2020

Weighted Fourier analysis and dispersive equations

BOSTON UNIVERSITY

GRADUATE SCHOOL OF ARTS AND SCIENCES

Dissertation

WEIGHTED FOURIER ANALYSIS AND DISPERSIVE EQUATIONS

by

BRIAN JONGWON CHOI

B.S., Boston College, 2014

Submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

2020

Approved by

First Reader

Mark A. Kon, PhD
Professor of Mathematics and Statistics

Second Reader

C. Eugene Wayne
Professor of Mathematics and Statistics

Third Reader

Samuel Isaacson
Associate Professor of Mathematics and Statistics

Acknowledgments

I thank my colleagues and my PhD advisor, Mark. A special acknowledgent goes to my family. A non-significant portion of this thesis comes from many memorable interactions with you all. Enjoy.

WEIGHTED FOURIER ANALYSIS AND DISPERSIVE EQUATIONS

BRIAN JONGWON CHOI

Boston University, Graduate School of Arts and Sciences, 2020

Major Professor: Mark A. Kon, PhD
Professor of Mathematics and Statistics

ABSTRACT

The goal of this thesis is to apply the theory of multilinear weighted Fourier estimates to nonlinear dispersive equations in order to tackle problems in regularity, well-posedness, and pointwise convergence of solutions. Dispersion of waves is a ubiquitous physical phenomenon that arises, among others, from problems in shallow-water propagation, nonlinear optics, quantum mechanics, and plasma physics. A natural tool for understanding the related physics is to study waves/signals simultaneously from both physical and spectral perspectives. Specifically, we will treat nonlinearities as multilinear operator perturbations, which (by the method of spacetime Fourier transforms), exhibit smoothing properties in norms defined to reflect the dispersive natures of the solutions. Our model equation is the quantum Zakharov system, which can be viewed as a variation on the cubic nonlinear Schrödinger equation (NLS). We investigate the model in various contexts (adiabatic limits, nonlinear Schrödinger limits, semi-classical limits). We additionally study a variation of Carleson's Fourier convergence problem in the context of pointwise convergence of the full Schrödinger operator with non-zero potential.

Contents

Chapter 1

Introduction

In this thesis, we consider nonlinear dispersive phenomena in the context of nonlinear partial differential equations. Broadly speaking, we say that a wave propagation is dispersive if waves of different frequencies travel at non-uniform speeds. One of the earliest examples of a dispersive equation is the Korteweg-de Vries (KdV) equation, which models nonlinearly the propagation of shallow water waves. This was later overshadowed by fundamental interest in the Schrödinger equation as a model for quantum mechanics. Here we mainly focus on the latter equation and its more current variations, including the Zakharov system, which has applications in quantum mechanics of plasma physics.

From the perspective of quantum mechanics, it is fundamentally important to construct a theory that keeps the inner product of two quantum states invariant under time evolution, an intuition that axiomatizes the unitarity of the time-evolution operator; see (Von Neumann, 2018) for a mathematically rigorous discussion of connections in quantum mechanics. Since the infinitesimal generator of a unitary action is given by a unique (anti) self-adjoint operator, we consider

$$i\partial_t u = Lu + N(u), \tag{1.1}$$

where $u : \mathbb{R} \to D$ is a continuous Banach-valued function with L a self-adjoint operator and N a nonlinear term.[1] Before we address nonlinearities, it gives some insight to build heuristics for understanding dispersion in the context of abstract harmonic analysis; see the introduction of (Tao, 2006) for more of these. A huge class of dispersive models, including the Schrödinger equation, admits solutions whose L^2-norm is conserved in time, and hence we can consider $u(t) \in L^2(M)$ where M is some appropriate spatial domain (or even an abelian Lie group with Riemannian manifold structure if appropriate). In this setting, L admits a unique map called the dispersion relation, $h : \widehat{M} \to \mathbb{R}$, defined on the Pontryagin dual of M. Let $\{e_\xi\}_{\xi \in \widehat{M}}$ be an orthonormal basis of the Hilbert space $L^2(M)$ such that $Le_\xi = h(\xi)e_\xi$ where $\xi \in \widehat{M}$ is interpreted as a frequency. By separation of variables, if $u(x,t) = \phi(t)e_\xi(x)$ where $x \in M$ and $\phi(0) = 1$, then (1.1) with $N = 0$ yields $u(x,t) = e^{-ih(\xi)t}e_\xi$. As such, pure states keep their amplitudes invariant over time, which explains the L^2-conservation (via Plancherel's theorem) for an arbitrary initial L^2-function. However, the time evolution of different pure states corresponding to different frequencies depends on h. Since the gradient ∇h is interpreted as the group velocity of waves (as opposed to phase velocity), we say that (1.1) is dispersive if ∇h is not a constant field. In contrast for example, the standard transport equation is not dispersive.

We wish to obtain a unique solution to (1.1) given any initial data with sufficiently low regularity, which forms an important problem first studied in the context of NLS with power nonlinearities in $L^2(\mathbb{R}^d)$ by (Tsutsumi, 1987). Broadly speaking, there seem to be at least two powerful approaches to solving a nonlinear PDE: 1. fixed point arguments; 2. compactness arguments. The latter approach involves mollification of solutions, either in the equation itself or the initial data, to obtain a sequence of approximate solutions, which by a compactness argument (perhaps via Banach-Alaoglu or Arzelà-Ascoli arguments), a convergent subsequence is obtained.

[1] See the following section for notation.

Then another compactness argument (for instance by a variation of the Lions-Aubins lemma; see (Temam, 2001)) must be used to show that the limit function indeed forms the desired solution.

Here we take the approach of fixed point arguments. For these, to obtain uniform bounds of approximate solutions in some norm, we are almost forced to exploit conservation laws other than canonical L^2-conservation. Although the NLS equations with their nonlinearities and the models that we consider are Hamiltonian PDEs, it is necessary to assume high regularity on the initial data to ensure that the energy functionals are well-defined, which in turn, broadly speaking defeats the purposes in this thesis. However for earlier work in this direction that takes advantage of the energy conservation laws, see (Kato, 1987; Ginibre and Velo, 1979; Strauss, 1970).

This in turn illustrates a key difference between dispersive and dissipative equations, where the latter admit spectra (of linear operators) on the real line (non-positive spectra for the heat equation, for example) that possibly allow decay of Fourier components, and hence smoothing in some norm, as is evident for parabolic equations. On the other hand, the former (dispersive) case admits spectrum on the imaginary axis that allows for L^2-norm conservation.

The heuristics behind the fixed point arguments are to use the Euler forward method to iterate the linear evolution of solutions, while assuming nonlinear contributions as small. In the limiting case (of the iterative scheme), one wishes the desired solution to satisfy the Duhamel principle

$$u(t) = S(t)u_0 - i \int_0^t S(t-t')N(u(t'))dt', \tag{1.2}$$

where $S(t) = e^{-itL}$ is defined via functional calculus and $u_0 \in L^2(M)$. This motivates:

Definition 1.0.1. *Consider the initial value problem (IVP) given by (1.1) with* $u(0) =$

$u_0 \in X$ where X is a Banach space, and let $T > 0$. We say $u \in C([0,T], X)$ is a ***strong solution*** *on $[0,T]$ if u satisfies (1.2) for each $t \in [0,T]$. We say the IVP is* ***locally well-posed*** *in X if for any $u_0 \in X$, there exists $T > 0$, an open ball $B \subseteq X$ that contains u_0, and a subset $X_T \subseteq C([0,T], X)$ such that if $\tilde{u}_0 \in B$, then there is a unique strong solution (to the IVP with $u(0) = \tilde{u}_0$). Furthermore, it is required that this data-to-solution map is continuous from B to X_T with respect to the topologies of X and $C([0,T], X)$, respectively. If T only depends on $\|u_0\|_X$, then we say the IVP is (locally) well-posed in the* ***subcritical sense****; otherwise, the IVP is well-posed in the* ***critical sense****. Given a locally well-posed IVP, if $T > 0$ can be arbitrarily large, then the IVP is* ***globally well-posed****.*

One of the earliest works to study (1.1) using the Duhamel principle dates back to (Segal, 1963), which was an attempt to better understand the solutions of hyperbolic equations in relation to relativistic physics. The main difficulty in closing the fixed point argument from (1.2) is in constructing the solution space $\mathcal{S}$ where both u and the Duhamel term $\int_0^t S(t-t')N(u(t'))dt'$ are in $\mathcal{S}$. In fact, we do not expect this in general, and thus are forced to construct another space $\mathcal{N}$ such that

$$F \mapsto \int\limits_0^t S(t-t')F(t')dt' \tag{1.3}$$

defines a bounded operator from $\mathcal{N}$ to $\mathcal{S}$. Then to close the argument, it suffices to show, with certain hypotheses on N, that $\|N(u)\|_{\mathcal{N}} \leq C\|u\|_S^k$ for any $u \in \mathcal{S}$, for some $C, k > 0$. This step requires the most work, which then forms the main body of this thesis. We will assume that there exists a multilinear operator $N_j : \mathcal{S} \times \cdots \times \mathcal{S} \to \mathcal{N}$ defined on the j-fold product of $\mathcal{S}$ such that $N_j(u, \ldots, u) = N(u)$, and thus our task reduces to showing that N_j is bounded. This method has been used successfully by (Kenig et al., 1996b; Kenig et al., 1996a; Ginibre et al., 1997) in applications to various dispersive equations such as the KdV, NLS, and the Zakharov system. For a precise statement of the abstract Duhamel principle above, see (Tao, 2006, Proposition 1.38).

To further understand how to define $\mathcal{S}$ and $\mathcal{N}$, we discuss the role of the spacetime Fourier transform to investigate the solutions to linear dispersive equations and the intuition of Strichartz that led to the the so-called Strichartz estimates (Strichartz et al., 1977). Here we let $M = \mathbb{R}^d$ and given $u \in L^2(\mathbb{R}^d_x \times \mathbb{R}_t)$, we use the convention

$$\widehat{f}(\xi, \tau) = \frac{1}{(2\pi)^{d+1}} \int f(x,t) e^{-i(x\cdot\xi + t\tau)} dx dt, \tag{1.4}$$

where $\cdot$ denotes the dot product in Euclidean space. Taking the spacetime Fourier transform of (1.1) with $N = 0$, we obtain

$$(\tau + p(\xi))\widehat{u} = 0, \tag{1.5}$$

where p denotes the symbol of L (or the negative of the dispersion relation h). We note that $\widehat{u}$ is supported in the variety $\{\tau + p(\xi) = 0\}$, or colloquially we say that linear dispersive solutions live in the dispersion relation. Since we assume ∇h is not a constant field, we expect the submanifold $S \coloneqq \{\tau + p(\xi) = 0\}$ to be non-flat. One can adopt the perspective of Fourier restriction theory and ask, given $u \in L^p(\mathbb{R}^d_x \times \mathbb{R}_t)$, $p \in [1, \infty]$, and $d\sigma$, the surface measure on S, for which $q \in [1, \infty]$ do we have $\widehat{u}|_S \in L^q(S, d\sigma)$? For $L = -\Delta$ and recalling that $\|u\|_{L^q_t L^r_x} \coloneqq \|\|u\|_{L^r_x}\|_{L^q_t}$, this question led to

Theorem 1.0.1 ((Strichartz et al., 1977)). *Given $q, r \in [2, \infty]$, $(q, r, d) \neq (2, \infty, 2)$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, we have*

$$\|e^{it\Delta} u_0\|_{L^q_t L^r_x} \leq C(d, q, r)\|u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.6}$$

Some of the earliest works to apply this tool were (Kato, 1987), which used it to establish well-posedness of NLS in $H^1(\mathbb{R}^d)$ with power-type nonlinearities, and (Yajima, 1987) to study the full Schrödinger equation with time-dependent potential; the end-point case of the theorem above was established in (Keel and Tao, 1998). The estimate (1.6) is useful in obtaining local well-posedness since the linear evolution is bounded above by a conserved quantity in some spacetime-averaged sense. The extent to which the estimate (1.6) can be extended to other domains, however, is critically dependent on the compactness of the spatial domain, and it is insightful to consider (1.6) as a consequence of the well-known TT^* duality argument and the dispersive inequality

$$\|e^{it\Delta}u_0\|_{L^{p'}(\mathbb{R}^d)} \leq Ct^{-d(\frac{1}{p}-\frac{1}{2})}\|u_0\|_{L^p(\mathbb{R}^d)}, \tag{1.7}$$

for $p \in [1,2]$ where $p' = \frac{p}{p-1}$ denotes the Hölder conjugate. To be consistent with the Hölder's inequality and L^2-conservation, the analogue of estimate (1.7) for compact domains is necessarily false for $p \in [1,2)$. Physically this reflects the intuition that energy cannot escape to infinity in a compact domain. Waves of different frequencies interfere with each other and it is crucial to understand the mechanism that yields destructive interference (that possibly yields smoothing) as opposed to constructive interference, which can cause a blow-up of solutions. Here we consider problems on $\mathbb{R}$ and $\mathbb{T}$, and it will be evident that we cannot use a unified tool to settle well-posedness problems for all domains.

It is the intuition of (Bourgain, 1993a) that taking the weighted averages in the spacetime Fourier space, as opposed to the physical space, does yield smoothing, an insightful perspective that we adopt in this thesis; moreover, a variation of this idea is found in (Beals, 1983; Klainerman and Machedon, 1993) in the context of the semilinear wave equations. Recalling that the (linear) solutions to linear dispersive equations are supported in the variety of the dispersion relation of L, we expect the (nonlinear) solutions to (1.1) to be supported in a small neighborhood of the variety at least for a short time. To establish well-posedness in $H^1(\mathbb{R}^d)$, for instance, we are led to consider the weight $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$, which is familiar from the classical Sobolev space theory, and the dispersive weight $\langle \tau + p(\xi)\rangle$, and consider $u \in C([0,T], H^1(\mathbb{R}^d))$ such that $\langle \xi\rangle\langle \tau+p(\xi)\rangle\widehat{f} \in L^2(\mathbb{R}^d\times\mathbb{R})$, which sets the foundation of how the solution space $\mathcal{S}$, the Bourgain space, should be constructed to successfully apply the abstract Duhamel principle. Hence we have the following algorithm, which we call the Bourgain norm method as is commonly called in the literature.

1. Construct $\mathcal{S}$ based on the dispersion relation determined by the self-adjoint operator.

2. Show that the linear time-evolution operator is bounded in $\mathcal{S}$.

3. Whenever the equation is subcritical with respect to the scale-invariance (to be explained further in subsequent chapters), use that subcriticality slack (also to be explained later) to extract a positive time during which our candidate solution persists.

4. Identify $\mathcal{N}$ on which the Duhamel operator in (1.2) is defined.

5. Show that the multilinear operator N_j is bounded via various tricks from calculus.

For more applications of this technique, see (Klainerman, 1997; Kenig et al., 1996b) applied to the Yang-Mills equation and the KdV, respectively. This is an instance where the physics of dispersion dictates how to construct a solution by penalizing spacetime Fourier components that deviate from the variety of the dispersion relation.

This dissertation is a concatenation of our work and as such, each chapter is self-contained. Nevertheless we give short introductions to each chapter to emphasize an underlying theme that weighted Fourier analysis applied to dispersive phenomena yields smoothing. In Chapter 2, we apply the Bourgain norm method to understand a variation of Carleson's Fourier convergence problem for the linear Schrödinger operator: for $H = -\Delta + V$, what is the lowest Sobolev exponent $s \geq 0$ such that $e^{-itH}u_0 \xrightarrow[t\to 0]{} u_0$ almost everywhere for any $u_0 \in H^s(\mathbb{R}^d)$? When $V = 0$, $d = 1$, (Carleson, 1980) settled the sufficiency ($s \geq \frac{1}{4}$) whereas (Dahlberg and Kenig, 1982), the necessity ($s < \frac{1}{4}$). Since then many relevant publications followed, with (Du and Zhang, 2018; Du et al., 2017) settling the sufficiency in higher dimensions ($s > \frac{d}{2(d+1)}$) and (Bourgain, 2016) providing an example that illustrates the condition $s \geq \frac{d}{2(d+1)}$ for necessity of almost-everywhere pointwise convergence. We consider the Hamiltonian operator H with a non-zero time-independent potential V. By the Bourgain norm method, we conclude that the Duhamel term $V \mapsto \int_0^t e^{i(t-t')\Delta}Vu(t')dt'$ is continuous for small time. Alternatively, the Trotter-Kato product formula provides a natural way to understand the exponential of a sum of two self-adjoint operators. We closely follow the intuition that for t sufficiently small, e^{-itH} splits into $e^{it\Delta}e^{-itV}$ modulo some error. On one hand, the previous intuition suggests the invariance of almost everywhere convergence under a huge class of potentials. On the other hand, we are able to derive the desired conclusion for potentials that are more restrictive than we originally thought. For now, we leave this apparent gap between intuition and proof unanswered. However, we remark that the theory of Schrödinger semi-groups is well-studied; see (Simon, 1982). For elliptic regularity of Schrödinger semi-group in L^p-based Sobolev spaces, see (Gulisashvili and Kon, 1996). In case of zero potential, a partial heuristic as to why the unitary group $e^{it\Delta}$ is less well-behaved than the corresponding semi-group is that $e^{t\Delta}$ defines an analytic semi-group in some sector symmetric around the positive axis that does not include the imaginary axis, thus forming a cone-shaped sector. Though strong continuity (of the semi-group) is preserved under the tangential limit $t \to 0$ from the imaginary axis, uniformity is not. Under analytic continuation of time, the heat equation becomes the Schrödinger equation; however the convergence of $e^{t\Delta}$ to the identity no longer holds in the uniform operator topology, since the region of analytic-continuation does not take place within some fixed obtuse cone-shaped sector whose closure contains the imaginary axis.

In subsequent chapters, we apply the so-called Bourgain norm method to study various dispersive models closely related with the NLS and the Zakharov system. In particular, we study a singularly perturbed cubic NLS on $\mathbb{R}$ and $\mathbb{T}$

$$i\partial u + \Delta u - \epsilon^2\Delta^2 u = -(I - \epsilon^2\Delta)^{-1}|u|^2 \cdot u, \tag{1.8}$$

which can also be realized as the (formal) adiabatic limit of the Quantum Zakharov system (QZS). Other than well-posedness, the purpose of this investigation is two-fold:

1. Although it is well-known that the (focusing) cubic NLS is ill-posed in negative Sobolev spaces as in (Kenig et al., 2001), disprove that the additional biharmonic operator lowers the threshold of Sobolev regularity exponent (below the L^2-regularity) in the context of local well-posedness.

2. Verify the informal principle that the qualitative behavior of solutions is governed by the highest derivative. As $\epsilon \to 0$, study the continuity of $\epsilon \mapsto u(\epsilon)$ at the origin in some norm.

Step 5 in the Bourgain norm method typically reduces to estimating a complicated integral or series. Our analysis of (1.8) on $\mathbb{R}$ is a straightforward estimation of a complicated integral via several calculus tricks, whereas we approach the analysis on $\mathbb{T}$ differently, since it is technically more challenging in this case to estimate a sum than an integral. We use modified Strichartz estimates to establish an embedding of Bourgain space into the (spacetime) mixed Lebesgue space.

