

Power Grid Decomposition Based on Vertex Cut Sets and Its Applications to Topology Control and Power Trading

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Abstract—It is well known that the reserves/redundancies built into the transmission grid in order to address a variety of contingencies over a long planning horizon may, in the short run, cause economic dispatch inefficiency. Accordingly, power grid optimization by means of short term line switching has been proposed and is typically formulated as a mixed integer programming problem by treating the state of the transmission lines as a binary decision variable, i.e. in-service or out-of-service, in the optimal power flow problem. To handle the combinatorial explosion, a number of heuristic approaches to grid topology reconfiguration have been proposed in the literature. This paper extends our recent results on the iterative heuristics and proposes a fast grid decomposition algorithm based on vertex cut sets with the purpose of further reducing the computational cost. The paper concludes with a discussion of the possible relationship between vertex cut sets in transmission networks and power trading.

I. INTRODUCTION

Siting and maintaining massive power infrastructure is not cheap [1] and therefore makes the optimal use of existing network a priority. Because of the fast time constants in changing the system state and the very low costs, corrective switching operations, including transmission line switching, bus-bar switching, and shunt element switching, etc., are often the first post-contingency corrective action to be considered and implemented [2].

The focus of corrective switching has been mainly on handling line overload [3], [4], voltage violations [5], [6], and most recently, on co-optimizing the generation along with the network topology [7], [8], [9]. Based on the DC optimal power flow (OPF) problem, such co-optimization is typically formulated as a mixed integer programming problem (MIP) with some binary variables denoting the choices of transmission line switches. There are a number of challenges in implementing effective transmission topology control, including understanding and avoiding voltage problems, transient instability, reactive power problems [10] and, most importantly, the ability to find a good solution within a time that is short enough to be of practical use.

Due to the combinatorial nature of the problem and the nonlinearities inherent to networks, the approaches proposed in the literature focus on the reduction of the search space of possible switching actions [11]. Accordingly, a set of fast

line switching heuristics have been proposed, and the running time records have been consistently set and then shattered [7], [8], [9]. In all of these studies, simulations show that significant savings are indeed possible from transmission switching. While optimality for this NP-hard problems is ever elusive, theorems together with simulation results on a set of iterative heuristics in [12], nevertheless, offer some justification that, with similar search space, the locally optimal switching at each stage of an iterative heuristic has the best chance in reducing generation cost while enforcing connectivity.

The present paper examines possible grid decomposition approaches that are able to reduce the search space of possible switching operations while at the same time retaining the locally optimal switchings to the maximum degree. The paper is organized as follows. In Section II, we first discuss the typical limitations with respect to the *composability* and *compositionality* of real transmission grids, showing the challenges in grid decomposition. A series of trade-offs is then proposed, and this forms the main thread of the paper. Section III proposes a decomposition method based on *vertex cut sets* and *pseudo biconnected components*. Rather than intuitively using *line outage distribution factor* to quantify the decomposition quality, Section IV makes a further trade-off by justifying the usage of *locational marginal price range* to quantify the potential of a pseudo biconnected component in reducing generation cost. The useful properties of the vertex cut set in LMP partitioning and power/FTR trading are discussed in detail in Section V. A fast grid decomposition algorithm is then proposed in Section VI, and its efficiency is tested in Section VII. The paper ends with concluding remarks in Section VIII.

II. COMPOSABILITY AND COMPOSITIONALITY

When proposing heuristics to deal with the “largest machine in the world”, the transmission network, one natural idea is a divide-and-conquer strategy. We ask whether this large and complex system can be usefully decomposed into smaller, more tractable subsystems, thereby reducing the search space of line switching. Here, we refer the ease of building a system out of subsystems as *composability*, and the ease of validation of system properties by using the related subsystems as *compositionality* [13].

A transmission network can be modeled as a graph $G = (V, E)$ such that each node in V is either a substation bus, or simply a bus connecting several transmission lines. There is an edge $e_{i,j} = (v_i, v_j) \in E$ between two nodes if there is a physical line connecting directly the elements represented by

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v_i and v_j . Note that the terms vertex/node/bus, link/line/edge and grid/graph/network will be used interchangeably.

In graph theory, a *biconnected component* in a graph is one where if any node is removed (with its boundary edges), the component will still be connected [14]. Reference [15] proves that the impact of topology control is strictly confined within the biconnected component in which the line switching is applied in a transmission network. The decomposition of a transmission network based on biconnected components is thus “clean” in the sense that if the sizes of all its biconnected components are relatively small, the network will have high level composability and compositionality.

Also proved in [15], the event occurs with probability 1 that disconnecting a transmission line will change every line flow within the biconnected component that contained the disconnected line. The N-1 reliability standard of today’s transmission network, requiring the system to move to a satisfactory state for any single line outage event, significantly increases the likelihood of the whole network or a large portion of the network being biconnected. Despite of the desirability of maintaining N-1 connectivity, the downside of large biconnected components is a significantly decreased level of the network composability and compositionality.

