is used:

$$I_0(\tau^{k/2}(h)) := \tau^{k/2}(h) = \mathbb{E}\mathring{I}(h).$$

A heuristic understanding of the Hu-Meyer formula (61) is as follows. In the integral

$$\mathring{I}_k(\cdot) = \int_{\mathbb{R}^k} \cdot B(dx_1) \dots B(dx_k)$$

which includes the diagonals (we do not have the prime ' on the integral symbol), let's restrict first the integration set to

$$\{x_1 = x_2 = x, x \neq x_p, x_p \neq x_q, p \neq q \in \{3, \dots, k\}\}.$$

The integrator  $B(dx_1)B(dx_2)$  then becomes  $B(dx)^2 = [B(dx)^2 - dx] + dx$ . The first term  $[B(dx)^2 - dx]$ , whose variance is  $2(dx)^2$ , yields the integral  $\int_{\mathbb{R}^{k-1}} h(x, x, x_3, \dots, x_k) [B(dx)^2 - dx] B(dx_3) \dots B(dx_k)$  with variance  $\int_{\mathbb{R}^{k-2}} \int_{\mathbb{R}} h(x, x, x_3, \dots, x_k)^2 [2(dx)^2] dx_3 \dots dx_k = 0$ , because we have a higher power of  $dx$  than needed. This means that if we integrate on the set indicated above, we end up with

$$\int_{\mathbb{R}^{k-2}} \int_{\mathbb{R}} h(x, x, x_3, \dots, x_k) dx B(dx_3) \dots B(dx_k).$$

If moreover, we integrate on the set  $x_1 = \dots = x_l = x$ ,  $l \geq 3$  and all  $x, x_{l+1}, \dots, x_k$  distinct, using the fact  $\mathbb{E}(B(dx)^{2n}) = (2n-1)!(dx)^n$ , it can be shown that one always ends up with higher power of  $dx$  than needed, and these terms are thus all zero. Hence the only way of getting terms that really contribute is to identify only pairs of the variables, which results in the form (61).

To express the limits in Theorem 6.2 in terms of Wiener-Stratonovich integrals, let

$$h_t(\mathbf{x}) = \int_0^t g(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds, \quad (62)$$

where  $g$  is a GHK(B) on  $\mathbb{R}_+^k$ . Suppose that  $2r < k$ , which is always the case when  $k$  is odd. Then

$$\begin{aligned} \tau^r h_t(\mathbf{x}) &= \int_{\mathbb{R}^r} d\mathbf{y} \int_0^t ds g(s - y_1, s - y_1, \dots, s - y_r, s - y_r, s - x_1, \dots, s - x_{k-2r}) 1_{\{s\mathbf{1} > \mathbf{x}\}} 1_{\{s\mathbf{1} > \mathbf{y}\}} \\ &= \int_0^t ds \int_{\mathbb{R}_+^r} d\mathbf{y} g(y_1, y_1, \dots, y_r, y_r, s - x_1, \dots, s - x_{k-2r}) 1_{\{s\mathbf{1} > \mathbf{x}\}} \\ &= \int_0^t g_r(s\mathbf{1} - \mathbf{x}) 1_{\{s\mathbf{1} > \mathbf{x}\}} ds, \end{aligned}$$

where  $g_r(s\mathbf{1} - \mathbf{x})$  is as given in (53), and the change of integration order is justified by Fubini as the proof of Lemma 4.2. (Observe that  $2r < k$  is an assumption of Lemma 4.2.)

In the special case when  $k$  is even and  $2r = k$ , the change of the integration order cannot be justified by Fubini, and

$$\tau^{k/2}(h_t) = \int_{\mathbb{R}^{k/2}} d\mathbf{y} \int_0^t g(s - y_1, s - y_1, \dots, s - y_{k/2}, s - y_{k/2}) 1_{\{s\mathbf{1} > \mathbf{y}\}} ds$$

may not exist, because for example if  $g(\mathbf{x}) = \|\mathbf{x}\|^\alpha$ , then

$$\int_{\mathbb{R}_+^{k/2}} \|(x_1, x_1, \dots, x_{k/2}, x_{k/2})\|^\alpha d\mathbf{x} = c \int_{\mathbb{R}_+^{k/2}} (x_1 + \dots + x_{k/2})^\alpha d\mathbf{x} = \infty.$$

Theorem 6.2, however, does not involve the term  $r = k/2$  (the zero-order term  $Z_0(t)$  was defined to be zero). So we cannot directly use the Hu-Meyer formula (61) to re-express the limit process  $Z(t)$  in (50) in terms of a multiple Wiener-Stratonovich integral. We have to consider instead the *centered Wiener-Stratonovich integral* which we define as

$$\mathring{I}_k^c(h) = \sum_{0 \leq r < k/2} d_{k,r} I_{k-2r}(\tau^r h), \quad (63)$$

where we do not include the 0-th order (constant) term which arises when  $r = k/2$ . Consequently, the integral has always 0 mean. Note that  $\mathring{I}_k^c$  coincides  $\mathring{I}_k$  when  $k$  is odd, but obviously admits a larger class of integrands  $h$  when  $k$  is even. With this modification, we are able to restate Theorem 6.2 for the long-memory Volterra process as follows.

**Corollary 7.1.** *Let  $X(n) = \sum_{\mathbf{i} \in \mathbb{Z}_+^k} g(\mathbf{i}) L(\mathbf{i}) \epsilon_{n-i_1} \dots \epsilon_{n-i_k}$  be as in Theorem 6.2. Then*

$$\frac{1}{NH} \sum_{n=1}^{\lfloor Nt \rfloor} [X(n) - \mathbb{E}X(n)] \Rightarrow Z(t) = \mathring{I}_k^c(h_t) = \sum_{0 \leq r < k/2} d_{k,r} I_{k-2r}(\tau^r h_t),$$

where  $h_t$  is defined in (62).