Motivated by the existence of solitons of the form $sech(x)$ for the focusing cubic NLS on $\mathbb{R}$, we establish, by considering solutions of the form $e^{i\tau t}Q_\epsilon(x)$, that (1.8) also admits solitons in $\mathbb{R}^d$ for $1 \leq d \leq 9$ by applying the well-known mountain-pass lemma to a resulting nonlinear elliptic PDE; see (Tao, 2006, appendix B) for a construction of soliton solutions of NLS with $N(u) = -|u|^{p-1}u$ where the nonlinearity is $\dot{H}^1$-subcritical, i.e., $\frac{d}{2} - \frac{2}{p-1} < 1$. For the unperturbed cubic case, we have $p = 3$ and hence $d \leq 3$. When $\epsilon > 0$, we obtain solutions in dimensions higher than 3, which is consistent with the intuition in (Simpson et al., 2009) that the biharmonic operator, with the proper sign, adds an extra degree of (fourth-order) dispersion and thus smoothing. We leave some interesting questions open, however. We do not answer whether Q_ϵ is radially symmetric. For $\epsilon = 0$, (Gidas et al., 1979) applied the method of moving planes to show radial symmetry of positive solutions to a second-order nonlinear elliptic PDE. It might be of interest to extend their result to higher-order differential operators.

We study the non-adiabatic Quantum Zakharov system in chapter 5:

$$\begin{cases} (i\partial_t + \alpha\Delta - \epsilon^2\Delta^2)u = un, \ (x,t) \in \mathbb{T} \times [0,T] \\ (\beta^{-2}\partial_{tt} - \Delta + \epsilon^2\Delta^2)n = \Delta(|u|^2). \end{cases} \tag{1.9}$$

Recently the QZS has garnered much attention from the physics community (Garcia et al., 2005) with interest in applying the Zakharov system (Zakharov et al., 1972), originally developed to model the nonlinear interaction between Langmuir turbulence wave and ion-acoustic wave, to plasma physics. The application is of particular interest when either the ion-plasma frequency is high or the electron temperature is low. This full system of equations is far from being scale-invariant since there are two-wave phenomena that behave very differently with respect to scaling. The Schrödinger wave u is governed by the first-order time derivative whereas n is dominated by the second-order time derivative. This poses an extra layer of mathematical difficulty compared to (1.8). Our goal in studying QZS on $\mathbb{T}$ is three-fold:

1. Considering the significance of resonance ($\frac{\beta}{\alpha} \in \mathbb{Z}$) and non-resonance ($\frac{\beta}{\alpha} \notin \mathbb{Z}$) in (Takaoka et al., 1999) regarding low-regularity well-posedness, prove that the additional smoothing provided by $\epsilon^2\Delta$ nullifies the distinction between the two.

2. Establish the sharp region (in $\mathbb{R}^2$) comprising pairs of Sobolev regularity exponents (s,l) such that any given initial data $(u_0,n_0) \in H^s(\mathbb{T}) \times H^l(\mathbb{T})$ yields a unique well-posed solution.

3. Derive the semi-classical limit as $\epsilon \to 0$ on $[0,T]$ for any $T \in (0,\infty)$. Show this fails for $T=\infty$.

1.1 Notation.

As is conventional, we first define Fourier transform of $f \in L^2(\mathbb{R}^d)$ and the inverse transform of $F \in L^2(\mathbb{R}^d)$:

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^d}\int f(x)e^{-i\xi\cdot x}dx;\ \check{F}(x) = \int F(\xi)e^{i\xi\cdot x}d\xi. \tag{1.10}$$

Fourier transform (and its inverse) is defined similarly for functions defined on $\mathbb{T}$. We use $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ and define a family of Sobolev spaces $W^{s,p}, \dot{W}^{s,p}$ (inhomogeneous and homogeneous, respectively) with $s \in \mathbb{R}$, $p \in (1,\infty)$ as[2]

$$\|f\|_{W^{s,p}} = \|\langle\nabla\rangle^s f\|_{L^p};\ \|f\|_{\dot{W}^{s,p}} = \||\nabla|^s f\|_{L^p}. \tag{1.11}$$

For $p=\infty$, we only use the integer-order Sobolev space. Of particular importance is when $p=2$ for which we write $H^s, \dot{H}^s$ as is usual, and the norms are defined via

[2] As is common, $p' = \frac{p}{p-1}$ is the Hölder conjugate.

Fourier multipliers:

$$\|f\|_{H^s} = \|\langle k\rangle^s \hat{f}(k)\|_{L^2};\ \|f\|_{\dot{H}^s} = \||k|^s \hat{f}(k)\|_{L^2}. \tag{1.12}$$

The spaces $\mathscr{S}(\mathbb{R}^d)$ and $C_c^\infty(\mathbb{R}^d)$ denote the Schwartz class of rapidly decaying smooth functions and the set of smooth functions with compact support, respectively. We generally let $\eta \in C_c^\infty(\mathbb{R})$[3] denote a smooth cutoff function that is identically one on $[-1,1]$ with a compact support in $[-2,2]$. For D a Banach space and for $T \in (0,\infty)$, we let $C([0,T],D)$ denote the Banach space of D-valued continuous (in time) functions with $\|u\|_{C_T D} := \sup\limits_{t\in[0,T]} \|u(t)\|_D < \infty$. When $T = \infty$, we consider $C_{loc}([0,\infty), D)$ where we only require continuity in t.

Let $\mathcal{H} = p.v.(\frac{1}{x})$ be the Hilbert transform on $\mathbb{R}$, or more precisely

$$\mathcal{H}f(x) = \lim_{\epsilon\to 0} \int\limits_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{f(x-y)}{y} dy,\ \forall f \in C_c^\infty(\mathbb{R}).$$

We use two facts regarding this:

1. $\mathcal{H}$ defines a unitary operator on $L^2(\mathbb{R})$.

2. $|\partial_x|$ has the polar decomposition $|\partial_x| = \partial_x \mathcal{H}$.

Given $f \in C_c^\infty(\mathbb{R}^{d+1})$, $s, b \in \mathbb{R}$, and $p(\xi)$, the symbol of L from (1.1), we define the space $X^{s,b}$ as the closure of $C_c^\infty(\mathbb{R}^{d+1})$ with respect to the norm

$$\|f\|_{X^{s,b}} = \|\langle \xi\rangle^s \langle \tau + p(\xi)\rangle^b \widehat{f}(\xi,\tau)\|_{L^2(\mathbb{R}^{d+1})}, \tag{1.13}$$

and for $T \in (0,\infty)$, define the quotient space $X_T^{s,b} = X^{s,b}/\sim$ where $f \sim g$ if $f - g$ vanishes for all $t \in [0,T]$, and accordingly $\|f\|_{X_T^{s,b}} = \inf\limits_{\tilde{f}=f,\, t\in[0,T]} \|\tilde{f}\|_{X^{s,b}}$. The construction

[3] Whenever different notations are used for a cut-off function, we will specify.

is similar for $f \in C_c^\infty(\mathbb{T}\times\mathbb{R})$. For $b=\frac{1}{2}$ and $\mathbb{T}$, we use the augmented space

$$\|f\|_{Y^s} = \|f\|_{X^{s,\frac{1}{2}}} + \|\langle k\rangle^s \widehat{f}\|_{l_k^2 L_\tau^1}, \tag{1.14}$$

and its companion space

$$\|f\|_{Z^s} = \|f\|_{X^{s,-\frac{1}{2}}} + \|\langle k\rangle^s \langle \tau + p(k)\rangle^{-1} \widehat{f}\|_{l_k^2 L_\tau^1}, \tag{1.15}$$

where the quotient space Y_T^s, Z_T^s are defined similarly.

We say $A \lesssim B$ or $A \gtrsim B$ if there exists some $C>0$ such that $A \leq CB$ or $A \geq CB$, and $A \simeq B$ if $A \lesssim B$ and $A \gtrsim B$. Given $A_\pm$, we denote $\sum_\pm A_\pm \coloneqq A_+ + A_-$. For $b \in \mathbb{R}$, we write $b\pm$ to denote $b \pm \epsilon'$ for some universal $\epsilon' \ll 1$.

1.2 Basic Estimates.

Here we state estimates that will be used repeatedly in subsequent chapters.

Lemma 1.2.1. *Let $T \in (0,1]$, $s, b \in \mathbb{R}$. Then*

1. *(Tao, 2006, Lemma 2.8): $\|\eta(t) e^{-itL} f\|_{X^{s,b}} \lesssim_{b,\eta} \|f\|_{H^s}$.*
2. *(Tao, 2006, Corollary 2.10): For every $b > \frac{1}{2}$, we have*

$$X_T^{s,b} \hookrightarrow C_t^0 H_x^s([-T,T]\times\mathbb{R}^d).$$

3. *(Tao, 2006, Lemma 2.11): Let $-\frac{1}{2} < b' \leq b < \frac{1}{2}$. Then,*

$$\|\eta(\frac{t}{T}) u\|_{X^{s,b'}} \lesssim_{b',b,\eta} T^{b-b'} \|u\|_{X^{s,b}}.$$

4. *(Tao, 2006, Proposition 2.12): Let $b > \frac{1}{2}$. Then,*

$$\left\| \eta(t) \int_0^t e^{-i(t-\tau)L} F(\tau) d\tau \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}.$$

5. *(Erdoğan and Tzirakis, 2013, Lemma 3.3) If* $\beta \geq \gamma \geq 0$ *and* $\beta + \gamma > 1$, *then*

$$\int \frac{dx}{\langle x - a_1 \rangle^{\beta} \langle x - a_2 \rangle^{\gamma}} \lesssim \langle a_1 - a_2 \rangle^{-\gamma} \phi_{\beta}(a_1 - a_2), \text{ where}$$

$$\phi_{\beta}(a) \sim \begin{cases} 1, & \beta > 1 \\ \log(1 + \langle a \rangle), & \beta = 1 \\ \langle a \rangle^{1-\beta}, & \beta < 1. \end{cases}$$

Remark 1.2.1. *Lemma 1.2.1 holds when the domain is replaced from* $\mathbb{R}^d$ *to* $\mathbb{T}$.

The following fundamental estimate, whose proof is by the triangle inequality, will be useful.

Lemma 1.2.2 (Peetre's Inequality)**.** *For all* $a, b \in \mathbb{R}$ *and* $t \in \mathbb{R}$, *we have*

$$\left(\frac{\langle a \rangle}{\langle b \rangle} \right)^{t} \leq 2^{\frac{|t|}{2}} \langle a - b \rangle^{|t|}.$$

Lemma 1.2.3 ((Taylor, 2007))**.** *For* $s \geq 0$ *and* $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$ *where* $p_1, q_2 \in (2, \infty]$, *there exists* $C = C(s, p_1, p_2, q_1, q_2, d) > 0$ *such that for all* $f, g \in C_C^{\infty}(M)$ *for* $M = \mathbb{R}^d, \mathbb{T}^d$, *we have*

$$\|fg\|_{H^s} \leq C(\|f\|_{L^{p_1}} \|\langle \nabla \rangle^s g\|_{L^{q_1}} + \|\langle \nabla \rangle^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}).$$

Chapter 2

Pointwise Convergence of the Full Schrödinger Operator

2.1 Introduction.

Consider the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = 0, (x,t) \in \mathbb{R}^n \times \mathbb{R} \\ u(0) = u_0 \in L^2(\mathbb{R}^n) \end{cases}$$

noting that $-\Delta$ is a non-negative operator. A straightforward computation with the Fourier transform yields

$$u(x,t) = e^{it\Delta}u_0(x) = \int \hat{u}_0(\xi)e^{-it|\xi|^2+i\xi\cdot x}d\xi.$$

In this paper we continue to build upon a question initially posed by (Carleson, 1966): what is the minimal Sobolev regularity s_* for which $e^{it\Delta}f \underset{t\to 0}{\longrightarrow} f$ almost everywhere (a.e.) with respect to Legesgue measure, for all $f \in H^{s_*}(\mathbb{R})$? Carleson originally proved a positive result, that any $f \in H^s(\mathbb{R})$ for $s \geq \frac{1}{4}$ exhibits almost everywhere (a.e.) convergence. Soon (Dahlberg and Kenig, 1982) showed that Carleson's result is sharp. In higher dimensions, this problem is closed except at the endpoint $s = \frac{n}{2(n+1)}$. In $n = 2$, (Du et al., 2017) showed sufficiency for $s > \frac{1}{3}$ while (Sjölin et al., 1987) and (Vega, 1988) independently showed sufficiency in $n \geq 3$ for

$s > \frac{1}{2}$. (Bourgain, 2013) showed sufficiency for $s > \frac{2n-1}{4n}$ for $n \geq 2$, and though it had long been believed that $s > \frac{1}{4}$ is the sharp sufficient condition in higher dimensions, (Bourgain, 2016) showed necessity for $s \geq \frac{n}{2(n+1)}$ in $n \geq 2$. Recently (Du et al., 2018) showed sufficiency for $s > \frac{n+1}{2(n+2)}$ for $n \geq 3$, which was subsequently improved to the sharp condition $s > \frac{n}{2(n+1)}$ by (Du and Zhang, 2018). Many of these results generalise nicely to $i\partial_t u + \Phi(D)u = 0$ where Φ is a Fourier multiplier satisfying $|D^\gamma \Phi(\xi)| \lesssim |\xi|^{\alpha-|\gamma|}$ and $|\nabla\Phi(\xi)| \gtrsim |\xi|^{\alpha-1}$ where $\alpha \geq 1$ and γ is a multi-index, which in particular involves the fractional Schrödinger operator $e^{-it(-\Delta)^{\frac{\alpha}{2}}}$; see (Lee, 2006) and (Cho and Ko, 2018).

Meanwhile further generalisations were established using geometric measure theory. Though Carleson's problem has an affirmative answer for a.e. convergence when $s \in [\frac{1}{4}, \frac{1}{2}]$ for $n = 1$, the divergence set (points $x \in \mathbb{R}$ where divergence occurs), which is of Lebesgue measure zero for such s, can still be big. (Barceló et al., 2011) show that the divergence set is of Hausdorff dimension at most $1 - 2s$ for $s \in [\frac{1}{4}, \frac{1}{2}]$. On the other hand, (Lucà and Rogers, 2017) generalises the necessity result of (Bourgain, 2016) from Lebesgue measure to the set of α-dimensional non-negative measures μ on $\mathbb{R}^n$ for $n \geq 2$; here a non-negative Borel measure μ is α-dimensional if $c_\alpha(\mu) = \sup_{x\in\mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^\alpha} < \infty$. It is shown that if $\alpha \in [\frac{3n+1}{4}, n]$ and $\mu\left\{x \in \mathbb{R}^n : e^{it\Delta}u_0 \xrightarrow[t\to 0]{} u_0 \text{ fails.}\right\} = 0$ for all $u_0 \in H^s(\mathbb{R})$, then $s \geq \frac{(n-1)(n-\alpha)}{2(n+1)} + \frac{n}{2(n+1)}$; since Lebesgue measure on $\mathbb{R}^n$ is n-dimensional, the result of (Bourgain, 2016) is recovered by letting $\alpha = n$. For recent results regarding the size of divergence set in higher dimensions, see (Du and Zhang, 2018).

It offers some insight to view this convergence problem in the context of summation methods. These originated in the study of alternative ways of summing Fourier series such as Abel or Riesz summability. Summation methods for Fourier series or transforms, in modern terms, involve a family of operators $\phi(-t\Delta)$ (with ϕ a Borel-measurable function satisfying $\phi(0) = 1$) forming an approximate identity as $t \to 0$. Questions of convergence in this context translate into strong convergence (as $t \to 0$) of such operator families. Abel summability corresponds to $\phi(x) = e^{-x}$, while other methods correspond to different choices of ϕ with $\phi(0) = 1$. Our current (Schrödinger) problem chooses $\phi(x) = e^{ix}$, while the original result of Carleson for a.e. convergence of Fourier series (Carleson, 1966) made the analogous statement for $\phi(x) = \frac{\sin(x)}{x}$.

The main purpose of this paper is to answer a variant of Carleson's problem, not for the free Schrödinger equation, but for the Schrödinger equation with a nonzero potential or nonlinearity. One motivation of this note comes from (Cowling, 1983) that whenever $|H|^\alpha u_0 \in L^2(X)$ where X is a measure space and H is some self-adjoint operator on $L^2(X)$ with $|H|$ given by the polar decomposition, we obtain $e^{-itH}u_0 \underset{t\to 0}{\longrightarrow} u_0$ a.e. if $\alpha \in (\frac{1}{2}, \infty)$. Another motivation comes from (Sjögren and Torrea, 2009), where given the following Cauchy problem (the quantum harmonic oscillator),

$$\begin{cases} i\partial_t u = -\partial_{xx} u + x^2 u, (x,t) \in \mathbb{R} \times \mathbb{R} \\ u(0) = u_0 \in H^s(\mathbb{R}), \end{cases}$$

pointwise convergence to initial data holds for every $s \geq \frac{1}{4}$ and fails for $s < \frac{1}{4}$. Typically, a standard strategy in proving such positive result is to show that the Schrödinger maximal operator satisfies either a strong-type or weak-type estimate, from which pointwise convergence follows by a now-standard approximation argument. For the quantum harmonic oscillator, (Sjögren and Torrea, 2009) takes advantage of the closed, analytic expression for the fundamental solution associated with the quadratic Schrödinger propagator, also known as the Mehler kernel:

$$K_{it}(x,y) = (2\pi i \sin 2t)^{-n/2} e^{\frac{i}{2}(\cot 2t \cdot |y - \frac{x}{\cos 2t}|^2 - \tan 2t \cdot |x|^2)}, \forall x, y \in \mathbb{R}^n, t \in \mathbb{R} \setminus \frac{\pi}{2}\mathbb{Z}.$$

For a general potential, we have to work with analytic properties of the unitary group generated by the Hamiltonian $-\Delta + V$; note that the semigroup generated by this operator has been studied extensively, for example, by (Simon, 1982). In fact, an orbit of a square-integrable function generated by $e^{t\Delta}$, viewed as a spacetime function, solves the heat equation, and by exploiting the exponential decay of the corresponding Green's function, one can easily show pointwise convergence to initial data (the Green's function corresponding to $e^{it\Delta}$ has no such spatial decay). More generally, $\left\{e^{t\Delta}\right\}_{t>0}$ defines a holomorphic C_0-semigroup, and the strong convergence $e^{t\Delta} \underset{t\to 0}{\longrightarrow} \mathbb{I}$ is an example of standard Abel summability traditionally studied for Fourier series on an interval. For complex $t \to 0$ such convergence occurs in a sector symmetric about the positive t axis. However under the Wick rotation $t \mapsto it$, our sector of convergence is now symmetric about the imaginary t axis, and our case of real $t \to 0$ constitutes a boundary case of the known region of Abel summability. Therefore Abel summation is an insufficient tool to answer our problem. To this end, we summarise the main results of this paper:

Theorem 2.1.1. *Suppose $V \in L^2(\mathbb{R})$. Then the solutions to the linear Schrödinger equation converge a.e. to initial data in $H^s(\mathbb{R})$ if and only if $s \geq \frac{1}{4}$.*

The main theorem of (Carleson, 1980) is contained in the previous statement by taking $V = 0$. Moreover this class of potentials V contains some well-studied examples in physics such as the finite square well.

The above results related to the linear Schrödinger equation are naturally related to corresponding non-linearizations ((Kenig et al., 1996a),(Bejenaru and Tao, 2006)), for which, perhaps as expected, the corresponding results hold.

Theorem 2.1.2. *The solutions to quadratic nonlinear Schrödinger equation (qNLS) with nonlinearities*

$$N_1(u,\overline{u}) = u^2;\ N_2(u,\overline{u}) = u\overline{u};\ N_3(u,\overline{u}) = \overline{u}^2$$

converge a.e. to initial data for $s \geq \frac{1}{4}$, $s > \frac{1}{4}$ *and* $s \geq \frac{1}{4}$, *respectively. On the other hand, the convergence fails for qNLS with nonlinearities* N_1 *and* N_3 *in* $H^s(\mathbb{R})$ *with* $s \in [0, \frac{1}{4})$.