While we should not neglect that every switchable line has the potential to alleviate the congestion in a biconnected component of a grid, simulation results in [8] showed that a fairly large portion of the switchable lines were never switched in most samples, and thereby might well go unnoticed. Such observations suggest the possibility that a biconnected component of a grid may still be effectively further decomposed even if the interactions between the sub-grids of the biconnected component are not zero.

Our work thus aims to find an optimal trade off between the size of sub-grids of the biconnected component and the realized composability/compositionality by the grid decomposition. The desired outcome should help us focus attention on a relatively small set of promising switchable lines such that both the computational efforts and the attainable savings can be reasonably controlled.

III. VERTEX CUT SETS & PSEUDO BICONNECTED COMPONENTS

The initially “clean” decomposition of a transmission grid involves a graph theory term called the *cut vertex* [14] which is any single vertex/node whose removal increases the number of connected components of a graph and, by definition, is the only element shared by two or more biconnected components of a graph.

Apparently, further decomposing a biconnected component needs something beyond the cut vertex, and we propose the following:

Definition 1: A *vertex cut set* is a set of vertices of a graph which, if removed together with any incident edges, disconnects the graph. The sub-graphs of a biconnected graph obtained by one or several vertex cut sets are called *pseudo biconnected components*. The vertices in the vertex cut sets are the only elements shared by two or more pseudo

biconnected components. In addition, the union of several vertex cut sets for a given graph is also a vertex cut set.

Using the concept of vertex cut sets, a two-step construction of pseudo biconnected components can be carried out. Given a graph and a vertex cut set, the disconnected subgraphs remaining when the vertex cut set, together with any incident edges, is removed are called the *pre pseudo biconnected components*. From these, the pseudo biconnected components are obtained by adjoining to each pre pseudo biconnected component the entire vertex cut set along with the incident edges that have been removed. Fig. 1 shows an example of the decomposition of a biconnected graph where the three red nodes in the middle of graph form the vertex cut set. Note that the decomposition based on one vertex cut set may not be unique. For example, the horizontal black edge in Fig. 1(a) can be either assigned to the black-edge sub-graph (Fig. 1(c1)) or to the blue-edge sub-graph (Fig. 1(c2)). Thus we do have some degree of discretion in decomposing the grid. To reduce the search space of line switching, we will use the convention throughout this paper that, whenever possible, any incident edges of the vertex cut set subject to our discretion will be assigned to a pseudo biconnected component that is free of congested lines.

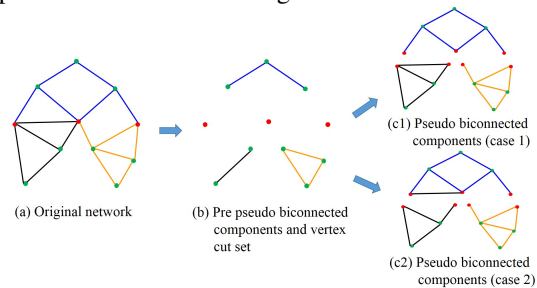


Fig. 1. One example of the decomposition based on a vertex cut set formed by the three red nodes in the middle.

Ideally, we would expect the decomposition in Fig 1 to satisfy the following criteria:

- (1) The interactions between any two lines belonging to two different pseudo biconnected components are insignificant;
- (2) The size of the targeting pseudo biconnected component is large enough to contain all promising switchings;
- (3) The size of the targeting pseudo biconnected component is small enough to be efficiently searched.

IV. QUANTIFYING DECOMPOSITION QUALITY

So far, the interactions mentioned in Criterion (1) at the end of Section III are imprecisely specified, and hence we look for some metrics to quantify the effects of switching lines out of service.

We introduce the well-known sensitivity, *line outage distribution factor* or LODF, denoted by $\zeta_i^{(j)}$, of the line flow i with respect to the removal of line j . In short, $\zeta_i^{(j)}$ denotes the percentage of line flow on line j that will show up on line i as a result of taking line j out of service. Similarly, we can compute the LODF, denoted by $\zeta_j^{(i)}$, of the power

flowing through line j with respect to the removal of line i . The pair of LODFs, $\zeta_i^{(j)}$ and $\zeta_j^{(i)}$, are usually different. The interactions between the line pair $\{i, j\}$ are said to be *insignificant* if $\zeta_i^{(j)}$ and $\zeta_j^{(i)}$ are both small. Similarly, the interactions between two pseudo biconnected components are said to be insignificant if the pair of LODFs for any two lines belonging to two different pseudo biconnected components are small.

To quantify the quality of grid decomposition based on vertex cut sets, we need to recalculate the LODFs for each possible line switching – which could be computationally intensive for large scale grids.

The computational overhead pushes us to make another trade off between the computational costs and the preciseness of the metrics measuring the interactions in a grid. Such trade off is achieved by introducing another well-known sensitivity, the *power transfer distribution factor* or PTDF, which gives the sensitivity of the flow on one line with respect to a power transfer between a pair of buses.