## 8 An extended hypercontractivity formula

Suppose that  $\epsilon_i = (\epsilon_i^{(1)}, \dots, \epsilon_i^{(k)})$  are i.i.d. vectors with 0 means and finite variances. Let  $a(\cdot)$  be a function defined on the tetrahedral  $\{(i_1, \dots, i_k) \in \mathbb{Z}_+^k, i_1 < \dots < i_k\}$ , and let

$$T_n^k(a) = \sum_{1 \leq i_1 < \dots < i_k \leq n} a(i_1, \dots, i_k) \epsilon_{i_1}^{(1)} \dots \epsilon_{i_k}^{(k)},$$

where the case  $k = 0$  is understood as a constant  $a$ . For  $k \geq 1$ , define

$$T_{i-1}^{k-1}(a) = \sum_{1 \leq i_1 < \dots < i_{k-1} \leq i-1} a(i_1, \dots, i_{k-1}, i) \epsilon_{i_1}^{(1)} \dots \epsilon_{i_{k-1}}^{(k-1)}.$$

Then

$$T_n^k(a) = \sum_{i=k}^n T_{i-1}^{k-1}(a) \epsilon_i^{(k)}. \quad (64)$$

Let  $\mathcal{F}_n = \sigma(\epsilon_i, i \leq n)$ . Then  $\{T_n^k(a), n \geq k\}$  is a martingale with respect to  $\mathcal{F}_n$  and (64) is a decomposition into martingale differences since

$$\mathbb{E}[T_{n-1}^{k-1}(a) \epsilon_n^{(k)} | \mathcal{F}_{n-1}] = T_{n-1}^{k-1}(a) \mathbb{E}[\epsilon_n^{(k)}] = 0.$$

**Lemma 8.1** (Burkholder [7]). *Let  $p > 2$ , and let  $X_i$  be martingale differences. Then for some constant  $C_p > 0$ , we have*

$$\left\| \sum_{i=1}^n X_i \right\|_p \leq C_p \left\| \left( \sum_{i=1}^n X_i^2 \right)^{1/2} \right\|_p.$$

**Lemma 8.2.** *Let  $p > 2$ , and let  $Y_i$ 's be random variables such that  $\mathbb{E}|Y_i|^p < \infty$ . Then*

$$\left\| \left( \sum_{i=1}^n Y_i^2 \right)^{1/2} \right\|_p \leq \left( \sum_{i=1}^n \|Y_i\|_p^2 \right)^{1/2}.$$



*Proof.* By Minkowski's inequality,

$$\left\| \left( \sum_{i=1}^n Y_i^2 \right)^{1/2} \right\|_p^2 = \left[ \mathbb{E} \left( \sum_{i=1}^n Y_i^2 \right)^{p/2} \right]^{2/p} = \left\| \sum_{i=1}^n Y_i^2 \right\|_{p/2} \leq \sum_{i=1}^n \|Y_i^2\|_{p/2} = \sum_{i=1}^n (\mathbb{E}|Y_i|^p)^{2/p} = \sum_{i=1}^n \|Y_i\|_p^2.$$

□

The following result is used in Lemma 5.1.

**Theorem 8.3.** *If  $p > 2$ , and  $\mathbb{E}|\epsilon_i^{(j)}|^p < \infty$ , then*

$$\|T_n^k(a)\|_p \leq c \|T_n^k(a)\|_2$$

where  $c$  is a constant that does not depend on  $a(\cdot)$  nor on  $n$ .

*Proof.* We prove it by induction. The case  $k = 0$  is trivial since  $T_n^0(a)$  is a constant. Suppose that the inequality holds for  $k - 1$ , where  $k \geq 1$ . Then by the forgoing lemmas,

$$\begin{aligned} \|T_n^k(a)\|_p &\leq C_p \left\| \left( \sum_{i=k}^n |T_{i-1}^{k-1}(a)\epsilon_i^{(k)}|^2 \right)^{1/2} \right\|_p \\ &\leq C_p \left( \sum_{i=k}^n \|T_{i-1}^{k-1}(a)\epsilon_i^{(k)}\|_p^2 \right)^{1/2} \\ &= C_p \left( \sum_{i=k}^n \|T_{i-1}^{k-1}(a)\|_p^2 \|\epsilon_i^{(k)}\|_p^2 \right)^{1/2}, \end{aligned}$$

by independence between  $T_{i-1}^{k-1}(a)$  and  $\epsilon_i^{(k)}$ . By the induction assumption,  $\|T_{i-1}^{k-1}(a)\|_p^2 \leq c_1 \|T_{i-1}^{k-1}(a)\|_2^2$  for some  $c_1 > 0$  which does not depend on  $a(\cdot)$  or  $n$ . In addition, trivially since the random vectors  $\{\epsilon_i, i \in \mathbb{Z}\}$  are identically distributed, one has  $\|\epsilon_i^{(k)}\|_p^2 \leq c_2 \|\epsilon_i^{(k)}\|_2^2$  for some  $c_2 > 0$  which does not depend on  $a(\cdot)$ ,  $n$  or  $i$ . The desired result is then immediate once noting that

$$\|T_n^k(a)\|_2^2 = \sum_{i=k}^n \|T_{i-1}^{k-1}(a)\|_2^2 \|\epsilon_i^{(k)}\|_2^2$$

due to the off-diagonality of  $T_n^k(a)$  and independence. □

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