At the cost of adding technical conditions on the potential, theorem 2.1.1 can be made more precise by treating if and only if statements separately.

Theorem 2.1.3. *Suppose* $s \geq \frac{1}{4}$ *and* $V \in L^2(\mathbb{R}) \cup \Big(W^{1,\infty}(\mathbb{R}) \cap \bigcup_{\rho \in [1,\infty)} L^\rho(\mathbb{R})\Big)$. *Then the solutions to the linear Schrödinger equation converge a.e. to initial data in* $H^s(\mathbb{R})$. *On the other hand, if* $s < \frac{1}{4}$ *and* $V \in L^2(\mathbb{R})$, *then there exists a compactly supported initial data* $f \in H^s(\mathbb{R})$ *and a measurable set (of positive measure)* $E_f \subseteq \mathbb{R} \setminus supp(f)$ *such that* $\overline{\lim_{t\to 0}} |e^{it\partial_{xx}} f| \geq c > 0$ *on* E_f.

We outline the organization of this chapter. In sections 2 and 3, we prove a positive pointwise convergence result for the linear Schrödinger equation with potential using restricted Fourier space methods and the Trotter-Kato product formula. In fact the class of potentials investigated does not include the quadratic case $V(x) = x^2$; our choice of potentials should be thought of as small perturbations to the free case $V = 0$. In section 4, we prove a similar result for the Schrödinger equation with a quadratic nonlinearity. In sections 5 and 6, we switch gears to prove the negative result that for $i\partial_t u = -\partial_{xx} u + Vu$, with an appropriate potential function, to exhibit pointwise convergence to initial data, it is necessary that $u_0 \in H^s(\mathbb{R})$ where $s \geq \frac{1}{4}$.

2.2 Positive Results.

Let $H = -\partial_{xx} + V$ denote the Hamiltonian operator on $\mathbb{R}$, where $V = V(x)$ is a real-valued time-independent multiplication operator. Note that H is self-adjoint on $D(H) = D(-\partial_{xx}) \cap D(V)$, if $V \in L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$, where $D(-\partial_{xx}) = H^2(\mathbb{R})$ and $D(V) = \{f \in L^2(\mathbb{R}) : Vf \in L^2(\mathbb{R})\}$; see (Hall, 2013, Theorem 9.38). Therefore, e^{-itH} gives a family of unitary actions on $L^2(\mathbb{R})$. It is of interest to ask whether known positive results for pointwise convergence of the free Schrödinger equation can be recovered with an addition of a potential.

Theorem 2.2.1. *Let $s \geq \frac{1}{4}$ and $1 \leq \rho < \infty$, and suppose a time-independent potential V satisfies the following hypothesis:*

$$V \in L^2(\mathbb{R}) \cup \Big(W^{1,\infty}(\mathbb{R}) \cap L^\rho(\mathbb{R})\Big).$$

Then for all $u_0 \in H^s(\mathbb{R})$, $e^{-itH}u_0 \to u_0$ as $t \to 0$ almost everywhere with respect to Lebesgue measure. More precisely,

$$\Big|\Big\{x \in \mathbb{R} : \overline{\lim_{t\to 0}}\, |e^{-itH}u_0 - u_0| > 0\Big\}\Big| = 0.$$

Remark 2.2.1. *By virtue of V being time-independent, the conclusion holds in the limit when $t \to t_0$ for any $t_0 \in \mathbb{R}$. This is a simple consequence of the time-translation symmetry in the equation, and this remark also holds for the nonlinear equation discussed in section 4.*

Remark 2.2.2. *Let $1 \leq \rho_1 < \rho_2 < \infty$. By the following interpolation on L^p spaces,*

$$\|V\|_{L^{\rho_2}} \leq \|V\|_{L^{\rho_1}}^{\frac{\rho_1}{\rho_2}} \|V\|_{L^\infty}^{1-\frac{\rho_1}{\rho_2}},$$

a low integrability of V is automatically upgraded to a high integrability, if V is bounded. Moreover, note that $V \in W^{1,\infty}(\mathbb{R}) \cap L^\rho(\mathbb{R})$ if and only if V is Lipschitz continuous with $|V(x)| \lesssim |x|^{-\epsilon_0}$ for all $x \in \mathbb{R}$ for some $\epsilon_0 > 0$.

By Stone's theorem on a Hilbert space, a time-evolution operator for non-relativistic quantum mechanics is in one-to-one correspondence with a self-adjoint operator. However, we remark that self-adjointness of H generally fails on $H^s(\mathbb{R})$ for $s > 0$, and therefore, e^{-itH} defines a family of unitary operators on $H^s(\mathbb{R})$ only if $s = 0$. In fact, it is not clear whether we have persistence of regularity for e^{-itH} on $H^s(\mathbb{R})$ for $s > 0$, and so this shall be proved. Some of these results are likely to be known; however the lemmas below contain some estimates that will be of use later. Treating the potential term as the nonlinearity, we recall

Definition 2.2.1. *For $\delta > 0$, $u \in C_t^0 H_x^s([-\delta,\delta],\mathbb{R})$ is a strong solution of*

$$\begin{cases} i\partial_t u & = -\partial_{xx} u + Vu, (x,t) \in \mathbb{R} \times [-\delta,\delta] \\ u(0) & = u_0 \in H^s(\mathbb{R}), \end{cases} \tag{2.1}$$

if u satisfies the following Duhamel integral formula for all $t \in [-\delta,\delta]$:

$$u(t) = e^{it\partial_{xx}} u_0 - i \int_0^t e^{i(t-t')\partial_{xx}} (Vu)(t') dt'.$$

Remark 2.2.3. *For $u_0 \in L^2(\mathbb{R})$, we claim that the notion of strong solution as in above, where we treat the potential term as a nonlinear perturbation, coincides with that of an orbit generated by the unitary group. Though this seems intuitive, some care is needed if V is not sufficiently regular. At least when $u_0 \in D(H)$, $u(t) = e^{-itH}u_0$ satisfies the Duhamel integral formula for each t, which is an easy consequence of the following product rule:*

$$\partial_t \Big(e^{-it\partial_{xx}} e^{-itH} u_0 \Big) = -ie^{-it\partial_{xx}} \Big(Ve^{-itH} u_0 \Big).$$

For $u_0 \in L^2(\mathbb{R}) \setminus D(H)$, let $u_0^{(n)} \to u_0$ as $n \to \infty$ where $u_0^{(n)} \in D(H)$. Then we have,

$$e^{-itH} u_0^{(n)} = e^{it\partial_{xx}} u_0^{(n)} - i \int_0^t e^{i(t-t')\partial_{xx}} (Ve^{-it'H} u_0^{(n)}) dt'.$$

As $n \to \infty$, we have $e^{-itH}u_0^{(n)} \to e^{-itH}u_0$ and $e^{it\partial_{xx}}u_0^{(n)} \to e^{it\partial_{xx}}u_0$ by unitarity. We claim

$$\int_0^t e^{i(t-t')\partial_{xx}} (Ve^{-it'H} u_0^{(n)}) dt' \to \int_0^t e^{i(t-t')\partial_{xx}} (Ve^{-it'H} u_0) dt',$$

in $L^2(\mathbb{R})$ as $n \to \infty$. Firstly for $V \in L^\infty(\mathbb{R})$, we have

$$\begin{aligned}\left\|\int_0^t e^{i(t-t')\partial_{xx}}\Big(V(e^{-it'H}u_0^{(n)} - e^{-it'H}u_0)\Big)dt'\right\|_{L^2} &\le \int_0^t \|\Big(V(e^{-it'H}u_0^{(n)} - e^{-it'H}u_0)\Big)\|_{L^2} \\ &\le \int_0^t \|V\|_{L^\infty}\|u_0^{(n)} - u_0\|_{L^2}dt' \xrightarrow[n\to\infty]{} 0,\end{aligned}$$

where the last inequality is by Hölder's inequality.

Secondly for $V \in L^2(\mathbb{R})$, we apply the following form of inhomogeneous Strichartz estimate (see (Tao, 2006, Theorem 2.3)):

$$\left\|\int_0^t e^{i(t-t')\partial_{xx}} F(t')dt'\right\|_{L_t^\infty L_x^2} \lesssim \|F\|_{L_t^{\frac{4}{3}} L_x^1}.$$

Let χ be a characteristic function on $[0,T]$ where $0 < t < T$. Then we have,

$$\begin{aligned}\left\|\int_0^t e^{i(t-t')\partial_{xx}}\Big(Ve^{-it'H}(u_0^{(n)} - u_0)\Big)dt'\right\|_{L_x^2} &= \left\|\int_0^t e^{i(t-t')\partial_{xx}}\Big(V\chi(t')e^{-it'H}(u_0^{(n)} - u_0)\Big)dt'\right\|_{L_x^2} \\ &\le \left\|\int_0^t e^{i(t-t')\partial_{xx}}\Big(V\chi(t')e^{-it'H}(u_0^{(n)} - u_0)\Big)dt'\right\|_{L_t^\infty L_x^2} \\ &\lesssim \|V\chi e^{-itH}(u_0^{(n)} - u_0)\|_{L_t^{\frac{4}{3}} L_x^1} \\ &\lesssim \|V\|_{L^2}\|u_0^{(n)} - u_0\|_{L^2} \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

We will see that these two interpretations of a solution account for the two different hypotheses on V. In particular, if $V \in L^2(\mathbb{R})$, then we have a complete control over $\widehat{V}$, and so it is reasonable to apply Fourier analysis. By Fourier restriction space method, we show the following:

Lemma 2.2.1. *Suppose $V \in L^2(\mathbb{R})$. The Cauchy problem (2.1) is globally well-posed in $H^s(\mathbb{R})$ for $s \in [0, \frac{3}{4})$. In particular if u is the strong solution with the initial data $u_0 \in H^s(\mathbb{R})$, then there exists $\delta = \delta(\|V\|_{L^2}) > 0$ such that $\|u\|_{X_\delta^{s,b}} \le$*

$C(s, b, \|V\|_{L^2})\|u_0\|_{H^s}$ for some $b \in (\frac{1}{2}, 1]$.

Lemma 2.2.2. *Let $s \in [0, \frac{3}{4})$. Then, there exists $b \in (\frac{1}{2}, 1]$, $\gamma \in [0, \frac{1}{2})$ and $a \in [0, \frac{1}{2})$ that satisfy*

$$\frac{s+a}{2} \leq \gamma < \min(\frac{s}{2} + \frac{1}{4}, \frac{1}{2});\ \max(\frac{s}{2} + \frac{1}{4}, \frac{1}{2}) < b < 1 - \gamma.$$

Furthermore for every such (s, b, γ, a), we have $\|Vu\|_{X_\delta^{s+a,-\gamma}} \lesssim_{s,b,\gamma,a} \|V\|_{L^2}\|u\|_{X_\delta^{s,b}}$.

proof of theorem 2.2.1 (sufficiency). We claim the theorem holds when $V \in L^2(\mathbb{R})$; when V is bounded, see the remainder of the proof in the next section. For initial data in $H^s(\mathbb{R})$ for $s > \frac{1}{2}$, the solution for each $t \in \mathbb{R}$ can be identified with a continuous function by Sobolev embedding, and therefore, the conclusion follows immediately. Suppose $u_0 \in H^s(\mathbb{R})$ for $s \in [\frac{1}{4}, \frac{1}{2}]$. We have

$$\begin{aligned}
&\left|\left\{x : \overline{\lim_{t\to 0}}\,|e^{-itH}u_0 - u_0| > 0\right\}\right| \\
\leq &\left|\left\{x : \overline{\lim_{t\to 0}}\,|e^{it\partial_{xx}}u_0 - u_0| > 0\right\}\right| + \left|\left\{x : \overline{\lim_{t\to 0}}\left|\int_0^t e^{i(t-t')\partial_{xx}}\Big(Vu\Big)(t')dt'\right| > 0\right\}\right| \\
= &\left|\left\{x : \overline{\lim_{t\to 0}}\left|\int_0^t e^{i(t-t')\partial_{xx}}\Big(Vu\Big)(t')dt'\right| > 0\right\}\right|,
\end{aligned}$$

where the equality holds due to (Carleson, 1980). Then there exists $a > 0, b > \frac{1}{2}$ and $\gamma < \frac{1}{2}$ such that

$$s + a > \frac{1}{2};\ s + a \leq 2\gamma;\ b - 1 < -\gamma,$$

and for $\delta \in (0, 1]$:

$$\|Vu\|_{X_\delta^{s+a,-\gamma}} \lesssim \|V\|_{L^2}\|u\|_{X_\delta^{s,b}}.$$

We use the previous lemmas to conclude

$$\left\|\int_0^t e^{i(t-t')\partial_{xx}}\Big(Vu\Big)(t')dt'\right\|_{C_t^0H_x^{s+a}([-\delta,\delta]\times\mathbb{R})} \lesssim \left\|\int_0^t e^{i(t-t')\partial_{xx}}\Big(Vu\Big)(t')dt'\right\|_{X_\delta^{s+a,b}}$$
$$\lesssim \|Vu\|_{X_\delta^{s+a,b-1}} \lesssim \|V\|_{L^2}\|u\|_{X_\delta^{s,b}} \lesssim \|V\|_{L^2}\|u_0\|_{H^s} < \infty.$$

Hence another application of Sobolev embedding implies

$$\left|\left\{x : \overline{\lim_{t\to 0}}\left|\int_0^t e^{i(t-t')\partial_{xx}}\Big(Vu\Big)(t')dt'\right| > 0\right\}\right| = 0.$$

□

proof of lemma 2.2.1. Assuming that lemma 2.2.2 holds, let s, b, γ be as in lemma 2.2.2, $\delta \in (0,1]$ and fix $C > 0$ that satisfies $\|e^{it\partial_{xx}}f\|_{X_\delta^{s,b}} \le C\|f\|_{H^s}$ for all $f \in H^s(\mathbb{R})$ by lemma 1.2.1. Let $X = \Big\{u \in X_\delta^{s,b} : \|u\|_{X_\delta^{s,b}} \le 2C\|u_0\|_{H^s}\Big\}$. Define $\Gamma u = e^{it\partial_{xx}}u_0 - i\int_0^t e^{i(t-t')\partial_{xx}}(Vu)(t')dt'$. Then by lemma 1.2.1 and 2.2.2 we have,

$$\begin{aligned}\|\Gamma u\|_{X_\delta^{s,b}} &\lesssim \|u_0\|_{H^s} + \|Vu\|_{X_\delta^{s,b-1}} \lesssim \|u_0\|_{H^s} + \delta^{1-(b+\gamma)}\|Vu\|_{X_\delta^{s,-\gamma}} \\ &\lesssim \|u_0\|_{H^s} + \delta^{1-(b+\gamma)}\|V\|_{L^2}\|u\|_{X_\delta^{s,b}}. \\ \Rightarrow \|\Gamma u\|_{X_\delta^{s,b}} &\le C\|u_0\|_{H^s} + \tilde{C}C\delta^{1-(b+\gamma)}\|V\|_{L^2}\|u_0\|_{H^s}.\end{aligned}$$

By choosing $\delta \le (\tilde{C}\|V\|_{L^2})^{-\frac{1}{1-(b+\gamma)}}$, it is shown that $\Gamma : X \to X$. Similarly, we obtain

$$\|\Gamma u - \Gamma v\|_{X_\delta^{s,b}} \le C_0\delta^{1-(b+\gamma)}\|V\|_{L^2}\|u-v\|_{X_\delta^{s,b}},$$

from which it is shown that Γ is a contraction map by shrinking $\delta > 0$ if necessary, and the resulting unique fixed point is the desired strong solution. Since the time step only depends on the norm of V, this local result can be iterated infinitely many times, and hence our solution is global in time.

Continuous dependence on initial data follows similarly, for if $T > 0$, $u_0^{(n)} \to u_0$ in $H^s(\mathbb{R})$ and $u^{(n)}, u$ denote the strong solution corresponding to $u_0^{(n)}, u_0$, respectively, then for $t \le T$,

$$\begin{aligned}\|u^{(n)}(t)-u(t)\|_{H^s(\mathbb{R})} &\le \|u_0^{(n)}-u_0\|_{H^s} + \left\| \int_0^t e^{i(t-t')\partial_{xx}} \Big(V(u^{(n)}-u)\Big)(t')dt' \right\|_{H^s} \\ &\lesssim \|u_0^{(n)}-u_0\|_{H^s} + T^{1-(b+\gamma)}\|V\|_{L^2}\|u^{(n)}-u\|_{X_T^{s,b}} \\ &\lesssim \|u_0^{(n)}-u_0\|_{H^s} + T^{1-(b+\gamma)}\|V\|_{L^2}\|u_0^{(n)}-u_0\|_{H^s},\end{aligned}$$

where the implicit constant may depend on T. Taking $\sup\limits_{t\in[0,T]}$ both sides and taking $n\to\infty$, we obtain the desired result. $\square$

proof of lemma 2.2.2. The first statement is a straightforward algebra exercise. As for the second, it suffices to prove the statement neglecting the δ-dependence, for if $\tilde{u}=u$ on $t\in[-\delta,\delta]$, we have

$$\|Vu\|_{X_\delta^{s,-\gamma}} \le \|\eta(\frac{t}{\delta})V\tilde{u}\|_{X^{s,-\gamma}} \lesssim_\eta \|V\tilde{u}\|_{X^{s,-\gamma}} \lesssim \|V\|_{L^2}\|\tilde{u}\|_{X^{s,b}}.$$

Taking infimum over $\tilde{u}$, we derive the desired result. We argue as in the proof of (Erdogan and Tzirakis, 2013, Proposition 1); see also (Kenig et al., 1996a) for a similar technique.