The PTDF, by definition, is essentially the difference between a pair of *shift factors*. In the DC power flow model, the *transmission sensitivity matrix* Ψ , also known as the *injection shift factor matrix*, gives the sensitivities in line flows due to changes in the nodal injections, with the reference bus assumed to ensure the power balance. Suppose Ψ_{i,j_1} denotes the sensitivity in power flowing through line i due to one unit increase in the power injection at bus j_1 and one unit decrease in power injection at the reference bus, i.e.

$$\Psi_{i,j_1} = \frac{\Delta \text{flow in line } i}{\Delta \text{injection at bus } j_1}.$$

Then the shift factor difference $|\Psi_{i,j_1} - \Psi_{i,j_2}|$ gives the PTDF of power flowing through line i with respect to one unit power transfer between bus j_1 and j_2 .

The network LODFs, mathematically, are equivalent to the PTDFs after the line in question has been removed. While the pre- and post-removal PTDFs are rarely equivalent, there usually exists a strong correlation between their values. Therefore, although the estimations of the decomposition quality based on pre-removal PTDFs maybe different from those based on LODFs, such differences are reasonably expected to be small.

Here, we would like to remark that the introduction of PTDF does reduce the computation cost associated with estimating decomposition quality. But such computation saving is still relatively small compared with solving the updated DCOPFs. The value of the following PTDF analysis, in fact, is more towards a fundamental understanding of some structural properties of the grid decomposition, thus reducing the computational effort spent on finding the optimal vertex cut sets.

We now need to deal with the most challenging step of the decomposition, i.e. minimizing the interactions (PTDFs) between pseudo biconnected components. Though highly desired, such decomposition quality is usually not achievable by just using vertex cut sets as the cushion zones between pseudo biconnected components. This is due to the

unavoidable strong LODFs between those edges belonging to different pseudo biconnected components but incident to same nodes in the vertex cut sets. Instead of complicating the decomposition by expanding the cushion zones, our decomposition instead aims to minimize the potentially “useful” portion of those PTDFs, i.e. the portion associated with the congestion effect.

When talking about the congestion effect, it is natural to wonder about the *locational marginal price* (LMP). In a lossless DC power flow model, the LMP at a given bus, say bus i , is equal to the sum of the LMP at the reference bus and the congestion cost which can be calculated using the following equation:

$$\lambda_i = \lambda_{ref} - \sum_j (\mu_j \times \Psi_{j,i}) \quad (1)$$

where

$$\begin{aligned} \lambda_i &= \text{LMP at bus } i \\ \lambda_{ref} &= \text{LMP at the reference bus} \\ \mu_j &= \text{Shadow price of line constraint } j \\ \Psi_{j,i} &= \text{Shift factor for real load at bus } i \text{ on line flow } j. \end{aligned}$$

The sensitivity analysis of the DCOPF model suggests that power transfer between buses with close LMPs usually can’t change the generation cost as much as that between buses with significantly different LMPs, and similarly line switches in areas with low range LMPs are typically of little use in reducing overall generation costs. Therefore, the goal of the partition problem based on vertex cuts is to decompose the grid into two sub-grids with one being of high range LMPs and the other being of low range LMPs.

V. THE OPTIMALITY OF VERTEX CUT SETS

In this section, we will show a set of useful properties of vertex cut sets with respect to the decomposition based on LMPs. The discussion starts with the following definition.

Definition 2: Suppose there is one line congestion in a transmission network and we are able to control certain number of nodal injections, then the smallest set (i.e. of smallest cardinality) of buses that is able to fully alleviate the given line congestion by changing least amount of nodal power injections is defined as the *optimal* set of buses for that congested line.

Remark 1: The optimal set of buses for a given congested line may not be unique.

We first show that for a given congestion in one pseudo biconnected component, the optimal set of buses in another pseudo biconnected component must be a subset of the vertex cut set. The complete proof involves two lemmas, one theorem, and one proposition.

We start the proof by introducing the well established correspondence between power grids and electric circuits. Mathematically, the DC power flow model is equivalent to a current driven network (an electric circuit comprised purely by resistors and current sources) [15], where power injections are equivalent to current sources; power flowing through lines is equivalent to current through edges, etc. See Table 1. Thus part of the following discussion will be focusing on

our previously studied resistance networks with nodal current injections controlled by a set of current sources [15].

network	potential	injection	admittance	equation
circuit	voltage V	current I	conductance G	$I = GV$
grid	phase θ	power P	susceptance B	$P = B\theta$

Table 1: The equivalence between current driven circuits and DC power flow models of transmission grids.

Lemma 1: If we are allowed to alter all controllable nodal power injections by arbitrary amount and all uncongested lines have enough line capacity, then the cardinality of the optimal set of buses for a given congested line must be 2, i.e. the optimal set of buses is a bus pair.