Define

$$\begin{aligned}F(\xi) &= |\hat{V}(\xi)|,\\ G(\xi,\tau) &= \langle\xi\rangle^{s+a}\langle\tau+\xi^2\rangle^b|\hat{u}(\xi,\tau)|,\\ W(\xi,\tau,\xi_1) &= \frac{\langle\xi\rangle^{s+a}\langle\tau+\xi^2\rangle^{-\gamma}}{\langle\xi_1\rangle^s\langle\tau+\xi_1^2\rangle^b}.\end{aligned}$$

Noting that $\mathcal{F}[Vu](\xi,\tau) = \int \hat{V}(\xi-\xi_1)\hat{u}(\xi_1,\tau)d\xi_1$, we have $\|Vu\|^2_{X^{s,-\gamma}}$

$$\begin{aligned}
&= \left\| \int \langle\xi\rangle^{s+a}\langle\tau+\xi^2\rangle^{-\gamma}\hat{V}(\xi-\xi_1)\hat{u}(\xi_1,\tau)d\xi_1 \right\|^2_{L^2_{\xi,\tau}} \\
&\le \left\| \int \frac{\langle\xi\rangle^{s+a}\langle\tau+\xi^2\rangle^{-\gamma}}{\langle\xi_1\rangle^{s}\langle\tau+\xi_1^2\rangle^{b}} F(\xi-\xi_1)G(\xi_1,\tau)d\xi_1 \right\|^2_{L^2_{\xi,\tau}} \\
&\le \left\| \Big(\int W^2 d\xi_1\Big)^{1/2}\Big(\int F(\xi-\xi_1)^2 G(\xi_1,\tau)^2 d\xi_1\Big)^{1/2} \right\|^2_{L^2_{\xi,\tau}} \\
&= \left\| \int W^2 d\xi_1 \cdot \int F(\xi-\xi_1)^2 G(\xi_1,\tau)^2 d\xi_1 \right\|_{L^1_{\xi,\tau}} \\
&\le \left\| \int W^2 d\xi_1 \right\|_{L^\infty_{\xi,\tau}} \cdot \|F^2 *_{\xi_1} G^2\|_{L^1_{\xi,\tau}} \\
&\le \left\| \int W^2 d\xi_1 \right\|_{L^\infty_{\xi,\tau}} \cdot \|V\|^2_{L^2}\|u\|^2_{X^{s,b}},
\end{aligned}$$

where we use the Cauchy-Schwarz inequality, the Hölder's inequality, and the Young's inequality at various steps. It remains to prove that $\|\int W^2 d\xi_1\|_{L^\infty_{\xi,\tau}}$ is finite. Changing variable $z = \xi_1^2$,

$$\begin{aligned}
\int W^2 d\xi_1 &\simeq \langle\xi\rangle^{2s+2a}\langle\tau+\xi^2\rangle^{-2\gamma}\int \frac{d\xi_1}{\langle\xi_1^2\rangle^{s}\langle\tau+\xi_1^2\rangle^{2b}} \\
&\simeq \langle\xi\rangle^{2s+2a}\langle\tau+\xi^2\rangle^{-2\gamma}\int_0^\infty \frac{dz}{\langle z\rangle^{s}\langle z+\tau\rangle^{2b} z^{1/2}}.
\end{aligned}$$

We note that $\sup\limits_{\tau\in\mathbb{R}}$ can be replaced by $\sup\limits_{|\tau|>1}$ without loss of generality, for if $|\tau| \le 1$, then $\langle\tau+\xi^2\rangle^{-2\gamma} \le \langle\xi^2-1\rangle^{-2\gamma}$ for $|\xi| \ge 1$, and

$$\sup_{|\tau|\le 1}\int_0^\infty \frac{dz}{\langle z\rangle^{s}\langle z+\tau\rangle^{2b} z^{1/2}} < \infty.$$

Hence $\sup\limits_{|\xi|\ge 1, |\tau|\le 1}\int W^2 d\xi_1 \lesssim \sup\limits_{|\xi|\ge 1}\langle\xi\rangle^{2s+2a-4\gamma} < \infty$, whereas $\sup\limits_{|\xi|\le 1, |\tau|\le 1}\int W^2 d\xi_1 < \infty$

follows from extreme value theorem. Now suppose $|\tau| > 1$. Then we have

$$\int_0^\infty \frac{dz}{\langle z\rangle^s \langle z+\tau\rangle^{2b} z^{1/2}} = \int_0^{\frac{1}{2}} \frac{dz}{\langle z\rangle^s \langle z+\tau\rangle^{2b} z^{1/2}} + \int_{\frac{1}{2}}^\infty \frac{dz}{\langle z\rangle^s \langle z+\tau\rangle^{2b} z^{1/2}}$$
$$\lesssim \langle \tau\rangle^{-2b} + \langle \tau\rangle^{-(s+\frac{1}{2})} \lesssim \langle \tau\rangle^{-(s+\frac{1}{2})},$$

since $s + \frac{1}{2} < 2b$. Moreover since $2\gamma < s + \frac{1}{2}$, we have

$$\sup_{\xi\in\mathbb{R}, |\tau|>1} \langle \xi\rangle^{2s+2a} \langle \tau + \xi^2\rangle^{-2\gamma} \langle \tau\rangle^{-(s+\frac{1}{2})} \lesssim \sup_{\xi\in\mathbb{R}} \langle \xi\rangle^{2s+2a-4\gamma} < \infty.$$

□

2.3 Linear Operator Estimates.

Note that the square-integrability of the potential was crucial in establishing the fixed point argument by exploiting the Schrödinger dispersion relation to obtain a smoothing estimate. Now we depart from this Duhamel picture of the solution and study the Sobolev space estimates of e^{-itH}. If V and ∂_{xx} commute, then

$$e^{-itH} = e^{it\partial_{xx}} e^{-itV}, \tag{2.2}$$

and therefore, the operator e^{-itH} would obey the same maximal operator estimate of $e^{it\partial_{xx}}$ as in (Carleson, 1980), and our problem would be trivial. Generally the exponential map does not take addition into multiplication. If t is small, however, it is feasible to believe that (2.2) holds approximately, and the following lemma quantifies this intuition:

Lemma 2.3.1. *(Simon, 1980, Theorem 8.30) Let A and B be self-adjoint operators on a Hilbert space $\mathscr{H}$. If $A + B$ is self-adjoint on $D(A) \cap D(B)$, then*

$$\lim_{n\to\infty} (e^{i\frac{t}{n}A} e^{i\frac{t}{n}B})^n \phi = e^{it(A+B)}\phi, \ \forall \phi \in \mathscr{H}.$$

We apply this *Trotter-Kato product formula* to obtain persistence of regularity when the derivative of V is bounded.

Lemma 2.3.2. *Suppose $t \in \mathbb{R}$ and $s \in [0,1]$. If $\|V\|_{\dot{W}^{1,\infty}(\mathbb{R}^n)} < \infty$, then we obtain*

$$\|e^{-itH} f\|_{H^s} \leq e^{st\sqrt{n}\|V\|_{\dot{W}^{1,\infty}}} \|f\|_{H^s},\ \forall f \in C_c^\infty(\mathbb{R}^n).$$

In the previous section, we established

$$\int_0^t e^{i(t-t')\partial_{xx}} \Big(Vu\Big)(t')dt' \in C_t^0 H_x^{\frac{1}{2}+}([-\delta,\delta]\times\mathbb{R})$$

for $\delta \in (0,1]$ by using the local theory; more precisely, $\|u\|_{X_\delta^{s,b}} \lesssim \|u_0\|_{H^s}$ for $s \in [\frac{1}{4},\frac{1}{2}]$ and $b = \frac{1}{2}+$. We obtain a similar estimate via Trotter-Kato product formula and fractional Gagliardo-Nirenberg interpolation. Let $s \in (0,1)$ and $\max(\frac{1-s}{s},2) < \rho < \infty$. Then for every $f \in C_c^\infty(\mathbb{R})$, we obtain[1]

$$\|f\|_{W^{s,\frac{\rho}{1-s}}} \lesssim \|f\|_{L^\rho}^{1-s} \|f\|_{W^{1,\infty}}^{s}. \tag{2.3}$$

Lemma 2.3.3. *Let $s \in (0,1), b \in (\frac{1}{2},1]$ and $\delta \in (0,1]$. Then,*

$$\|e^{-itH} f\|_{X_\delta^{s,b}} \lesssim_{s,b,\eta} \|f\|_{H^s},\ \forall f \in H^s(\mathbb{R}).$$

Remark 2.3.1. *When $V=0$, this lemma reduces to lemma 1.2.1 where the proof heavily depends on the fact that the time-evolution operator defines a Fourier multiplier. However, if V is not identically zero, then the linear group action by e^{-itH} defines a Fourier integral operator. The linear estimate as above, therefore, is not entirely obvious for e^{-itH}.*

proof of theorem 2.2.1. Fix an open cover $\big\{(\frac{k}{2},\frac{k}{2}+1)\big\}_{k\in\mathbb{Z}}$ of $\mathbb{R}$ and let $\{\psi_k\}_k$ be a smooth partition of unity subordinate to the open cover. For $V \in W^{1,\infty}(\mathbb{R}) \cap L^\rho(\mathbb{R})$, we have $V = \sum\limits_k V_k$ where $V_k = V\psi_k \in L^2(\mathbb{R})$. For $u_0 \in H^s(\mathbb{R})$ where $s \in [\frac{1}{4},\frac{1}{2}]$, we

[1] For a more general statement, see (Brezis and Mironescu, 2018, Theorem 1).

have

$$\begin{aligned}
&\left|\left\{x : \overline{\lim_{t\to 0}}\,|e^{-itH}u_0 - u_0| > 0\right\}\right| \\
&\le \left|\left\{x : \overline{\lim_{t\to 0}}\,|e^{it\partial_{xx}}u_0 - u_0| > 0\right\}\right| + \sum_{k\in\mathbb{Z}}\left|\left\{x : \overline{\lim_{t\to 0}}\left|\int_0^t e^{i(t-t')\partial_{xx}}\left(V_k e^{-it'H}u_0\right)dt'\right| > 0\right\}\right| \\
&= \sum_{k\in\mathbb{Z}}\left|\left\{x : \overline{\lim_{t\to 0}}\left|\int_0^t e^{i(t-t')\partial_{xx}}\left(V_k e^{-it'H}u_0\right)dt'\right| > 0\right\}\right|.
\end{aligned}$$

For $\delta \in (0,1]$ and $b = \frac{1}{2}+$, we obtain

$$\begin{aligned}
\left\|\int_0^t e^{i(t-t')\partial_{xx}}\left(V_k e^{-it'H}u_0\right)(t')dt'\right\|_{C_t^0 H_x^{\frac{1}{2}+}([-\delta,\delta]\times\mathbb{R})} &\lesssim \|V_k\|_{L^2}\|e^{-itH}u_0\|_{X_\delta^{\frac{1}{4},b}} \\
&\lesssim \|V_k\|_{L^2}\|u_0\|_{H^{\frac{1}{4}}} < \infty,
\end{aligned}$$

where the second inequality follows from lemma 2.3.3. By Sobolev embedding,

$$\left|\left\{x : \overline{\lim_{t\to 0}}\left|\int_0^t e^{i(t-t')\partial_{xx}}\left(V_k e^{-it'H}u_0\right)dt'\right| > 0\right\}\right| = 0,$$

for all $k \in \mathbb{Z}$ and this completes the proof. $\square$

proof of lemma 2.3.2. We first show $\|e^{-itV}\|_{H^1\to H^1} \le 1 + t\sqrt{n}\|V\|_{\dot{W}^{1,\infty}}$, $\forall t \in \mathbb{R}$. Let $f \in C_c^\infty(\mathbb{R}^n)$. Then we have

$$\begin{aligned}
\|e^{-itV}f\|_{H^1}^2 &= \|e^{-itV}f\|_{L^2}^2 + \sum_{j=1}^n \|\partial_j(e^{-itV}f)\|_{L^2}^2 \\
&\le \|f\|_{L^2}^2 + \sum_{j=1}^n (t\|\partial_j V\cdot f\|_{L^2} + \|\partial_j f\|_{L^2})^2 \le (1 + t\sqrt{n}\|V\|_{\dot{W}^{1,\infty}})^2\|f\|_{H^1}^2.
\end{aligned}$$

Hence, the best constant $C(t) \le 1 + t\sqrt{n}\|V\|_{\dot{W}^{1,\infty}}, \forall t \in \mathbb{R}$.

Let $\phi = e^{-itH}u_0$ for a fixed $u_0 \in H^1(\mathbb{R}^n)$ and $\phi_m = (e^{-i\frac{t}{m}V}e^{i\frac{t}{m}\partial_{xx}})^m u_0$. Then, $\phi_m \to \phi$ in $L^2(\mathbb{R}^n)$ by Trotter-Kato product formula. By the estimate on $\|e^{-itV}\|_{H^1\to H^1}$, we obtain $\|e^{-itV}e^{it\partial_{xx}}f\|_{H^1} \leq (1+t\sqrt{n}\|V\|_{\dot{W}^{1,\infty}})\|f\|_{H^1}$. Then we have

$$\|\phi_m\|_{H^1} \leq (1+\frac{\sqrt{n}\|V\|_{\dot{W}^{1,\infty}}t}{m})^m\|u_0\|_{H^1}.$$

Hence for $t \in [0,T]$ for $T > 0$, we have a bounded sequence $\{\phi_m\}_m \subset H^1(\mathbb{R}^n)$, a reflexive Banach space. Then, there exists a weakly convergent subsequence $\{\phi_{m_k}\}_k$ where $\phi_{m_k} \rightharpoonup \tilde{\phi} \in H^1(\mathbb{R}^n)$. Since $H^1(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$, $\phi_{m_k} \rightharpoonup \tilde{\phi}$ in $L^2(\mathbb{R}^n)$ and since $\phi_m \to \phi$ in $L^2(\mathbb{R}^n)$, the convergence holds in weak topology, and by the uniqueness of weak-limit in Banach space, $\phi = \tilde{\phi}$; in particular, $e^{-itH}u_0 \in H^1(\mathbb{R}^n)$. Since norm is lower semicontinuous with respect to weak topology, we have

$$\|e^{-itH}u_0\|_{H^1} \leq \varliminf_{k\to\infty}\|\phi_{m_k}\|_{H^1} \leq e^{t\sqrt{n}\|V\|_{\dot{W}^{1,\infty}}}\|u_0\|_{H^1}.$$

Since the bound above holds for all $t \in [0,T]$ uniformly in T, it holds for all $t \in \mathbb{R}$. Then by complex interpolation, it follows that for $s \in [0,1]$,

$$\|e^{-itH}f\|_{H^s} \leq e^{st\sqrt{n}\|V\|_{\dot{W}^{1,\infty}}}\|f\|_{H^s}, \forall f \in H^s(\mathbb{R}^n).$$

□

proof of lemma 2.3.3. Let $F(\xi,\tau) = \mathcal{F}[\eta(\cdot)e^{-i\cdot H}f](\xi,\tau)$ and $\tilde{F}(\xi,t) = \mathcal{F}_\tau^{-1}F$ where $\mathcal{F}_\tau^{-1}$ is the inverse Fourier transform in τ variable, and let $\mathcal{F}_x$ be defined similarly. Moreover, denote $u(t) = e^{-itH}f$ as a shorthand. Then we obtain

$$\begin{aligned}\|e^{-itH}f\|_{X_\delta^{s,b}} &\leq \|\eta(t)e^{-itH}f\|_{X^{s,b}} = \|\langle\xi\rangle^s\langle\tau+\xi^2\rangle^bF(\xi,\tau)\|_{L^2_{\xi,\tau}} \\ &= \|\langle\xi\rangle^s\|\langle\tau\rangle^bF(\xi,\tau-\xi^2)\|_{L^2_\tau}\|_{L^2_\xi} = \|\langle\xi\rangle^s\|e^{it\xi^2}\tilde{F}(\xi,t)\|_{H^b_t}\|_{L^2_\xi} \\ &\leq \|\langle\xi\rangle^s\|e^{it\xi^2}\tilde{F}(\xi,t)\|_{H^1_t}\|_{L^2_\xi} \\ &\lesssim_s \|\langle\xi\rangle^s\tilde{F}(\xi,t)\|_{L^2_{\xi,t}} + \|\langle\xi\rangle^s\Big(|\partial_t|(e^{it\xi^2}\tilde{F}(\xi,t))\Big)\|_{L^2_{\xi,t}}.\end{aligned}$$

For the first term, integrate in ξ variable first using Plancherel's theorem, followed by the estimate for the operator norm $\|e^{-itH}\|_{H^s\to H^s}$ and followed by the t-integral as follows:

$$\|\langle\xi\rangle^s\tilde{F}(\xi,t)\|_{L^2_{\xi,t}} = \|\eta(t)\cdot\|e^{-itH}f\|_{H^s_x}\|_{L^2_t} \leq \|\eta(t)e^{s\|V'\|_{L^\infty}t}\|_{L^2_t}\cdot\|f\|_{H^s} \lesssim_\eta \|f\|_{H^s}.$$

As for the second term,

$$\begin{aligned}
\|\langle\xi\rangle^s\Big(|\partial_t|(e^{it\xi^2}\tilde{F}(\xi,t))\Big)\|_{L^2_{\xi,t}} &= \|\langle\xi\rangle^s\cdot\||\partial_t|(e^{it\xi^2}\tilde{F}(\xi,t))\|_{L^2_t}\|_{L^2_\xi}\\
&= \|\langle\xi\rangle^s\cdot\|\mathcal{H}\partial_t e^{it\xi^2}\tilde{F}(\xi,t)\|_{L^2_t}\|_{L^2_\xi}\\
&= \|\langle\xi\rangle^s\cdot\|\partial_t\Big(\eta(t)e^{it\xi^2}\mathcal{F}_x[e^{-itH}f](\xi,t)\Big)\|_{L^2_t}\|_{L^2_\xi}\\
\le \|\langle\xi\rangle^s\cdot\|\partial_t\eta\cdot e^{it\xi^2}\mathcal{F}_x[e^{-itH}f]\|_{L^2_t}\|_{L^2_\xi} &+ \|\langle\xi\rangle^s\cdot\|\eta(t)\partial_t\Big(e^{it\xi^2}\mathcal{F}_x[e^{-itH}f]\Big)\|_{L^2_t}\|_{L^2_\xi}.
\end{aligned}$$

For the first term, switching the order of integration and recalling that the family $e^{it\partial_{xx}}$ is unitary on $H^s(\mathbb{R})$,

$$\begin{aligned}
\|\langle\xi\rangle^s\cdot\|\partial_t\eta\cdot e^{it\xi^2}\mathcal{F}_x[e^{-itH}f]\|_{L^2_t}\|_{L^2_\xi} &= \|\partial_t\eta\cdot\|e^{-it\partial_{xx}}u(t)\|_{H^s_x}\|_{L^2_t}\\
&= \|\partial_t\eta\cdot\|u(t)\|_{H^s_x}\|_{L^2_t}\\
&\le \|\partial_t\eta\cdot e^{s\|V'\|_{L^\infty}t}\|_{L^2_t}\cdot\|f\|_{H^s}\lesssim_\eta \|f\|_{H^s}.
\end{aligned}$$

For the second term, use product rule in t to obtain

$$\begin{aligned}
\|\langle\xi\rangle^s\cdot\|\eta(t)\partial_t\Big(e^{it\xi^2}\mathcal{F}_x[e^{-itH}f]\Big)\|_{L^2_t}\|_{L^2_\xi} &= \|\eta(t)\cdot\|\partial_t\Big(e^{-it\partial_{xx}}e^{-itH}f\Big)\|_{H^s_x}\|_{L^2_t}\\
&= \|\eta(t)\cdot\|Ve^{-itH}f\|_{H^s_x}\|_{L^2_t}.
\end{aligned}$$

where the second equality follows from

$$\partial_t\Big(e^{-it\partial_{xx}}e^{-itH}f\Big) = -ie^{-it\partial_{xx}}\Big(Ve^{-itH}f\Big).$$

Then with $q\in(2,\infty)$ defined as follows,

$$\frac{1}{q}+\frac{1-s}{\rho}=\frac{1}{2},$$

apply the following particular form of Leibniz rule for Sobolev space to obtain[2]

$$\begin{aligned}
\|Ve^{-itH}f\|_{H^s} &\lesssim_s \|V\|_{L^\infty}\|e^{-itH}f\|_{H^s}+\|V\|_{W^{s,\frac{\rho}{1-s}}}\|e^{-itH}f\|_{L^q}\\
&\lesssim \Big(\|V\|_{L^\infty}+\|V\|_{W^{s,\frac{\rho}{1-s}}}\Big)\|e^{-itH}f\|_{H^s}\\
&\le (\|V\|_{L^\infty}+\|V\|_{W^{s,\frac{\rho}{1-s}}})e^{s\|V'\|_{L^\infty}t}\|f\|_{H^s}.
\end{aligned}$$

[2]Unfortunately, the Leibniz rule generally fails when the L^∞ norm is applied to the Bessel potential term. Had this been true, the decay condition on V could have been removed.

Since the first factor of the RHS is finite by (2.3), the proof is complete by integrating the upper bound in t against the smooth bump η. $\square$

2.4 Quadratic Nonlinearities.

We consider the following qNLS Cauchy problem:

$$\begin{cases} i\partial_t u + \partial_{xx} u = N_i(u, \overline{u}) \\ N_1(u, \overline{u}) = u^2; N_2(u, \overline{u}) = u\overline{u}; N_3(u, \overline{u}) = \overline{u}^2 \\ u(0) = u_0 \in H^s(\mathbb{R}). \end{cases}$$

The well-posedness of qNLS above is studied in (Kenig et al., 1996a). By the $X^{s,b}$ method, they prove that qNLS for N_1 and N_3 are well-posed in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$ whereas that for N_2 is well-posedness for $s > -\frac{1}{4}$; the well-posedness associated to N_1 was improved to $H^{-1}(\mathbb{R})$ and was shown to be sharp in (Bejenaru and Tao, 2006). Writing the solution in the integral form,

$$u(t) = e^{it\partial_{xx}} u_0 - i \int_0^t e^{i(t-\tau)\partial_{xx}} N_i(u)(\tau) d\tau, t \in [-\delta, \delta],$$

the technique used to control the potential term in the previous section would work on nonlinearities as well. The goal is to prove analogous smoothing estimates for N_i, $i = 1, 2, 3$ as in lemma 2.2.2 from which convergence to initial data follows by Sobolev embedding.

Proposition 2.4.1. *Let u be the (local) strong solution of the qNLS corresponding to N_i, $i = 1, 3$. Then convergence a.e. to initial data holds if and only if $s \geq \frac{1}{4}$.*

Proposition 2.4.2. *Let u be the (local) strong solution of the qNLS corresponding to N_2. Then convergence a.e. to initial data holds for $s > \frac{1}{4}$.*

Lemma 2.4.1. *Let $s \geq 0$, $a \in [0, \frac{1}{2})$ and $\delta \in (0,1]$. Then there exists $b = \frac{1}{2}+$ and $\gamma = \frac{1}{2}-$ such that $b < 1-\gamma$ and the following estimates hold for $i = 1, 3$:*

$$\|N_i(u,\overline{u})\|_{X_\delta^{s+a,-\gamma}} \lesssim_{s,a,b,\gamma} \|u\|^2_{X_\delta^{s,b}}.$$

Lemma 2.4.2. *Let $s > \frac{1}{4}$, $a \in [0, \frac{1}{2}]$ and $\delta \in (0,1]$. Then there exists $b = \frac{1}{2}+$, $\gamma = \frac{1}{2}-$ such that $b < 1-\gamma$ and the following estimate holds:*

$$\|N_2(u,\overline{u})\|_{X_\delta^{s+a,-\gamma}} \lesssim_{s,a,b,\gamma} \|u\|^2_{X_\delta^{s,b}}.$$

Remark 2.4.1. *As for proposition 2.4.2, the condition $s > \frac{1}{4}$ is needed to make certain integrals converge; in fact if $\xi = \tau = 0$, then the expression inside the $\sup\limits_{\xi,\tau}$ (see (2.4)) is*

$$\int \frac{d\xi_1 d\tau_1}{\langle \xi_1 \rangle^{4(\frac{1}{4})} \langle \tau_1 - \xi_1^2 \rangle^{4b}} = \infty.$$

proof of proposition 2.4.1 and 2.4.2. The positive statements are consequences of (Carleson, 1980) and Duhamel nonlinear terms being continuous in space and time via the smoothing estimates followed by Sobolev embedding. We focus on the negative part of proposition 2.4.1.

For $s \in (0, \frac{1}{4})$ we know from (Dahlberg and Kenig, 1982) that there exists $u_0 \in H^s(\mathbb{R})$ such that convergence to initial data (for the linear evolution) fails on some set E of positive measure. By Lemma 2.4.1, we choose $a = \frac{1}{2}-$ to obtain

$$DN(x,t) \coloneqq \int_0^t e^{i(t-t')\partial_{xx}} N_i(u,\overline{u})(t')dt' \in C_t^0 H_x^{\frac{1}{2}+}([0,\delta] \times \mathbb{R}).$$

By triangle inequality,

$$|u(t) - u_0| \geq |e^{it\partial_{xx}} u_0 - u_0| - |DN(x,t)|.$$

By continuity, $\lim\limits_{t\to 0} |DN| = 0$ a.e., and therefore

$$\left| \left\{ x \in E : \overline{\lim_{t\to 0}} |u(t) - u_0| > 0 \right\} \right| \geq \left| \left\{ x \in E : \overline{\lim_{t\to 0}} |e^{it\partial_{xx}} u_0 - u_0| > 0 \right\} \right| > 0.$$

Since $H^s(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ for $s \in (0, \frac{1}{4})$, a.e. pointwise convergence cannot hold for initial data in $L^2(\mathbb{R})$, and this finishes the proof. $\square$

proof of lemma 2.4.1. The N_3-estimate will be shown to be an easy consequence of the N_1-estimate, and therefore we focus on the former. Denote

$$F(\xi, \tau) = |\hat{u}(\xi, \tau)|\langle\xi\rangle^s\langle\tau + \xi^2\rangle^b$$
$$W(\xi, \tau, \xi_1, \tau_1) = \frac{\langle\xi\rangle^{s+a}\langle\tau + \xi^2\rangle^{-\gamma}}{\langle\xi - \xi_1\rangle^s\langle\tau - \tau_1 + (\xi - \xi_1)^2\rangle^b\langle\xi_1\rangle^s\langle\tau_1 + \xi_1^2\rangle^b}.$$

Neglecting δ-dependence as before, we have

$$\begin{aligned}
\|u^2\|^2_{X^{s+a,-\gamma}} &= \left\| \int \langle\xi\rangle^{s+a}\langle\tau + \xi^2\rangle^{-\gamma}\hat{u}(\xi - \xi_1, \tau - \tau_1)\hat{u}(\xi_1, \tau_1)d\xi_1 d\tau_1 \right\|^2_{L^2_{\xi,\tau}} \\
&\leq \left\| \int \frac{\langle\xi\rangle^{s+a}\langle\tau + \xi^2\rangle^{-\gamma}F(\xi - \xi_1, \tau - \tau_1)F(\xi_1, \tau_1)d\xi_1 d\tau_1}{\langle\xi - \xi_1\rangle^s\langle\tau - \tau_1 + (\xi - \xi_1)^2\rangle^b\langle\xi_1\rangle^s\langle\tau_1 + \xi_1^2\rangle^b} \right\|^2_{L^2_{\xi,\tau}} \\
&= \left\| \int W^2 d\xi_1 d\tau_1 \cdot \int F(\xi - \xi_1, \tau - \tau_1)^2 F(\xi_1, \tau_1)^2 d\xi_1 d\tau_1 \right\|_{L^1_{\xi,\tau}} \\
&\leq \left\| \int W^2 d\xi_1 d\tau_1 \right\|_{L^\infty_{\xi,\tau}} \cdot \|F^2 * F^2\|_{L^1_{\xi,\tau}} \\
&= \left\| \int W^2 d\xi_1 d\tau_1 \right\|_{L^\infty_{\xi,\tau}} \cdot \|u\|^4_{X^{s,b}}.
\end{aligned}$$

Hence, it suffices to prove that $\|\int W^2 d\xi_1 d\tau_1\|_{L^\infty_{\xi,\tau}} < \infty$.

By lemma 1.2.1, we have

$$\int \frac{d\tau_1}{\langle\tau - \tau_1 + (\xi - \xi_1)^2\rangle^{2b}\langle\tau_1 + \xi_1^2\rangle^{2b}} \lesssim \langle\tau + (\xi_1 - \xi)^2 + \xi_1^2\rangle^{-2b}.$$

Similarly,

$$\langle\tau + (\xi_1 - \xi)^2 + \xi_1^2\rangle^{-2b}\langle\tau + \xi^2\rangle^{-2\gamma} \lesssim \langle\xi_1(\xi_1 - \xi)\rangle^{-2\gamma},$$

and

$$\frac{\langle\xi\rangle^{2s+2a}}{\langle\xi_1 - \xi\rangle^{2s}\langle\xi_1\rangle^{2s}} \lesssim \langle\xi\rangle^{2a}.$$

Altogether we have

$$\sup_{\xi,\tau}\int W^2 d\xi_1 d\tau_1 \lesssim \sup_{\xi}\Big(\langle\xi\rangle^{2a}\int\frac{d\xi_1}{\langle\xi_1(\xi_1-\xi)\rangle^{2\gamma}}\Big).$$

Note that the integral is symmetric with respect to $\xi_1=\frac{\xi}{2}$, and therefore $\int\frac{d\xi_1}{\langle\xi_1(\xi_1-\xi)\rangle^{2\gamma}}=2\int_{\xi/2}^{\infty}\frac{d\xi_1}{\langle\xi_1(\xi_1-\xi)\rangle^{2\gamma}}$. Henceforth, assume $\xi\geq 0$ without loss of generality. On the region of integration, change variable $\eta=\xi_1(\xi_1-\xi)=\xi_1^2-\xi\xi_1$ to obtain:

$$\xi_1=\frac{\xi+\sqrt{\xi^2+4\eta}}{2};\ d\xi_1=\frac{d\eta}{\sqrt{\xi^2+4\eta}}.$$

and so the integral becomes

$$2\int_{\xi/2}^{\infty}\frac{d\xi_1}{\langle\xi_1(\xi_1-\xi)\rangle^{2\gamma}}=2\int_{-\xi^2/4}^{\infty}\frac{d\eta}{\sqrt{\xi^2+4\eta}\langle\eta\rangle^{2\gamma}}=\int_0^{\infty}\frac{d\eta}{\sqrt{\eta}\langle\eta-\frac{\xi^2}{4}\rangle^{2\gamma}}.$$

Since this integral is bounded for all $\xi\in[0,1)$, it suffices to assume $\xi\geq 1$ and show $\int_0^{\infty}\frac{d\eta}{\sqrt{\eta}\langle\eta-\frac{\xi^2}{4}\rangle^{2\gamma}}\lesssim\frac{1}{\langle\xi\rangle^{4\gamma-1}}$. Then, with $a<\frac{1}{2}$, it follows immediately that $\|\int W^2 d\xi_1 d\tau_1\|_{L^{\infty}_{\xi,\tau}}<\infty$, provided $b>\frac{1}{2}$ is chosen sufficiently small.

Let $c=\frac{\xi^2}{4}$ and estimate the integral in three different regions: i) $\eta\in[2c,\infty)$; ii) $\eta\in[\frac{c}{2},2c)$; iii) $\eta\in(0,\frac{c}{2})$.

$$i):\int_{\eta\geq 2c}\lesssim\int_{2c}^{\infty}\frac{d\eta}{\sqrt{\eta}\eta^{2\gamma}}\simeq\frac{1}{\xi^{4\gamma-1}}.$$

$$ii):\int_{\frac{c}{2}}^{2c}\lesssim\int_{\frac{c}{2}}^{2c}\frac{d\eta}{\sqrt{\eta}|\eta-c|^{2\gamma}}\lesssim c^{-1/2}\int_{\frac{c}{2}}^{2c}\frac{d\eta}{|\eta-c|^{2\gamma}}\lesssim c^{-1/2}\cdot c^{1-2\gamma}\simeq\frac{1}{\xi^{4\gamma-1}}.$$

$$iii):\int_0^{\frac{c}{2}}\frac{d\eta}{\sqrt{\eta}\langle\eta-c\rangle^{2\gamma}}\lesssim\frac{1}{\langle c\rangle^{2\gamma}}\int_0^{\frac{c}{2}}\frac{d\eta}{\sqrt{\eta}}\lesssim\frac{c^{1/2}}{\langle c\rangle^{2\gamma}}\simeq\frac{1}{\xi^{4\gamma-1}}.$$

Bringing all three cases together, we obtain the desired estimate, and this proves the first smoothing estimate.

As for the second estimate, for a general spacetime function u, we have

$$\|\overline{u}\|_{X^{s.b}} = \|\hat{u}(\xi,\tau)\langle\xi\rangle^s\langle\tau-\xi^2\rangle^b\|_{L^2_{\xi,\tau}}.$$

Arguing as before, one obtains

$$\|\overline{u}^2\|_{X^{s+a,-\gamma}} \leq \left\|\int \Omega^2 d\xi_1 d\tau_1\right\|_{L^\infty_{\xi,\tau}} \|u\|^4_{X^{s,b}}.$$

where

$$\Omega(\xi,\tau,\xi_1,\tau_1) = \frac{\langle\xi\rangle^{s+a}\langle\tau-\xi^2\rangle^{-\gamma}}{\langle\xi-\xi_1\rangle^s\langle\tau-\tau_1+(\xi-\xi_1)^2\rangle^b\langle\xi_1\rangle^s\langle\tau_1+\xi_1^2\rangle^b},$$

and therefore it suffices to show $\|\int\Omega^2 d\xi_1 d\tau_1\|_{L^\infty_{\xi,\tau}} < \infty$. As before,

$$\begin{aligned}\left\|\int \Omega^2 d\xi_1 d\tau_1\right\|_{L^\infty_{\xi,\tau}} &\lesssim \left\|\int \frac{\langle\xi\rangle^{2s+2a}\langle\tau-\xi^2\rangle^{-2\gamma}}{\langle\xi_1-\xi\rangle^{2s}\langle\xi_1\rangle^{2s}\langle\tau+(\xi_1-\xi)^2+\xi_1^2\rangle^{2b}} d\xi_1\right\|_{L^\infty_\xi} \\ &\leq \left\|\int \frac{\langle\xi\rangle^{2s+2a}}{\langle\xi_1-\xi\rangle^{2s}\langle\xi_1\rangle^{2s}\langle\xi_1^2-\xi_1\xi+\xi^2\rangle^{2\gamma}} d\xi_1\right\|_{L^\infty_\xi} \\ &\lesssim \left\|\langle\xi\rangle^{2a}\int \frac{d\xi_1}{\langle\xi_1^2-\xi_1\xi+\xi^2\rangle^{2\gamma}}\right\|_{L^\infty_\xi},\end{aligned}$$

where these inequalities are direct applications of lemma 1.2.1. Then by a direct computation, $|\xi_1^2-\xi_1\xi+\xi^2| \geq |\xi_1(\xi_1-\xi)|$, and therefore

$$\langle\xi_1^2-\xi_1\xi+\xi^2\rangle \geq \langle\xi_1(\xi_1-\xi)\rangle,\ \forall\xi,\xi_1\in\mathbb{R}.$$

Then $\|\int\Omega^2 d\xi_1 d\tau_1\|_{L^\infty_{\xi,\tau}} < \infty$ follows from our previous result:

$$\left\|\langle\xi\rangle^{2a}\int \frac{d\xi_1}{\langle\xi_1(\xi_1-\xi)\rangle^{2\gamma}}\right\|_{L^\infty_\xi} < \infty.$$

$\square$

proof of lemma 2.4.2. Arguing as before, it suffices to prove

$$\sup_{\xi,\tau}\left(\langle\xi\rangle^{2s+2a}\langle\tau+\xi^2\rangle^{-2\gamma}\int\frac{d\xi_1 d\tau_1}{\langle\xi-\xi_1\rangle^{2s}\langle\xi_1\rangle^{2s}\langle\tau_1-(\tau+(\xi-\xi_1)^2)\rangle^{2b}\langle\tau_1-\xi_1^2\rangle^{2b}}\right)<\infty. \tag{2.4}$$

For $|\xi|<1$,

$$\begin{aligned}&\langle\xi\rangle^{2s+2a}\langle\tau+\xi^2\rangle^{-2\gamma}\int\frac{d\xi_1 d\tau_1}{\langle\xi-\xi_1\rangle^{2s}\langle\xi_1\rangle^{2s}\langle\tau_1-(\tau+(\xi-\xi_1)^2)\rangle^{2b}\langle\tau_1-\xi_1^2\rangle^{2b}}\\ &\lesssim\int\frac{d\xi_1}{\langle\xi_1\rangle^{4s}\langle\tau+\xi^2-2\xi\xi_1\rangle^{2b}}\lesssim_{s,a}\int\frac{d\xi_1}{\langle\xi_1\rangle^{4s}}\leq C<\infty,\end{aligned}$$

where the upper bound C is independent of τ. For $|\xi|\geq 1$, changing variable $z=2\xi\xi_1-(\tau+\xi^2)$,

$$\begin{aligned}&\langle\xi\rangle^{2s+2a}\langle\tau+\xi^2\rangle^{-2\gamma}\int\frac{d\xi_1 d\tau_1}{\langle\xi-\xi_1\rangle^{2s}\langle\xi_1\rangle^{2s}\langle\tau_1-(\tau+(\xi-\xi_1)^2)\rangle^{2b}\langle\tau_1-\xi_1^2\rangle^{2b}}\\ &\lesssim\frac{\langle\xi\rangle^{2a}}{\langle\tau+\xi^2\rangle^{2\gamma}}\int\frac{d\xi_1}{\langle 2\xi\xi_1-(\tau+\xi^2)\rangle^{2b}}\simeq\frac{\langle\xi\rangle^{2a-1}}{\langle\tau+\xi^2\rangle^{2\gamma}}\int\frac{dz}{\langle z\rangle^{2b}}\lesssim\langle\xi\rangle^{2a-1}.\end{aligned}$$

□

2.5 Hamiltonian Flow, Integral Kernel and Negative Results for $s<\frac{1}{4}$.

We ask a similar question as before: if we perturb $-\partial_{xx}$ to $H=-\partial_{xx}+V$, do we obtain $e^{-itH}\nrightarrow\mathbb{I}$ for $s<\frac{1}{4}$? To motivate what is to come, consider the following where both V and ϕ are Schwartz functions:

$$\frac{e^{-itH}-e^{it\partial_{xx}}e^{-itV}}{t}\phi=\frac{e^{-itH}-\mathbb{I}}{t}\phi-e^{it\partial_{xx}}\frac{e^{-itV}-\mathbb{I}}{t}\phi-\frac{e^{it\partial_{xx}}-\mathbb{I}}{t}\phi\xrightarrow[t\to 0]{L^2}0,$$

and hence $e^{-itH}=e^{it\partial_{xx}}e^{-itV}+o(t)$ in strong operator topology as $t\to 0$. Hence it is reasonable to believe that the *failure* of $e^{it\partial_{xx}}$ to converge to the identity as $t\to 0$ in a.e. sense would directly contribute to that of e^{-itH}, provided that V is a *small* *perturbation*; in fact, we will be interested in the case when V is squre-integrable. Now we remind the reader that the time-evolution operator can not only be understood in terms of Duhamel formula, but also by Schwartz kernel representation in the physical space. We take a slight detour from our pointwise convergence problem, and study the dispersive estimate for e^{-itH} where the hypotheses on V are as follows:

Assumption: $V \in C^\infty(\mathbb{R})$ satisfies $|\partial^\alpha V| \leq C_\alpha$ for all $|\alpha| \geq 2$.

For the class of V as above, H is essentially self-adjoint on $\mathscr{S}(\mathbb{R})$, and therefore, has a unique self-adjoint extension for this class of potentials (see (Faris and Lavine, 1974)). Therefore we shall refer to this unique extension whenever we mention the infinitesimal generator of the time-evolution operator. One motivation for this hypothesis on V comes from (Koch et al., 2005) that studies the L^p eigenfunction bounds associated to $-\Delta + V$. Here our goal is to give an alternative proof of (Koch et al., 2005, Thereom 1.b.) by directly applying results of Fujiwara based on Feynmann path integrals.