Proof: Shift factors do not change when the injection/withdrawal amount increases for any set of buses. Suppose line i is overloaded by Δ_i units of power and the power injections at bus j_1 and j_2 are controllable. In order to alleviate the congestion of line i , the power injections at bus j_1 and j_2 need to be changed by

$$\Delta_{j_1 \& j_2} = \frac{\Delta_i}{|\Psi_{i,j_1} - \Psi_{i,j_2}|}.$$

Thus the pair of controllable buses with the largest shift factor difference associated with the congested line must be the optimal set of buses for that congested line. ■

Theorem 1: ([16]) Any n -terminal connected resistance network is electrically equivalent to a circuit whose graph is a complete graph consisting of $n(n-1)/2$ resistances. Such n -terminal equivalent graph can be obtained by using Kron Reduction to eliminate all interior (non-terminal) nodes.

Remark 2: The above theorem can be easily extended to any disconnected graph by adding some trivial lines of infinite resistance to make the equivalent graph connected and complete. The current flowing through those trivial lines must always be zero no matter how the nodal current injections are changed.

Lemma 2: Suppose line i is connected to bus pair $\{i_1, i_2\}$ and line j is connected to bus pair $\{j_1, j_2\}$ in a connected transmission network, and the reference directions of the power flowing through line i and line j are defined as from bus i_2 to bus i_1 and from bus j_2 to bus j_1 , respectively. Then the shift factor differences $\Psi_{i,j_2} - \Psi_{i,j_1}$ and $\Psi_{j,i_2} - \Psi_{j,i_1}$ must be of the same sign.

Proof: By keeping applying Kron reduction [16] to the original transmission network (shown in the right side of Fig. 2) until all nodes but $\{i_1, i_2, j_1, j_2\}$ are eliminated, we can create a 4-terminal equivalent grid with only four buses $\{i_1, i_2, j_1, j_2\}$ which is shown in the left side of Fig. 2. Note that the two red lines in Fig. 2 may be of different susceptance but the direction of power flowing the lines, P_j and $P_{j'}$, must be the same. The same is true of the two blue lines. Thus we focus our attention on the 4-terminal equivalent grid.

By the equivalence in Table 1, we can actually treat the 4-terminal equivalent grid as a 4-terminal resistance network and discuss the problem in a circuit environment. Using the

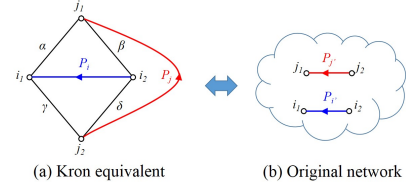


Fig. 2. The 4-terminal equivalent network of the original network.

circuit equivalent, we denote the line resistances of the four black lines in the left side of Fig. 2 by α , β , γ and δ , respectively, and suppose there are no current injections and withdrawals in the network except one unit nodal injection at node i_2 and one unit nodal withdrawal at node i_1 . If the direction of red line flow in the left figure is from node j_2 to node j_1 , we know $\Psi_{j,i_2} - \Psi_{j,i_1} > 0$, indicating

$$\frac{\delta}{\gamma} < \frac{\beta}{\alpha},$$

that we must have $\Psi_{i,j_2} - \Psi_{i,j_1} > 0$. If the direction of red line flow in the left figure is from node j_1 to node j_2 , we know $\Psi_{j,i_2} - \Psi_{j,i_1} < 0$, indicating

$$\frac{\delta}{\gamma} > \frac{\beta}{\alpha},$$

that we must have $\Psi_{i,j_2} - \Psi_{i,j_1} < 0$. ■

Proposition 1: Suppose we use a vertex cut set to decompose a connected transmission network into n ($n \geq 2$) sub-graphs. Assume that we can arbitrarily change the power injection at all buses in the j -th sub-graph to alleviate a given congested line in the i -th sub-graph ($i \neq j$), then one optimal pair of buses must be a subset of the vertex cut set.

Proof: Although the value of each entry in the injection shift factor matrix varies as the reference bus changes, the PTDF or the difference between a pair of shift factors is independent of the choice of the reference bus. Suppose line i is congested and is connected to bus i_1 and i_2 , and we pick up i_1 as the reference bus and assuming the direction from i_2 to i_1 as the positive power direction, we can get an injection shift factor matrix with all i -th row entries being non-negative.

Suppose there are M buses $\{n_1, \dots, n_M\}$ that are shared by the j -th sub-graph and the rest of the graph. Clearly, $\{n_1, \dots, n_M\}$ is a subset of the vertex cut set. Then in order to prove the proposition, we just need to show that the shift factor differences $\{\Psi_{i,p} - \Psi_{i,n_1}, \dots, \Psi_{i,p} - \Psi_{i,n_M}\}$ cannot be all positive or all negative for any bus p that is owned exclusively by the j -th sub-graph, i.e.

$$\min\{\Psi_{i,n_1}, \dots, \Psi_{i,n_M}\} \leq \Psi_{i,p} \leq \max\{\Psi_{i,n_1}, \dots, \Psi_{i,n_M}\}.$$

We first create two sub-networks to simplify the proof. The first sub-network (shown at the bottom middle of Fig. 3) is created from the original network by removing all lines in the j -th sub-graph, and it is called the nearby sub-network. The second sub-network (shown at the top middle of Fig. 3) is just the j -th sub-graph, and it is called the remote sub-network. It is easy to see that the two sub-networks share

nothing except the buses $\{n_1, \dots, n_M\}$. Buses $\{i_1, i_2\}$ are interior buses of the nearby sub-network, and bus p is interior bus of the remote sub-network.