To carry out our short-time analysis, it turns out to be useful to study the integral kernel corresponding to time-evolution unitary operators. As a shorthand, let $U(t), U_0(t)$ be the unitary groups generated by H and $-\partial_{xx}$ respectively. Let $K(t,x,y), K_0(t,x,y)$ be the corresponding Schwartz kernel, i.e.,

$$U(t)f(x) = \int K(t,x,y)f(y)dy;\ U_0(t)f(x) = \int K_0(t,x,y)f(y)dy.$$

The goal is to use some known properties of K and K_0 to study local-in-time properties of $U(t)$. First of all, it is well known that

$$K_0(t,x,y) = (4\pi it)^{-1/2} e^{i\frac{|x-y|^2}{4t}}.$$

$K(t,x,y)$ has a similar kernel representation as follows (see (Fujiwara, 1983)), as long as we are willing to restrict the time parameter:

$$K(t,x,y) = (4\pi it)^{-1/2}k(t,x,y)e^{iS(t,x,y)},$$

for $0 < |t| \leq \delta$ for some $\delta > 0$ that depends only on V. For our purposes, $k(t,x,y)$ is smooth in the space variable $\mathbb{R} \times \mathbb{R}$, measurable in t and satisfies

$$\sup_{x,y} |k(t,x,y) - 1| \lesssim |t|^2, \tag{2.5}$$

for $0 < |t| \leq \delta$ (see (Fujiwara, 1983, Theorem 2.2)). We fix this $\delta > 0$ in this section.

On the other hand, let S be the action of a classical path going from y at time 0 to x at time t. More precisely, consider the following Hamiltonian flow generated by $h(x,\xi) = \xi^2 + V(x)$:

$$\begin{cases} \dot{x} = \frac{\partial h}{\partial \xi} = 2\xi \\ \dot{\xi} = -\frac{\partial h}{\partial x} = -V'(x). \end{cases}$$

Since the Hamiltonian vector field is smooth and globally Lipschitz, thanks to $|V''| \lesssim 1$, every orbit is globally defined and is smooth. By studying the regularity of $x = x(t,y,\eta)$ and $\xi = \xi(t,y,\eta)$, where η is the initial condition for ξ, one can show that there exists $\delta > 0$ such that $(y,\eta) \mapsto (y, x = x(t,y,\eta))$ defines a C^∞ diffeomorphism on $\mathbb{R} \times \mathbb{R}$ for all $t \in [-\delta,\delta]$. Then by implicit function theorem, we can solve η for t,x,y, i.e., $\eta = \eta(t,x,y)$ for $t \in [-\delta,\delta]$. Then, define $x(\tau) = x(\tau,y,\eta(t,x,y))$. Then, $x(\tau)$ is the unique path that starts at y at time 0 and ends at x at time t; for a more thorough discussion, see (Fujiwara, 1983). The classical action functional corresponding to this path is defined as follows:

$$S(t,x,y) = \int_0^t L(\tau, x(\tau), \dot{x}(\tau))d\tau,$$

where $L = \xi^2 - V(x)$ is the Lagrangian functional corresponding to this flow. In particular for $V = 0$, we recover the action for free Schrödinger equation. For readers curious about regularity properties of S, we state the following facts from (Fujiwara, 1983):

Lemma 2.5.1. *Let $|t| \in (0, \delta]$.*

1. *$S(t,x,y)$ is Lipschitz continuous in t and smooth in $(x,y) \in \mathbb{R} \times \mathbb{R}$.*
2. *S satisfies*

$$\begin{cases} \partial_x S(t,x,y) = \xi(t,y,\eta(t,x,y)) \\ \partial_y S(t,x,y) = -\eta(t,x,y), \end{cases}$$

   or, i.e., S is the generating function of the canonical diffeomorphism $(y,\eta) \mapsto (x,\xi)$.

3. *Define $w(t,x,y)$ as follows: $S(t,x,y) = S_0(t,x,y) + t \cdot w(t,x,y)$ where S_0 is the action for free Schrödinger equation. Then, $|\partial_x^\alpha \partial_y^\beta w(t,x,y)| \lesssim_{\alpha,\beta} 1$ uniformly in space for all multi-indices $|\alpha| + |\beta| \geq 2$.*

In fact, the main idea of (Fujiwara, 1983) is to introduce a parametrix

$$E(t)f(x) = \int (4\pi i t)^{-1/2} e^{iS(t,x,y)} f(y)dy,$$

and discretize time as in the Euler's forward method. More precisely, let $\Delta_n = \{0 = t_0, t_1, ..., t = t_n\}$ be a partition where $t_j < t_{j+1}$. Define

$$E(\Delta_n, t) = E(t_n, t_{n-1})E(t_{n-1}, t_{n-2}) \cdots E(t_1, 0),$$

where $E(t,s)f(x) = \int (4\pi i|t-s|)^{-1/2} e^{iS(t,s,x,y)} f(y)dy$, and where $S(t,s,x,y)$ is the action of a unique path that starts at y at time s and ends at x at time t defined similarly as before. Then as the partition becomes finer, the sequence of parametrices converges to the unitary operator as $n \to \infty$:

$$E(\Delta_n, t)f \to U(t)f, \ \forall f \in L^2(\mathbb{R}).$$

The estimate (2.5) says that, for t small, the amplitude of the integral kernel is controlled uniformly by that of free Schrödinger propagator, which is identically one. This, along with complex interpolation with the linear estimate of $\|e^{-itH}\|_{L^1 \to L^\infty}$, gives a quick proof that e^{-itH} is bounded on $L^p(\mathbb{R})$ only if $p = 2$ for small t. Moreover the following statement is false if the spatial domain were a one-dimensional torus instead of $\mathbb{R}$ due to a lack of full dispersion on a compact domain; see the section on *remarks and extensions* of (Taylor, 2003).

Proposition 2.5.1. *Suppose e^{-itH} is bounded on $L^p(\mathbb{R})$ for $0 < |t| \leq \delta$. Then, $p = 2$.*

Remark 2.5.1. *Since the proof relies on the amplitude function k being well-defined at t, this proof does not generalise to global $t \in \mathbb{R}$; note that the Mehler kernel blows up at $t = \frac{\pi}{2}\mathbb{Z}$.*

proof of proposition 2.5.1. For $0 < |t| \leq \delta$,

$$e^{-itH} f(x) = (4\pi i t)^{-1/2} \int k(t,x,y) e^{iS(t,x,y)} f(y)dy, \ \forall f \in \mathscr{S}(\mathbb{R}).$$

By triangle inequality and estimate (2.5),

$$\begin{aligned} |e^{-itH} f(x)| &\leq (4\pi|t|)^{-1/2} \|k(t,\cdot,\cdot)\|_{L^\infty_{x,y}} \|f\|_{L^1} \\ &\leq (4\pi|t|)^{-1/2} (\|k(t,\cdot,\cdot) - 1\|_{L^\infty_{x,y}} + 1) \|f\|_{L^1} \leq c(t)\|f\|_{L^1}. \end{aligned}$$

Hence, $\|e^{-itH}\|_{L^1 \to L^\infty} \leq c(t) \simeq |t|^{-\frac{1}{2}}$ and recall that e^{-itH} is unitary on $L^2(\mathbb{R})$. By complex interpolation for $p \in [2, \infty]$, we obtain

$$\|e^{-itH}\|_{L^{p'} \to L^p} \leq c(t)^{\frac{1}{p'} - \frac{1}{p}}.$$

If e^{-itH} is L^p-bounded for $p \in (2, \infty]$, then for $f \in \mathscr{S}(\mathbb{R})$,

$$\|f\|_{L^p} = \|e^{itH}e^{-itH}f\|_{L^p} \lesssim \|e^{-itH}f\|_{L^p} \le c(t)^{\frac{1}{p'}-\frac{1}{p}}\|f\|_{L^{p'}}.$$

Hence the previous estimate implies $L^{p'}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$, a contradiction. By duality of L^p space, this proves that L^2-boundedness is the unique L^p-boundedness for e^{-itH} for a short time. $\square$

Note that if a.e. pointwise convergence does not hold for $s < \frac{1}{4}$, then it also fails for $0 \le s' \le s$. Before we state our next proposition, define $D(s)$ to be the collection of $f \in H^s(\mathbb{R})$ with a compact support such that $\overline{\lim_{t\to 0}}|U(t)f| \gtrsim 1$ uniformly on some measurable set (of positive measure) $E_f \subseteq \mathbb{R} \setminus supp(f)$. Define $D_0(s)$ similarly via $U_0(t)$. One motivation for considering functions in $D(s)$ comes from Sjölin's work on localization of Schrödinger means (see next section). If there exists $f \in D(s)$, this agrees, in a pointwise sense, with the fact that Schrödinger flow admits infinite speed of propagation. The goal is to show $D_0(s) = D(s)$ for $s < \frac{1}{4}$, or i.e., that in the short-time limit, potentials play no role in the convergence of solutions. We rule out the vacuous case when $D_0(s) = \emptyset$ for $s < \frac{1}{4}$ by citing an explicit construction given as follows:

Lemma 2.5.2 ((Sjölin, 2013))**.** *Let $s < \frac{1}{4}$. There exists $f = \sum_{n=1}^{\infty} f_n \in H^s(\mathbb{R})$ supported in $(-\frac{\delta}{4}, \frac{\delta}{4})$ for some $\delta > 0$ where f_n's are smooth and $\overline{\lim_{t\to 0}}|e^{it\partial_{xx}}f(x)| \ge c > 0$ uniformly on a measurable set $E \subseteq (\frac{\delta}{2}, \delta)$ of positive measure.*

Remark 2.5.2. *Given $K \subseteq \mathbb{R}$, a compact subset, one can modify the arguments of the lemma above to explicitly construct $f \in H^s(\mathbb{R})$ with its support in K such that $e^{it\partial_{xx}}f \to f$ as $t \to 0$ fails in a.e. sense outside of K.*

Proposition 2.5.2. *Let $V \in L^2(\mathbb{R})$ and $0 < s < \frac{1}{4}$. Then, $D_0(s) = D(s)$.*

Remark 2.5.3. *In the following proof, note that our smoothing estimate is insufficient to conclude $D(0) = D_0(0)$.*

proof of proposition 2.5.2. Writing $u(t) = e^{-itH}f$, the Duhamel formula yields

$$u(t) - f = e^{it\partial_{xx}}f - f - i\int_0^t e^{i(t-\tau)\partial_{xx}}(Vu)(\tau)d\tau.$$

We apply the smoothing estimate (lemma 2.2.2) on Vu by choosing $a = \frac{1}{2}-$ and the well-posedness result (lemma 2.2.1) to obtain that the Duhamel integral term is continuous in time and $H^{\frac{1}{2}+}$ in space, from which $D_0(s) = D(s)$ follows immediately. $\square$

We end this section with a discussion regarding the integral kernel representation of the Schrödinger operator. Let $f \in D_0(s)$. Then by definition, there exists a set of positive measure $E_f \subset \mathbb{R} \setminus supp(f)$ such that $\overline{\lim_{t\to 0}}|U_0(t)f(x)| \gtrsim 1$ uniformly on E_f. Fix $x \in E_f$. Since by triangle inequality,

$$|U(t)f(x)| \geq |U_0(t)f(x)| - |U(t)f(x) - U_0(t)f(x)|,$$

it suffices to show $|U(t)f(x) - U_0(t)f(x)| \xrightarrow[t\to 0]{} 0$ to show $f \in D(s)$; by symmetry, this would show $D_0(s) = D(s)$. The integral kernel representation gives

$$\begin{aligned}|U(t)f(x) - U_0(t)f(x)| &\leq \int_{supp(f)} |K(t,x,y) - K_0(t,x,y)| \cdot |f(y)|dy \\ &\leq \|K(t,x,y) - K_0(t,x,y)\|_{L^\infty_{y\in supp(f)}} \|f\|_{L^1_y}.\end{aligned}$$

Let $\tilde{K}(t,x,y) = (4\pi it)^{-1/2}e^{iS(t,x,y)}$. Then we obtain,

$$\begin{aligned}|K(t,x,y) - K_0(t,x,y)| &\leq |K(t,x,y) - \tilde{K}(t,x,y)| + |\tilde{K}(t,x,y) - K_0(t,x,y)| \\ &\equiv I + II,\end{aligned}$$

where

$$
\begin{aligned}
I &= |(4\pi it)^{-1/2} e^{iS(t,x,y)}(k(t,x,y)-1)| \lesssim |t|^{-\frac{1}{2}} \|k(t,x,y)-1\|_{L^\infty_{x,y}} \\
&\lesssim t^{-\frac{1}{2}+2}, \\
II &= |(4\pi it)^{-1/2} e^{iS_0(t,x,y)}(e^{it\cdot w(t,x,y)}-1)| \\
&\lesssim |t|^{\frac{1}{2}} \|w(t,x,y)\|_{L^\infty_{y\in supp(f)}}.
\end{aligned}
$$

where the estimate of I is by (2.5). The problem would be done if

$$
\|w(t,x,y)\|_{L^\infty_{y\in supp(f)}} = o(|t|^{-\frac{1}{2}})
$$

as $t \to 0$; in fact, this would allow us to include a more general class of potentials such as $V \in C^\infty$ with $|\partial^k V| \lesssim_k 1$ for $k \geq 2$. Unfortunately lemma 2.5.1 yields a uniform control on the *derivatives* of w, not w itself. However it is shown by a direct computation in (Bongioanni and Torrea, 2006, Theorem 11) that

$$
\|K(t,x,y) - K_0(t,x,y)\|_{L^\infty_{y\in supp(f)}} \xrightarrow[t\to 0]{} 0,
$$

when $V(x) = x^2$.

2.6 Negative Results: Baire Category Approach.

Motivated from the previous section, we continue to find a function f such that $e^{-itH}f$ fails to converge to the initial data a.e. where we assume $0 < s < \frac{1}{4}$ in this section. Motivated by proposition 2.5.2, we restrict this space of counterexamples to $f \in D(s)$, and therefore e^{-itH} can be replaced by $e^{it\partial_{xx}}$. We fix $J = (-1,1), \phi \in C_c^\infty(K)$ where $K \subseteq \mathbb{R} \setminus J$ is compact. It turns out that it is not an easy task to explicitly find such examples; nevertheless see (Sjölin, 2013) for an explicit construction. Another more commonly-used approach is via the Stein-Nikisin maximal principle ((Nikishin, 1972)), which states the following:

Lemma 2.6.1. *$e^{it\Delta}f \xrightarrow[t\to 0]{a.e.} f$ for all $f \in H^s(\mathbb{R}^n)$ if and only if*

$$\left\|\sup_{0<t<1} |e^{it\Delta} f|\right\|_{L^2(B(0,1))} \lesssim_{s,n} \|f\|_{H^s(\mathbb{R}^n)}, \forall f \in C_c^\infty(\mathbb{R}^n). \tag{2.6}$$

Note that the $L^2(B(0,1))$ on the LHS of (2.6) cannot be upgraded to $L^p(B(0,1))$ for $p \in (2,\infty]$ when $s < \frac{1}{4}$, $n = 1$ due to Hölder's inequality. For such p, we ask whether the f on the left-hand side of (2.6) can be replaced by $f\phi$, i.e., whether the H^s norm controls the even-more localised version of the maximal operator, or in short, the *ϕ-localised maximal operator*. It turns out that this fails for a big class of functions.

Proposition 2.6.1. *For $s < \frac{1}{4}$ and $p \in (2,\infty)$, the following strong-type estimate fails:*

$$\left\|\sup_{0<t<1} |e^{it\partial_{xx}}(f\phi)|\right\|_{L^p(J)} \lesssim \|f\|_{H^s(\mathbb{R})}. \tag{2.7}$$

Note that if (2.7) fails for p, then it fails for $\tilde{p} \geq p$. On the other hand, Sjölin raised the following interesting question: what is the minimal regularity s such that for every $f \in H^s(\mathbb{R}^n)$ with a compact support, $e^{it\Delta}f \xrightarrow[t\to 0]{} 0$ *for all* $x \in \mathbb{R}^n \setminus supp(f)$? (Sjölin, 2012) shows that this is possible if and only if $s \geq \frac{n}{2}$. Since the free Schrödinger operator is given by a convolution $e^{it\Delta}f = K_t * f$ where $K_t(x) = (4\pi i t)^{-\frac{n}{2}} e^{i\frac{|x|^2}{4t}}$, we see that $e^{it\Delta}f \in C_x^\infty(\mathbb{R}^n)$ for each $t \in \mathbb{R}\setminus\{0\}$ since $K_t \in C_x^\infty(\mathbb{R})$ and f has a compact support, and hence it makes sense to evaluate $e^{it\Delta}f$ pointwise. Sjölin shows, via a Baire category approach, that for $s < \frac{n}{2}$ there exists $f \in H^s(\mathbb{R}^n)$ with compact support in $S = \{|x| \in (1,2)\}$ such that $e^{it\Delta}f(0) \to \infty$ as $t \to 0$. Hence $\|e^{it\Delta}f\|_{L^\infty(B(0,1))} \xrightarrow[t\to 0]{} \infty$ since $e^{it\Delta}f$ is smooth. Here we are interested in the L^p-behaviour of solutions in the short-time limit. For $p \in [1,2]$, $\|e^{it\Delta} f\|_{L^p(B(0,1))}$ stays bounded due to the L^2-conservation of solutions and Hölder's inequality. For $p \in (2,\infty)$, it is unclear whether the solution blows up or stays bounded; the rate at which the width of a wavefunction shrinks can be faster than that at which its amplitude blows up as $t \to 0$. We show a weaker result that the L^p-norm of solutions diverges in some time-averaged sense:

Proposition 2.6.2. *Let $\{t_k\}_{k=1}^{\infty}$ be a real-sequence contained in $(0,1]$ that tends to zero as $k \to \infty$ and $p \in (2,\infty)$. Then there exists a dense, G_δ residual set $\mathcal{C} \subseteq H^s(\mathbb{R})$ such that for every $f \in \mathcal{C}$, we obtain $\left\{e^{it_k \partial_{xx}}(f\phi)\right\} \notin l^q L^p(\mathbb{N} \times J)$ for all $q \in [1,p]$.*[3]

Our proof is a simple application of the Banach-Steinhaus theorem. Given a sequence $\{t_k\}_{k=1}^{\infty} \to 0$, define $S_n f = \sup_{k \le n} |e^{it_k \partial_{xx}}(f\phi)|$ and the maximal operator $Sf = \sup_k |e^{it_k \partial_{xx}}(f\phi)|$ for $f \in \mathscr{S}(\mathbb{R})$. It is straightforward to verify

$$S_n(f+g) \le S_n f + S_n g;\ S_n(\lambda f) = |\lambda| S_n f, \forall \lambda \in \mathbb{C},$$

pointwise on $\mathbb{R}$. There are many versions of Banach-Steinhaus theorem that studies equicontinuity of a family of linear operators, but since we will be interested in sublinear operators, we shall give a proof of the following statement at the end in the spirit of (Rudin, 2006, Theorem 5.8):

Lemma 2.6.2. *Let $\{T_\alpha\}_{\alpha \in A}$ be a family of continuous sublinear operators on X into $Z = L^p(Y,\nu)$ for $p \in [1,\infty]$ where X is a Banach space, (Y,ν) is a σ-finite measure space and A is some directed set.*[4] *Suppose for all $x, y \in X$ and $\alpha \in A$:*

$$\|T_\alpha(x+y)\| \le \|T_\alpha x\| + \|T_\alpha y\|;\ \|T_\alpha(\lambda x)\| = \lambda \|T_\alpha x\|,\ \forall \lambda \ge 0. \tag{2.8}$$

[3] Recall that a measurable set is G_δ if it can be realized as a countable intersection of open sets. A set is *meager* if it can be realised as a countable union of nowhere dense sets, and its complement is called a *residual*.