Then an equivalent sub-network of the remote sub-network can be created by applying Kron reduction to the remote sub-network until all nodes but $\{n_1, \dots, n_M, p\}$ are removed. The equivalent sub-network obtained in this way is a K_{M+1} complete graph as shown at the top right corner of Fig. 3. An equivalent network of the original network can then be created by putting together the nearby sub-network and the K_{M+1} equivalent sub-network. The process described above is visualized in Fig. 3.

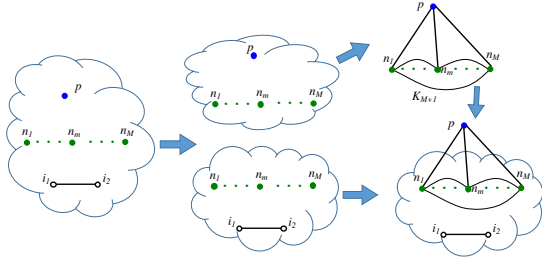


Fig. 3. The network simplification process.

Suppose the reference direction of the power flowing through the line connecting bus p and n_m ($m = 1, \dots, M$) in the equivalent network (shown at the bottom right corner of Fig. 3) is defined as from bus p to bus n_m . Then by Lemma 2, we know that the change of power flowing through the line connecting bus p and n_m ($m = 1, \dots, M$) in the equivalent network due to one unit increase in nodal injections at bus i_2 and one unit decrease in nodal injections at bus i_1 is of the same sign as that of the shift factor difference $\Psi_{i,p} - \Psi_{i,n_m}$.

Finally, we denote the current flowing through the lines connecting node p and node n_m ($m = 1, \dots, M$) as $\{I_{p,n_1}, \dots, I_{p,n_M}\}$ due to one unit nodal injection at bus i_2 and one unit nodal withdrawal at bus i_1 . By Kirchoff's current law, we must have

$$\sum_{m=1}^M I_{p,n_m} = 0$$

meaning

$$\sum_{m=1}^M c_m (\Psi_{i,p} - \Psi_{i,n_m}) = 0$$

where $I_{p,n_m} = c_m \Psi_{i,n_m}$ and the c_m 's are some positive coefficients by Lemma 2. Thus $\{\Psi_{i,p} - \Psi_{i,n_1}, \dots, \Psi_{i,p} - \Psi_{i,n_M}\}$ cannot be all positive or all negative. ■

Rather than limiting the discussion to single line congestion, we'd like to know if phenomena similar to Proposition 1 can be found for scenarios with multiple congestions. Things become a little complicated here as the changes of line flows of the congested lines due to the nodal injection regulations may not be all favorable, i.e. the alleviation of some congestions may come inevitably at the expense of worsening other congestions, requiring all congestions being

assigned proper weights to evaluate the overall benefit of the nodal injection regulations. Since our main consideration is to reduce generation cost, the *shadow prices* of line constraints are naturally used as the weights in the following discussion.

We will show that for multiple given congestions in one pseudo biconnected component, the optimal set of buses in another pseudo biconnected component must also be a subset of the vertex cut sets. The complete proof involves one proposition and two corollaries.

Proposition 2: Suppose we use a vertex cut set to decompose a connected transmission network into n ($n \geq 2$) sub-graphs. Assume that we can arbitrarily change the power injections at buses in the j -th sub-graph to alter the line flows in the i -th sub-graph ($i \neq j$), then such impact on the i -th sub-graph can be precisely replicated/canceled by regulating the power injections of the buses in the vertex cut set.

Proof: By the superposition principle in the DC power flow model, we just need to prove that all line flows in the i -th sub-graph can always be set to zero if the net power injections at all interior buses of the i -th sub-graph are strictly zero and if we can arbitrarily regulate the power injections at the buses in the vertex cut set.

We first show that the reduced nodal susceptance matrix B of any transmission network is positive definite. Since any resistance network is a passive system that consumes energy, and the absorbed power is given by

$$P = v^T G v \quad (2)$$

where v is the nodal voltage vector and G is the reduced conductance matrix of the resistance network. It follows that the energy consumption will always be positive only if all eigenvalues of G are positive. Thus, a criterion for passivity is that G be positive definite. Since the DC model of power flow is equivalent to a current driven network as shown in Table 1, we know the reduced susceptance matrix B in the DC power flow model is positive definite, meaning its inverse matrix B^{-1} is also positive definite.

Suppose, in the bus vector, the first l buses are owned exclusively by the i -th sub-graph, the middle m buses are owned by the vertex cut set, and the last $n+1$ buses including the reference bus are owned exclusively by other sub-graphs. We then have the DC power flow equation

$$\theta = B^{-1} p \quad (3)$$

where p is the net nodal power injection vector, and θ is the nodal voltage angle vector. Based on the classification of the buses, the reduced voltage angle vector θ can be decomposed into 3 sub-vectors

$$\theta^T = [\theta_l^T \quad \theta_m^T \quad \theta_n^T],$$

and the positive definite matrix B^{-1} can be decomposed into 9 sub-matrices

$$B^{-1} = \begin{array}{ccc} \begin{array}{ccc} in & at & out \\ \downarrow & \downarrow & \downarrow \\ A_{l \times l} & B_{l \times m} & C_{l \times n} \end{array} & \begin{array}{l} \leftarrow in \\ \leftarrow at \\ \leftarrow out \end{array} \end{array} \quad (4)$$

and we know the sub-matrix $E_{m \times m}$ must have full rank and be positive definite. Thus we have

$$\theta_m = [D_{m \times l} \quad E_{m \times m} \quad F_{m \times n}] p.$$

It follows that we can always set the voltage angles of the vertex cut set θ_m to be equal if we can arbitrarily regulate the power injections of the buses in the vertex cut set.

Next we are going to prove that the line flows in the i -th sub-graph must be all zero if we set the voltage angles θ_m of the vertex cut set to be equal and if the net power injections at the interior buses of the i -th sub-graph are all zero. If under such scenario there still exist some non-zero line flows in the i -th sub-graph, then we know that the voltage angles of the interior buses of the i -th sub-graph can not be equal, meaning at least one of the interior buses is of non-zero net power injection. This contradicts our assumption. ■

Corollary 1: Suppose we use a vertex cut set to decompose a connected transmission network into n ($n \geq 2$) connected sub-graphs. We first classify the columns of the injection shift factor matrix into 3 sets with the first set being associated with the buses owned exclusively by the i -th sub-graph, and the second set being associated with the buses in the vertex cut set, and the third set being associated with the buses owned exclusively by other sub-graphs. Assume their cardinalities are l , m , and n , respectively. We then classify the rows of the injection shift factor matrix into 2 sets with the first set being associated with the lines owned exclusively by the i -th sub-graph, and the second set being associated with the lines owned exclusively by other sub-graphs, and assume their cardinalities are x , and y , respectively. A visualization of the decomposition is shown below.

$$\Psi = \begin{array}{ccc} \begin{array}{ccc} in & at & out \\ \downarrow & \downarrow & \downarrow \\ A_{x \times l} & B_{x \times m} & C_{x \times n} \end{array} & \begin{array}{l} \leftarrow in \\ \leftarrow out \end{array} \end{array} \quad (5)$$

Then the ranks of the sub-matrix

$$[B_{x \times m} \quad C_{x \times n}]$$

and

$$[D_{y \times l} \quad E_{y \times m}]$$

must both be $\leq m$.

Proof: By Proposition 2, we know that the effect of one unit net power injection at a bus in the j -th sub-graph on lines in the i -th sub-graph is a linear combination of the effects of one unit net power injections at the buses in the vertex cut set on lines in the i -th sub-graph. Thus we know each column of

the sub-matrix $[C_{x \times n}]$ is a linear combination of the columns of the sub-matrix $[B_{x \times m}]$, meaning the rank the sub-matrix $[B_{x \times m} \quad C_{x \times n}]$ is $\leq m$. Similarly, we can prove that each column of the sub-matrix $[D_{y \times l}]$ is a linear combination of the columns of the sub-matrix $[E_{y \times m}]$, meaning the rank the sub-matrix $[D_{y \times l} \quad E_{y \times m}]$ is $\leq m$. ■

Corollary 2: Suppose we use a vertex cut set to decompose a connected transmission network into n ($n \geq 2$) sub-graphs and all congested lines are contained in the i -th sub-graph (which may be unconnected), then one pair of buses with maximum LMP difference in the j -th sub-graph ($i \neq j$) must be in the vertex cut set.

Proof: Here, we just need to show that the LMP at any bus in the j -th sub-graph is a convex combination of the LMPs at the buses in the vertex cut set.