[4] By sublinear operator, we mean $|T_\alpha(x+y)| \le |T_\alpha x| + |T_\alpha y|$ and $|T_\alpha(\lambda x)| = |\lambda||T_\alpha x|, \forall x, y \in X, \lambda \in \mathbb{C}$ in ν-a.e. sense. In our application, $L^p(\nu)$ is a collection of real-valued functions, but this construction works for complex-valued functions as well.

Then either $\lim_{x\to 0}\|T_\alpha x\| = 0$ *uniformly in* α, *i.e.,* $\{T_\alpha\}$ *is equicontinuous at the origin, or*
$\{x \in X : T_\alpha x$ *is unbounded in* $Z\}$ *forms a residual set that is dense* G_δ *in* X.

proof of proposition 2.6.1 and 2.6.2. We first claim that $\{S_n\}$ defines a family of continuous sublinear operators on $H^s(\mathbb{R})$ into $L^p(J)$ that satisfies the hypotheses of lemma 2.6.2. By the triangle inequality, one can show

$$|S_n f - S_n g| \leq S_n(f-g).$$

Hence it suffices to show that S_n is a bounded map to show continuity. By the continuous embedding $l^p(\mathbb{N}) \hookrightarrow l^\infty(\mathbb{N})$ and the dispersive inequality (1.7), we obtain

$$\begin{aligned} &\|S_n f\|_{L^p(J)} = \|e^{it_k\partial_{xx}}(f\phi)\|_{L^p(J)l^\infty_{k\leq n}} \leq \|e^{it_k\partial_{xx}}(f\phi)\|_{l^p_{k\leq n}L^p(J)} \qquad (2.9)\\ \leq &\left|\left|\|e^{it_k\partial_{xx}}(f\phi)\|_{L^p(\mathbb{R})}\right|\right|_{l^p_{k\leq n}} \lesssim \left|\left||t_k|^{-(\frac{1}{2}-\frac{1}{p})}\|f\phi\|_{L^{p'}(\mathbb{R})}\right|\right|_{l^p_{k\leq n}} \simeq_{p,n} \|f\phi\|_{L^{p'}(K)} \\ \lesssim &\|f\phi\|_{L^2(\mathbb{R})} \lesssim_{p,\phi,n,K} \|f\|_{H^s(\mathbb{R})}, \end{aligned}$$

and hence the continuity of S_n. From sublinearity and triangle inequality, one can easily verify (2.8). We claim, by contradiction, that the $\{S_n\}$ cannot be equicontinuous at the origin. Assume it is. Then we claim S is continuous in measure at the origin. Suppose $f_j \to 0$ in $H^s(\mathbb{R})$ as $j \to \infty$ and let $\lambda > 0$. Let $\epsilon > 0$ for which there exists $\delta > 0$ such that $\|S_n f\|_{L^p(J)} < \epsilon^{\frac{1}{p}}\lambda$ for all but finitely many $n \in \mathbb{N}$ and all f such that $\|f\|_{H^s} < \delta$. Then let $j \geq N$, some $N \in \mathbb{N}$ sufficiently big, such that $\|f_j\|_{H^s} < \delta$ for all $j \geq N$, and let n big enough such that $\left|\{|Sf_j - S_n f_j| \geq \frac{\lambda}{2}\}\right| \lesssim \epsilon$; recall that $S_n f_j \xrightarrow{a.e.} Sf_j$ implies $S_n f_j \to Sf_j$ in measure on a finite measure space as $n \to \infty$. Then we obtain

$$|\{|Sf_j| \geq \lambda\}| \leq \left|\left\{|Sf_j - S_n f_j| \geq \frac{\lambda}{2}\right\}\right| + \left|\left\{|S_n f_j| \geq \frac{\lambda}{2}\right\}\right| \lesssim \epsilon,$$

where the second term is bounded above by ϵ up to a constant by Chebyshev's inequality.

Now we show that convergence a.e. to initial data holds for all $f \in H^s(\mathbb{R})$ with a compact support, which is a contradiction since $s < \frac{1}{4}$ and due to the explicit construction of an initial data with a compact support in (Sjölin, 2013). Pick $f_n \xrightarrow[n\to\infty]{}$

$f \in H^s(\mathbb{R})$ where $f_n \in \mathscr{S}(\mathbb{R})$. Then we obtain

$$\begin{aligned}\left|\left\{x \in J : \overline{\lim_{k\to\infty}} |e^{it_k\partial_{xx}}(f\phi)| > \lambda\right\}\right| &= \left|\left\{x \in J : \overline{\lim_{k\to\infty}} |e^{it_k\partial_{xx}}((f-f_n)\phi)| > \lambda\right\}\right| \\ &\leq |\{x \in J : S(f-f_n) > \lambda\}| \xrightarrow[n\to 0]{} 0,\end{aligned}$$

for all $\lambda > 0$ where the last limit follows from the continuity in measure of S. Hence the supposed equicontinuity fails and there exists a dense G_δ set $\mathcal{C} \subseteq H^s(\mathbb{R})$ such that if $f \in \mathcal{C}$, then $\{S_n f\}$ is unbounded in $L^p(J)$. By monotonicity, $\|S_n f\|_{L^p(J)} \leq \|Sf\|_{L^p(J)} \leq \left\|\sup_{0<t<1} |e^{it\partial_{xx}}(f\phi)|\right\|_{L^p(J)}$, and therefore, (2.7) cannot hold for every $f \in \mathcal{C}$. By the right-most estimate in (2.9), we obtain that $\left\{e^{it_k\partial_{xx}}(f\phi)\right\} \notin l^p L^p(\mathbb{N} \times J)$ for all $f \in \mathcal{C}$. □

proof of lemma 2.6.2. Let $E_n = \left\{x \in X : \sup_\alpha \|T_\alpha x\| \leq n\right\}$ for $n \in \mathbb{N}$. By continuity of T_α, E_n's are closed. Note that $\bigcup_n E_n = \{x \in X : T_\alpha x \text{ is bounded in Z}\}$. There are two cases. First, assume that not all E_n's are nowhere dense. Then, there exists $n_0 \in \mathbb{N}$ such that E_{n_0} contains a closed ball $\overline{B(x_0, r_0)}$. Then, we claim $\overline{B(0, r_0)} \subseteq E_{2n_0}$. Let $\|x\| \leq r_0$. Then, $x + x_0 \in \overline{B(x_0, r_0)}$, and therefore

$$\|T_\alpha x\| = \|T_\alpha(x + x_0 - x_0)\| \leq \|T_\alpha(x + x_0)\| + \|T_\alpha x_0\| \leq 2n_0.$$

Hence given $\epsilon > 0$, choose $\lambda > 0$ sufficiently big such that $\frac{2n_0}{\lambda} < \epsilon$. Then, choose $\delta > 0$ such that $\delta < \frac{r_0}{\lambda}$. Then for all $\|x\| < \delta$,

$$\|T_\alpha x\| = \|T_\alpha(\lambda^{-1}\lambda x)\| = \lambda^{-1}\|T_\alpha(\lambda x)\| \leq \frac{2n_0}{\lambda} < \epsilon.$$

On the other hand, assume all E_n's are nowhere dense. Then, $\bigcap_n E_n^c = \{x \in X : T_\alpha x \text{ is unbounded in Z}\}$ is dense G_δ by Baire Category theorem. Lastly since $\bigcup_n E_n$ is meager, $\{x \in X : T_\alpha x \text{ is unbounded in Z}\}$ is a residual. □

Chapter 3

Adiabatic Limit of Quantum Zakharov System

3.1 Introduction.

In this chapter, we study well-posedness as well as ill-posedness of the adiabatic limit of the quantum Zakharov system and its nonlinear Schrödinger limit as the quantum parameter tends to zero:

$$\begin{cases} i\partial_t u + \Delta u - \epsilon^2 \Delta^2 u = -(I - \epsilon^2 \Delta)^{-1}(|u|^2)u, \ (x,t) \in \mathbb{R}^d \times \mathbb{R} \\ u(0) = u_0 \in H^s(\mathbb{R}^d). \end{cases} \tag{3.1}$$

We are motivated by the classical Zakharov system (Zakharov et al., 1972):

$$\begin{cases} i\partial_t E + \Delta E = nE, \ (x,t) \in \mathbb{R}^d \times \mathbb{R} \\ \frac{1}{\lambda^2}\partial_{tt} n - \Delta n = \Delta(|E|^2), \\ E(0) = E_0, \ n(0) = n_0, \ \partial_t n(0) = n_1, \end{cases} \tag{3.2}$$

which describes the propagation of Langmuir waves in an ionized plasma. Here a complex-valued $E(x,t)$ describes a slowly-varying envelope of a rapidly oscillating electric field, and a real-valued $n(x,t)$ describes the deviation of the ion density from its mean. Classical solutions satisfy the conservation of mass and energy as follows:

$$\mathrm{MASS} = \|E(t)\|_{L^2}^2 = \text{constant}$$
$$\mathrm{ENERGY} = \|\nabla E(t)\|_{L^2}^2 + \frac{\|n(t)\|_{L^2}^2}{2} + \frac{\|\partial_t n\|_{\dot{H}^{-1}}}{2\lambda^2} + \int n(t)|E(t)|^2 = \text{constant}.$$

The ion acoustic speed is proportional to $\lambda > 0$, and the *adiabatic limit* $\lambda \to \infty$ was studied by Schochet-Weinstein (Zakharov et al., 1972). In that regime, the second time-derivative term becomes negligible, at least formally, and under the assumption that $n + |E|^2$ vanishes at the infinity, the limiting function E satisfies the cubic (focusing) nonlinear Schrödinger equation (NLS):

$$i\partial_t E + \Delta E = -|E|^2 E. \tag{3.3}$$

Physically the adiabatic limit corresponds to the assumption that the fluctuation in electric fields instantaneously affects that of plasma. Under some hypotheses, they proved (Schochet and Weinstein, 1986) that (3.2) is locally well-posed in some time interval $[0, T]$ (with T independent of λ), and also identified rigorously the nonlinear Schrödinger limit, both in the strong and weak sense, as $\lambda \to \infty$. Later Ozawa-Tsutsumi found the optimal convergence rate for this nonlinear Schrödinger limit (Ozawa et al., 1992).

On the other hand, the following more detailed model accounts for the *quantum effects* on the nonlinear interaction between Langmuir waves and ion-acoustic waves:

$$\begin{cases} i\partial_t E + \Delta E - \epsilon^2 \Delta^2 E = nE, \ (x,t) \in \mathbb{R}^d \times \mathbb{R} \\ \frac{1}{\lambda^2}\partial_{tt} n - \Delta n + \epsilon^2 \Delta^2 n = \Delta(|E|^2), \\ E(0) = E_0, \ n(0) = n_0, \ \partial_t n(0) = n_1, \end{cases} \tag{3.4}$$

where $\epsilon = \frac{\hbar w_i}{k_B T_e} > 0$; $\hbar$ is the Planck's constant divided by 2π, w_i, ion plasma frequency, k_B, Boltzmann's constant, and T_e, the electrons' temperature. As before, the classical solutions of (3.4) satisfy the mass and energy conservation as follows:

$$\begin{aligned} \text{MASS} &= \|E(t)\|_{L^2}^2 = \text{constant} \\ \text{ENERGY} &= \epsilon^2 \|\Delta E(t)\|_{L^2}^2 + \|\nabla E(t)\|_{L^2}^2 \\ &\quad + \frac{\epsilon^2}{2}\|\nabla n(t)\|_{L^2}^2 + \frac{\|n(t)\|_{L^2}^2}{2} + \frac{\|\partial_t n\|_{\dot{H}^{-1}}}{2\lambda^2} + \int n(t)|E(t)|^2 = \text{constant}; \end{aligned}$$

for the variational treatment on conservation laws, see (Jiang et al., 2014). The consideration of ϵ would for example be experimentally relevant in the case of dense and cold plasmas, which occur in astrophysical scenarios; see (Garcia et al., 2005; Haas, 2011; Haas and Shukla, 2009) for more physical background.

We formally take $\lambda \to \infty$ in (3.4), assuming $n + (I - \epsilon^2\Delta)^{-1}|E|^2 \xrightarrow[|x|\to\infty]{} 0$. We re-label the quantity of interest from E to u, thereby obtaining (3.1). This can be thought of as a modified NLS (mNLS). More precisely, the extra biharmonic term is expected to play a significant role for a long-time behaviour of solutions. On the other hand, the linear operator $J_\epsilon := (I - \epsilon^2\Delta)^{-1}$ is a non-local Fourier multiplier that converges to the identity strongly on $L^2(\mathbb{R}^d)$ as $\epsilon \to 0$. By now it is a folklore in the community of nonlinear equations that a PDE posed at a subcritical regime is well-posed. Indeed the equation above is locally well-posed in $L^2(\mathbb{R})$ and moreover globally well-posed in $H^s(\mathbb{R})$ for $s \geq 2$, thanks to mass/energy conservation, Strichartz estimates and the Gagliardo-Nirenberg inequality (see (Fang et al., 2016, Proposition 2.5)). Nevertheless, we have not been able to find a global result regarding low regularity well-posedness in the literature- this is what we aim to achieve here. We briefly review the well-posedness theory of (3.2) and (3.4).

The study of well-posedness theory of (3.2) by now is a mature subject - we give a list of (some) references in chronological order for completeness: (Ozawa and Tsutsumi, 1992; Kenig et al., 1995; Bourgain and Colliander, 1996; Ginibre et al., 1997; Colliander et al., 2008; Bejenaru et al., 2009; Bejenaru and Herr, 2011). On the other hand, a lot of rigorous study of (3.4), has taken place since 2010. Guo-Zhang-Guo proved that (3.4) is globally well-posed for data $(E_0, n_0, n_1) \in H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) \times H^{k-3}(\mathbb{R}^d)$ where $k \geq 2$ and $d = 1, 2, 3$, and that the classical limit holds as the quantum parameter tends to zero (Guo et al., 2013a). Jiang-Lin-Shao further obtained well-posedness results at lower regularities by carefully estimating the non-linear interactions of waves of different frequencies in the presence of the biharmonic operator (Jiang et al., 2014). Fang-Lin-Segata proved that solutions of (3.4) converge to those of (3.2) at an optimal rate as the wave speed approaches infinity (Fang et al., 2016). Chen-Fang-Wang proved global well-posedness of (3.4) in $1D$ when the electric field component is only assumed to be square-integrable; this at least formally generalises the classical result of Colliander-Holmer-Tzirakis with the quantum parameter tending to zero (Chen et al., 2017; Colliander et al., 2008). Fang-Shih-Wang further contributed in obtaining low-regularity well-posedness results, which again formally recovered another classical result by Gibibre-Tsutsumi-Velo (Ginibre et al., 1997). Finally Fang-Kuo-Shih-Wang generalized the result of (Guo et al., 2013a) by obtaining a refined version of the semi-classical limit, with a convergence rate (Fang et al., 2019).

A difficulty in obtaining global existence results for mNLS occurs in $H^s(\mathbb{R})$ when $s \in (0, \frac{1}{2}]$; for $s = 0$, the global result is immediate due to mass conservation. For $s > \frac{1}{2}$, one can exploit the fact that $H^s(\mathbb{R})$ defines an algebra to obtain an exponential bound on the growth of Sobolev norm, which prevents finite time blow-up. To overcome this obstacle, we adopt the method of Fourier restricted norms, first initiated by Bourgain (Bourgain, 1993a). This method was also used to obtain well-posedness of KdV on $\mathbb{R}$ for $s > -\frac{3}{4}$ by Kenig-Ponce-Vega (Kenig et al., 1996b), and a similar bilinear estimate was obtained in a negative Sobolev space for $1D$ NLS with quadratic nonlinearities (Kenig et al., 1996a). The goal is to obtain a sufficient degree of smoothing, measured in Fourier restriction norm, for non-local nonlinearities. Though the argument follows closely those of the references above, a careful analysis on the quartic dispersion relation introduced by Δ^2 is needed. To this end, we state our main result:

Theorem 3.1.1. *The modified NLS (3.1) is globally well-posed in $H^s(\mathbb{R})$ for every $s \geq 0$. Furthermore, the data-to-solution map is Lipschitz in any closed ball in $H^s(\mathbb{R})$.*

Then we consider the semi-classical limit of (3.1). The smoothing estimate obtained above is to no avail since the implicit constant is not uniform with respect to $\epsilon > 0$ as $\epsilon \to 0$; for the same reason, Strichartz estimates are not a useful tool in this context. However a direct estimate on the Duhamel formula yields the desired result if we are in the Sobolev algebra regime, i.e., $s > \frac{1}{2}$. We are able to obtain the desired convergence result locally in time, but we are only able to obtain a partial convergence result when the solution flow is not restricted to a finite-time interval. This long-time asymptotics case is much more difficult and interesting since it is known that the solutions to (3.3) and (3.1) do not scatter. For the topic of soliton existence for NLS, see the appendix of (Tao, 2006), and for the existence and stability of solitions for mNLS in $d = 1, 2, 3$, see (Fang et al., 2018). By adopting the variational method used in (Zhao et al., 2015), we extend the result of (Fang et al., 2018) by showing that solitons exist in higher dimensions as well. As for the first part of the next statement, it is of interest to ask whether the Sobolev algebra assumption can be removed.

Theorem 3.1.2. *Let $s > \frac{1}{2}$ and $\left\{u_0^{(\epsilon)}\right\}_{\epsilon>0} \subseteq B_{H^s(\mathbb{R})}(0, R)$ for some $R > 0$ where $u_0^{(\epsilon)} \xrightarrow[\epsilon\to 0]{H^s} u_0$, and let $u^{(\epsilon)}$ and u be the global solutions corresponding to $u_0^{(\epsilon)}$ and u_0, respectively, given by theorem 3.1.1. Then for every $T \in (0, \infty)$, $u^{(\epsilon)} \xrightarrow[\epsilon\to 0]{} u$ in $C([0, T], H^s(\mathbb{R}))$. Furthermore let $U(t) = e^{it\Delta}$ and $U_\epsilon(t) = e^{it(\Delta-\epsilon^2\Delta^2)}$ be the solution maps for the linearized equations of (3.3) and (3.1), respectively. Then for every*

$s \in \mathbb{R}$, there exists some $u_0 \in H^s(\mathbb{R})$ such that $U_\epsilon(t)u_0 \nrightarrow U(t)u_0$ in $C([0,\infty), H^s(\mathbb{R}))$ as $\epsilon \to 0$.

We also study the ill-posedness of (3.1) in negative Sobolev spaces. Due to Galilean symmetry, the solution map for (3.3) fails to be uniformly continuous for $s \in (-\infty, 0)$ and even worse, the map exhibits norm inflation[1] for $s \leq -\frac{1}{2}$; see (Kenig et al., 2001; Christ et al., 2003a; Kishimoto, 2018). However it is unknown whether there is Galilean invariance in the presence of biharmonic operator, and therefore a direct proof of the failure of uniform well-posedness is unavailable. Instead we show that $u^{(\epsilon)} - u$ stays small for a short time. More precisely, we prove the following:

Theorem 3.1.3. *Let $s < 0$ and $R, T > 0$. Then there exists $\epsilon_0(s, R, T) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, if the solution $u^{(\epsilon)}$ has a well-defined data-to-solution map from $B_{H^s(\mathbb{R})}(0, R)$ to $C([0,T], H^s(\mathbb{R}))$, then such map fails to be uniformly continuous (with respect to the Banach space topology and the topology of uniform convergence, respectively).*

In particular, the contraction argument cannot yield well-posedness of (3.1) in negative Sobolev spaces. Here, it is of interest to investigate whether the smoothing operator J_ϵ adds any regularity to the data-to-solution map. When $s \leq -\frac{1}{2}$, for instance, do we expect (3.1) to exhibit the norm inflation as in its formal limit, i.e., the (focusing) cubic NLS? We leave these questions unanswered for now.