Suppose there are m buses in the vertex cut set and their LMPs are $\{\lambda_1, \dots, \lambda_m\}$, and we randomly pick a bus, say bus $j1$ with LMP being λ_{j1} , from the j -th sub-graph. By Proposition 2, we know that the effect of one unit net power injection at a bus $j1$ on lines in the i -th sub-graph is a linear combination of the effect of one unit net power injections at the m buses in the vertex cut set on lines in the i -th sub-graph, and we assume the coefficients of the linear combination are $\{a_1, \dots, a_m\}$, where the power balance equation ensures that $a_1 + a_2 + \dots + a_m = 1$. Thus we have

$$\vec{\Psi}_{i,j1} = [\vec{\Psi}_{i,v_1}, \dots, \vec{\Psi}_{i,v_m}] [a_1, \dots, a_m]^T,$$

where $\vec{\Psi}_{i,j1}$ denotes the shift factor vector for real power at bus $j1$ on lines in the i -th sub-graph, and $\vec{\Psi}_{i,v_k}$ ($k = 1, \dots, m$) denotes the shift factor vector for real power at the k -th vertex cut set bus on lines in the i -th sub-graph.

Since we assume all congestions are in the i -th sub-graph, we know the shadow prices for line constraints that are not in the i -th sub-graph are all zero. Denoting the shadow price vector for all line constraints in the i -th sub-graph by $\vec{\mu}_i$, we have

$$\begin{aligned} \lambda_{j1} &= \lambda_{ref} - \vec{\mu}_i^T \vec{\Psi}_{i,j1} \\ &= \lambda_{ref} - \vec{\mu}_i^T [\vec{\Psi}_{i,v_1}, \dots, \vec{\Psi}_{i,v_m}] [a_1, \dots, a_m]^T \quad (6) \\ &= a_1 \lambda_1 + a_2 \lambda_2 + \dots + a_m \lambda_m \end{aligned}$$

Next we will show that $a_i \geq 0$ ($i = 1, \dots, m$), i.e. to cancel the effect of one unit net power injection at bus $j1$ on lines in the i -th sub-graph requires a non-negative net power withdrawn at each bus in the vertex cut set. Apparently, the bus with the lowest voltage angle must be one of the buses with positive net power withdrawn (or negative net power injection) and its voltage angle must be strictly lower than any bus with positive net power injection. Suppose we only inject and withdraw power at bus $j1$ (which has one unit net power injection) and the buses in the vertex cut set, and their effects on the i -th sub-graph precisely offset each other. Therefore, the bus with lowest voltage angle must be in the vertex cut set. If one of the buses in the vertex cut set has positive net power injection, meaning its voltage angle is different from the lowest nodal voltage angle, then there exists a nodal voltage angle difference in the vertex cut set.

In such case, the absorbed power in the i -th sub-graph

$$P_i = [\theta_i^T \theta_m^T] B_i [\theta_i^T \theta_m^T]^T$$

must be positive where θ_i is the voltage angle vector of all interior buses in the i -th sub-graph, θ_m is the voltage angle vector of all buses in the vertex cut set and B_i is the reduced susceptance matrix of the i -th sub-graph. This means some line power flows in the i -th sub-graph must be non-zero, indicating a contradiction to the assumption that the effects of power injections/withdrawals at bus $j1$ and at the buses in the vertex cut set on the i -th sub-graph precisely offset each other. ■

Remark 3: Corollary 2 points out a useful property of vertex cut set in the trading of power or financial transmission rights (FTRs). Essentially, the main task in power trading is to find those pairs of buses between which the LMP differences are significant. If a vertex cut set can decompose the grid such that all congestions are isolated to one or several sub-grids, then the range of the LMPs of buses in the vertex cut that form the boundary of the congested sub-grid(s) will dominate the LMP range of all buses in the non-congested sub-grids. Loosely speaking, the closer the vertex cut gets to the congestions, the higher trading values of the buses in the vertex cut set. Or in other words, if the congested sub-grid(s) defined by one given vertex cut set is a subset of the congested sub-grid(s) defined by another vertex cut set, then the buses in the given vertex cut must be of higher or at least equal trading value compared with those in the other vertex cut set, suggesting an arbitrage opportunity.

VI. GRID DECOMPOSITION HEURISTIC

This section presents a general algorithm structure for grid decomposition heuristics. The objective of the algorithm is eliminating the portion(s) of the grid with relatively small range of LMPs from the consideration of topology control with little computational effort. The algorithm is specified in Fig. 4. This simple algorithm structure first solves an OPF and, given the OPF results, if the OPF is feasible, finds an appropriate vertex cut set to decompose the grid into two parts: one with relatively high range of LMPs and the other with relatively low range of LMPs.

The initial vertex cut set that includes the ending buses of all congested lines forms the boundary of the smallest sub-grid containing all congestions. The algorithm then calculates the mean LMP and the absolute deviations from the mean LMP for buses in the initial vertex cut set, and locates the bus, say bus i , with the largest LMP absolute deviation. It then “pushes out” the initial vertex cut from bus i by swapping out bus i (and therefore bus i becomes an interior bus of the congested sub-grid) and bringing bus i ’s neighboring buses into the vertex cut set.

A pre-set LMP range threshold is the characterizing element for the stopping criteria which determines the number of iterations applied, and can include a number of conditions. The algorithm terminates if the range of the LMP of the vertex cut set falls below the pre-set threshold or if all buses have been brought into the congested sub-grid. Additionally,

the algorithm may have a pre-set maximum number of iterations.

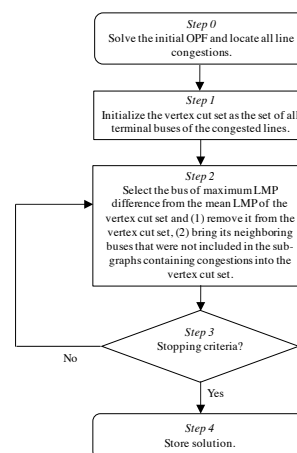


Fig. 4. Flow chart describing general algorithm structure of grid decomposition based on vertex cut set.

VII. SIMULATION RESULTS

To test the validity of the grid decomposition based on vertex cut set, a new topology control heuristic was proposed and tested on the IEEE 118-bus test systems. The so called Local Greedy Heuristic is the same as the Standard Greedy Heuristic in [12] except that the switchable lines are limited to the neighboring area of the congested line, i.e the congested sub-grid obtained by the grid decomposition algorithm shown in Fig. 4.

A. Test Network Overview

The heuristics were tested on the IEEE 118-bus test system. This test system represents a portion of the Midwestern US Power System as of December, 1962. The data of the test system employed is downloaded from the website of UW¹. The generator cost information used in the IEEE 118 network is extracted from [17]. The test system consists of 118 buses, 54 generators, and 194 lines, all of which are assumed to be connected in the initial topology.

B. Standard Greedy Heuristic Overview

The algorithm is specified in Fig. 5. The Standard Greedy Heuristic redoes the DCOPFs for each allowable line outage and then selects the single line outage that leads to maximum savings in each iteration. Please refer [12] for more details.

Clearly, optimality is not guaranteed, since the transmission topology is not co-optimized with power dispatch.

C. Simulation Results and Analysis

To generate the test scenarios for the heuristics, a fixed load is maintained and we perform a Monte Carlo simulation where the generator costs are randomly varied. The sample size used for the Monte Carlo simulation is 50. For the Local Greedy Heuristic based on vertex cut sets, the LMP range

¹<https://www2.ee.washington.edu/research/pstca/pf118/pg.tca118bus.htm>

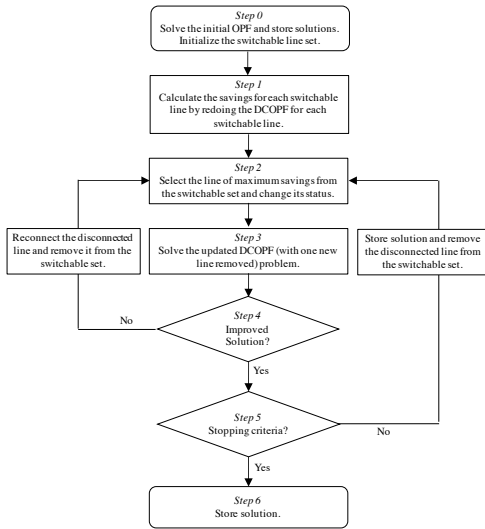


Fig. 5. Flow chart describing general algorithm structure of the Standard Greedy Heuristic.

threshold is set to be 10% of the maximum LMP difference across the grid.

The percent savings rather than the dollar value are the focus of the paper. Two benchmark cases are used to evaluate the savings of the heuristic: 1) the one with initial topology and line constraints whose DCOPF sets an upper bound of the total generation costs, and 2) the unconstrained case whose DCOPF provides a lower bound of the generation costs. The cost difference between the two benchmark cases is called the *maximum attainable savings (MAS)*.

The two heuristics are implemented in Matlab, using Matpower [18] as the DCOPF solver. The performance comparison between the heuristics is detailed in Table 2.

Heuristic	Saving/MAS	Lines removed	Mean effort
Local	0.653 ± 0.055	12.910 ± 4.312	0.216
Standard	0.672 ± 0.051	10.18 ± 3.98	1

Table 2: Performance comparison.

In the worst case, the two heuristics would be of the same level computational complexity if the grid decomposition algorithm brings all buses into the congested sub-grid. This is not the case for the 118-bus system we studied, and there are considerable savings in computation using our LMP-based decomposition. Though the computational costs of the two heuristics with and without the decomposition step differ by almost five times, their attainable savings are quite close, showing the promise in applying grid decomposition to topology control.

VIII. CONCLUSION

In spite of the seemingly promising simulation results of the heuristic approaches pursued by the research community and a certain level of proven predictability [19], [15], [20] of topology control, satisfactory grid-scale solutions are not guaranteed. While optimality for this NP-hard problem is not obtained, our approach based on a general spatial

grid decomposition nevertheless offers some hope that it is possible to design algorithms that isolate congestion effects to a relatively small part of the network. By only focusing attention to the neighboring area of the congestions, the search space of possible switching operations and the computational cost is significantly reduced, and the locally optimal switchings at each iteration are still retained to a large degree.

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