We briefly outline how this paper is organized. In section 2, we prove theorem 3.1.1 by obtaining a smoothing estimate and a polynomial bound on the growth of Sobolev norm of the solutions. In section 3, we consider the semi-classical limit. Using an estimate obtained in this section, we obtain an ill-posedness result in negative Sobolev spaces. In section 4, we apply the variational method to obtain soliton solutions in higher dimensions.

[1] We say that *norm inflation* occurs in $H^s(\mathbb{R}^d)$ if for every $\delta > 0$, there exists an initial datum $f \in B_{H^s}(0, \delta)$ and $0 < T < \delta$ such that the solution corresponding to f exists on $[0, T]$ and $\|u[f](T)\|_{H^s} > \delta^{-1}$.

3.2 Low Regularity Well-posedness.

The quartic dispersion relation plays an important role in our analysis. We define $d_\epsilon(\xi) \coloneqq \xi^2 + \epsilon^2\xi^4$ for $\epsilon \geq 0$. To run a fixed point argument, we work on the Fourier restriction space $X^{s,b}_{\pm(\epsilon)}$ (also known as dispersive Sobolev space) where $s, b \in \mathbb{R}, \epsilon \geq 0$ and $u \in X^{s,b}_{\pm(\epsilon)}$ if

$$\|u\|_{X^{s,b}_{\pm(\epsilon)}} \coloneqq \|\langle\xi\rangle^s\langle\tau \pm d_\epsilon(\xi)\rangle^b\hat{u}(\xi,\tau)\|_{L^2_{\xi\tau}} < \infty.$$

In practice, we show that the operator $u \mapsto U_\epsilon(t)u_0 + i\int_0^t U_\epsilon(t-\tau)[J_\epsilon(|u|^2)u(\tau)]d\tau$ defines a contraction on some Fourier restriction space, provided that we obtain appropriate nonlinear estimates on $N(u) \coloneqq J_\epsilon(|u|^2)u$. We state an elementary calculus fact that are used repeatedly:

Lemma 3.2.1. *Let $A > 0$ and $0 < a < 1$. Then, $\int_A^\infty \frac{dz}{z(z-A)^a} \lesssim_a A^{-a}$.*

proof of Lemma 3.2.1. As for the second statement, let $t = z - A$. Then

$$\int_A^\infty \frac{dz}{z(z-A)^a} = \int_0^\infty \frac{dt}{(t+A)t^a} = \int_0^A \frac{dt}{(t+A)t^a} + \int_A^\infty \frac{dt}{(t+A)t^a} \leq (\frac{1}{1-a} + \frac{1}{a})A^{-a}.$$

□

Lemma 3.2.2. *Let $s \in \mathbb{R}, \epsilon > 0$ and $X = H^s(\mathbb{R}^d)$ or $L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$. Then $\|J_\epsilon f\|_X \leq \|f\|_X$ for all $f \in X$. Moreover $J_\epsilon : H^s(\mathbb{R}^d) \to H^{s+2}(\mathbb{R}^d)$ is bounded with the best constant $\leq \epsilon^{-2}$.*

proof of Lemma 3.2.2. See section 1.5 of (Cazenave, 2003). □

The majority of this section is devoted to obtaining the following estimate:

Proposition 3.2.1. *Let $s \in [0,\infty), \gamma \in [\frac{1}{3},\frac{1}{2}), \delta \in (0,1]$ and $a \in [0,\frac{4}{3})$. Then for every $\epsilon > 0$ and $b \in (\frac{1}{2}, 1-\gamma)$, we have*

$$\|(J_\epsilon(u\overline{v})w\|_{X^{s+a,-\gamma}_{(\epsilon),\delta}} \lesssim_{s,a,\gamma,b,\epsilon} \|u\|_{X^{s,b}_{(\epsilon),\delta}}\|v\|_{X^{s,b}_{(\epsilon),\delta}}\|w\|_{X^{s,b}_{(\epsilon),\delta}}.$$

We note that the implicit constant is not bounded as $\epsilon \to 0$. By letting $u = v = w$, the proposition above immediately yields:

Corollary 3.2.1. *Let $s, \gamma, \delta, a, \epsilon$ and b be as above. Then* $\|N(u)\|_{X^{s+a,-\gamma}_{(\epsilon),\delta}} \lesssim \|u\|^3_{X^{s,b}_{(\epsilon),\delta}}$.

A straightforward application of fixed point argument yields:

Corollary 3.2.2. *The mNLS is locally well-posed (in subcritical sense) in $H^s(\mathbb{R})$ for every $s \geq 0$. Furthermore the data-to-solution map is Lipschitz in any closed ball in $H^s(\mathbb{R})$.*

Remark 3.2.1. *Local well-posedness can be established by an application of Strichartz estimates, which is by now well understood. Recall that a biharmonic admissible (B-admissible) pair is a pair (q, r) such that*

$$q, r \in [2, \infty],\ (q, r, d) \neq (2, \infty, 4) \text{ and } \frac{4}{q} + \frac{d}{r} = \frac{d}{2}.$$

If (q, r) and $(\tilde{q}, \tilde{r})$ are B-admissible pairs and $u \in C([0, T], H^{-4}(\mathbb{R}^d))$ is a solution to

$$\begin{cases} i\partial_t u + \Delta u - \epsilon^2 \Delta^2 u = F,\ (x, t) \in \mathbb{R}^d \times \mathbb{R} \\ u(0) = u_0 \in L^2(\mathbb{R}^d), \end{cases}$$

then there exists a constant $C = C(q, r, \tilde{q}, \tilde{r}, d) > 0$ such that

$$\|u\|_{L^q([0,T], L^r(\mathbb{R}^d))} \leq C(\epsilon^{-\frac{2}{q}} \|u_0\|_{L^2} + \epsilon^{-2(\frac{1}{q} + \frac{1}{\tilde{q}})} \|F\|_{L^{\tilde{q}'}([0,T], L^{\tilde{r}'}(\mathbb{R}^d))}).$$

Unfortunately the Strichartz estimates do not directly yield the global existence result for infinite energy data. Moreover the estimate blows up as $\epsilon \to 0$, and therefore this is an insufficient tool to study the semi-classical limit. For references on B-admissible Strichartz estimate, see (Fang et al., 2019; Ben-Artzi et al., 2000).

proof of corollary 3.2.2. Let $s, \gamma, \delta, a, \epsilon$ and b be as above. For $g \in H^s(\mathbb{R})$, define $X = \left\{ u \in X^{s,b}_{(\epsilon),\delta} : \|u\|_{X^{s,b}_{(\epsilon),\delta}} \leq 2C\|g\|_{H^s} \right\}$ where C is a fixed implicit constant that satisfies $\|U_\epsilon(t)\tilde{g}\|_{X^{s,b}_{(\epsilon),\delta}} \lesssim \|\tilde{g}\|_{H^s}$ for all $\tilde{g} \in H^s(\mathbb{R})$. Define $\Gamma u = U_\epsilon(t)g + i\int_0^t U_\epsilon(t-\tau)N(u)(\tau)d\tau$.

Then,

$$\begin{aligned}\|\Gamma u\|_{X^{s,b}_{(\epsilon),\delta}} &\lesssim \|g\|_{H^s} + \|N(u)\|_{X^{s,b-1}_{(\epsilon),\delta}} \lesssim \|g\|_{H^s} + \delta^{1-(b+\gamma)}\|N(u)\|_{X^{s,-\gamma}_{(\epsilon),\delta}} \\ &\lesssim \|g\|_{H^s} + \delta^{1-(b+\gamma)}\|u\|^3_{X^{s,b}_{(\epsilon),\delta}} \lesssim \|g\|_{H^s} + \delta^{1-(b+\gamma)}\|g\|^3_{H^s} \\ \Rightarrow \|\Gamma u\|_{X^{s,b}_{(\epsilon),\delta}} &\leq C\|g\|_{H^s} + C_1\delta^{1-(b+\gamma)}\|g\|^3_{H^s}.\end{aligned}$$

Let $\delta = \delta(\|g\|_{H^s}) > 0$ such that $\delta < \frac{C}{C_1}\|g\|_{H^s}^{-\frac{2}{1-(b+\gamma)}}$. Then, $\Gamma : X \to X$. Now we show that Γ is a contraction on X, and hence there exists a unique fixed point, the desired strong solution. Note that uniqueness in X implies that in $X^{s,b}_{(\epsilon),\delta}$ by shrinking δ if necessary.

First, note the following algebraic manipulation of the nonlinearity:

$$N(u) - N(v) = J_\epsilon(|u|^2)(u-v) + J_\epsilon(u\cdot(\overline{u-v}))v + J_\epsilon((u-v)\overline{v})v.$$

Let $u, v \in X$. Then,

$$\begin{aligned}\|\Gamma u - \Gamma v\|_{X^{s,b}_{(\epsilon),\delta}} &= \left\| \int_0^t U_\epsilon(t-\tau)(N(u)-N(v))(\tau)d\tau \right\| \lesssim \|N(u)-N(v)\|_{X^{s,b-1}_{(\epsilon),\delta}} \\ &\lesssim \delta^{1-(b+\gamma)}\|N(u)-N(v)\|_{X^{s,-\gamma}_{(\epsilon),\delta}} \lesssim \delta^{1-(b+\gamma)}\|g\|^2_{H^s}\|u-v\|_{X^{s,b}_{(\epsilon),\delta}},\end{aligned}$$

and therefore by shrinking δ if necessary, Γ is a contraction on X.

To show continuous dependence on initial data, let $g, g_n \in H^s(\mathbb{R})$ such that $g_n \xrightarrow[n\to\infty]{} g$ and let u, u_n the corresponding solutions, respectively. Let u be defined on $[-\delta,\delta]$ and fix $0 < T < \delta$. Then there exists a sufficiently large $N \geq 1$ such that u_n is well-defined on $[-T,T]$ for all $n \geq N$. Arguing as above,

$$\begin{aligned}\|u-u_n\|_{X^{s,b}_{(\epsilon),T}} &\lesssim \|g-g_n\|_{H^s} + T^{1-(b+\gamma)}\|N(u)-N(u_n)\|_{X^{s,-\gamma}_{(\epsilon),T}} \\ &\lesssim \|g-g_n\|_{H^s} + T^{1-(b+\gamma)}\|g\|_{H^s}\cdot\|g_n\|_{H^s}\cdot\|g-g_n\|_{H^s},\end{aligned}$$

which proves the claim as $n \to \infty$. The Lipschitz regularity of data-to-solution map is proved similarly. $\square$

Similar to the Zakharov system, (3.1) admits the conservation of L^2-norm and the energy defined as follows:

$$ENERGY[u(t)] = \frac{\epsilon^2}{2}\|\partial_{xx}u\|_{L^2}^2 + \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{4}\|J_\epsilon(|u|^2)|u|^2\|_{L^1}.$$

Hence the solutions for mNLS are globally well-posed in $L^2(\mathbb{R})$. On the other hand, by the Gagliardo-Nirenberg inequality, one can show that smooth solutions in $H^2(\mathbb{R})$ are bounded for all times, from which global wellposedness in $H^2(\mathbb{R})$ is deduced. For $s \geq 2$, one can further show that the Sobolev norm of solutions grow at most exponentially in time by applying Gronwall's inequality to the Duhamel integral formula. In the following proposition, we show that the Sobolev norm of solutions grow at most polynomially in time for all $s \geq 0$ as an application of proposition 3.2.1. We note, however, that our growth rate is not sharp.

Proposition 3.2.2. *For all $s \geq 0$ and $\epsilon > 0$, let $g \in H^s(\mathbb{R})$ and $u^{(\epsilon)}$ be the corresponding strong local solution as in Corollary 3.2.2. Then there exists a non-decreasing function $C_s : [0,\infty) \to [0,\infty)$ such that $\|u^{(\epsilon)}(t)\|_{H^s} \leq C_s(\|g\|_{H^s})\langle t\rangle^{\frac{1}{2}\left(3^{\lfloor \frac{3s}{4} \rfloor + 1} - 1\right)}$ for all $t \in \mathbb{R}$.*

proof of theorem 3.1.1. From corollary 3.2.2, the mNLS is locally-wellposed in subcritical sense. Hence if $[0,T]$ is the maximum interval of existence for $u^{(\epsilon)}[g]$ for $T < \infty$, then $\overline{\lim_{t\to T-}}\|u^{(\epsilon)}[g](t)\|_{H^s} = \infty$, which contradicts proposition 3.2.2. Hence $T = \infty$. $\square$

To prove proposition 3.2.1, we need a technical lemma, whose proof is contained in the appendix, that gives a lower and upper bound to the unique negative root $r(\xi_1)$ of a cubic polynomial in ξ_2:

$$P(\xi_2) \coloneqq 4\epsilon^2\xi_2^3 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2 + 2)\xi_2 + |(1 + \epsilon^2(\xi_1 - \xi)^2)(\xi_1 - \xi)^2 + \tau|.$$

Lemma 3.2.3. *Let $\xi > 1, \tau \in (\frac{d_\epsilon(\xi)}{2}, 2d_\epsilon(\xi))$ and let $r(\xi_1)$ denote the unique negative root of P. Then,*

1. $r(\xi_1) = -\frac{|\xi_1|^{1/3}(\epsilon^2\xi_1^2+2)^{1/2}}{\sqrt{3}\epsilon}\sinh\left(\frac{1}{3}\sinh^{-1}\left(3\sqrt{3}\epsilon\frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}}\right)\right).$

2. $|r(\xi_1)| \gtrsim_\epsilon |\xi|^{\frac{4}{3}}$ *for all* $\xi_1 \in (-\infty, \infty)$ *where the implicit constant does not depend on* ξ.

3. $|r(\xi_1)| \lesssim_\epsilon \begin{cases} d_\epsilon(\xi)^{1/3}, & \textit{if } |\xi_1| \in [0, \frac{\xi}{2}] \\ |\xi_1|^{1/3}\langle\epsilon\xi_1\rangle, & \textit{otherwise.} \end{cases}$

Remark 3.2.2. *The main idea of the next proof adopts that of (Kenig et al., 1996b, Lemma 2.4) where the analysis of KdV (cubic dispersion) leads to an algebraic manipulation of a second-order polynomial; on the other hand, our model equation (quartic dispersion) demands an appropriate bound on a third-order polynomial.*

proof of proposition 3.2.1. We can neglect the δ-dependence in our proof as the following shows. Suppose $\|N(u)\|_{X^{s+a,-\gamma}_{(\epsilon)}} \lesssim \|u\|^3_{X^{s,b}_{(\epsilon)}}$ for all $u \in X^{s,b}_{(\epsilon)}$. Fix $u \in X^{s,b}_{(\epsilon)}$ and let $\tilde{u} \in X^{s,b}_{(\epsilon)}$ such that $u = \tilde{u}$ on $t \in [0, \delta]$. Then,

$$\|N(u)\|_{X^{s+a,-\gamma}_{(\epsilon),\delta}} \leq \|\eta(t/\delta)N(\tilde{u})\|_{X^{s+a,-\gamma}_{(\epsilon)}} \lesssim_\eta \|N(\tilde{u})\|_{X^{s+a,-\gamma}_{(\epsilon)}} \lesssim \|\tilde{u}\|^3_{X^{s,b}_{(\epsilon)}}.$$

Noting that the implicit constant is independent of δ, we take infimum over all possible $\tilde{u}$, thereby obtaining the desired result. Henceforth we ignore the δ-dependence and prove the estimate by duality. Let $u, v, w \in X^{s,b}_{(\epsilon)}$ and $\phi \in (X^{s+a,-\gamma}_{(\epsilon)})^*$. Define

$$f(\xi,\tau) = |\hat{u}(\xi,\tau)|\langle\xi\rangle^s\langle\tau + d_\epsilon(\xi)\rangle^b,\ g(\xi,\tau) = |\hat{v}(\xi,\tau)|\langle\xi\rangle^s\langle\tau + d_\epsilon(\xi)\rangle^b;$$
$$h(\xi,\tau) = |\hat{w}(\xi,\tau)|\langle\xi\rangle^s\langle\tau + d_\epsilon(\xi)\rangle^b,\ \psi(\xi,\tau) = |\hat{\phi}(\xi,\tau)|\langle\xi\rangle^{-(s+a)}\langle\tau - d_\epsilon(\xi)\rangle^\gamma.$$

By the Plancherel's theorem, we have

$$|\langle(I - \epsilon^2\partial_{xx})^{-1}(u\overline{v})w, \phi\rangle_{L^2_{xt}}| = |\langle\mathcal{F}[((I - \epsilon^2\partial_{xx})^{-1}(u\overline{v}))w], \mathcal{F}[\phi]\rangle_{L^2_{\xi\tau}}|,$$

followed by the triangle inequality, which yields

$$\leq \int W \cdot f(\xi - \xi_1 - \xi_2, \tau - \tau_1 - \tau_2)g(-\xi_2, -\tau_2)h(\xi_1, \tau_1)\psi(\xi,\tau)d\xi_1 d\xi_2 d\xi d\tau_1 d\tau_2 d\tau,$$

where

$$\begin{aligned}&W(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)\\=&\frac{\langle\epsilon(\xi-\xi_1)\rangle^{-2}\langle\xi\rangle^{s+a}\langle\xi-\xi_1-\xi_2\rangle^{-s}\langle\xi_2\rangle^{-s}\langle\xi_1\rangle^{-s}}{\langle\tau-\tau_1-\tau_2+d_\epsilon(\xi-\xi_1-\xi_2)\rangle^{b}\langle\tau_2-d_\epsilon(\xi_2)\rangle^{b}\langle\tau_1+d_\epsilon(\xi_1)\rangle^{b}\langle\tau-d_\epsilon(\xi)\rangle^{\gamma}}.\end{aligned}$$

We apply the Cauchy-Schwarz inequality (in variables $\xi_1,\xi_2,\tau_1,\tau_2$) on the product

$$f(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)g(-\xi_2,-\tau_2)h(\xi_1,\tau_1)$$

and the rest of the integrand. The former, followed by Young's inequality, yields

$$\|f^2*\overline{g^2}*h^2\|^{1/2}_{L^1_{\xi\tau}}\le\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}=\|u\|_{X^{s,b}_{(\epsilon)}}\|v\|_{X^{s,b}_{(\epsilon)}}\|w\|_{X^{s,b}_{(\epsilon)}}.$$

The remaining part of Cauchy-Schwarz inequality, followed by the L^1-L^∞ Hölder's inequality yields

$$\|\psi\|_{L^2}\cdot\sup_{\xi,\tau}\Big(\int W^2d\xi_1d\xi_2d\tau_1d\tau_2\Big)^{\frac12}.$$

After changing the variable $\xi_1\mapsto\xi_1+\xi$, it suffices to show

$$\sup_{\xi,\tau}\int\frac{\langle\epsilon\xi_1\rangle^{-4}\langle\xi\rangle^{2s+2a}\langle\tau-d_\epsilon(\xi)\rangle^{-2\gamma}\langle\xi_1+\xi_2\rangle^{-2s}\langle\xi_2\rangle^{-2s}d\xi_1d\xi_2d\tau_1d\tau_2}{\langle\tau-\tau_1-\tau_2+d_\epsilon(\xi_1+\xi_2)\rangle^{2b}\langle\tau_2-d_\epsilon(\xi_2)\rangle^{2b}\langle\xi_1+\xi\rangle^{2s}\langle\tau_1+d_\epsilon(\xi_1+\xi)\rangle^{2b}}<\infty.$$

We further reduce this task by integrating in τ_1 and τ_2:

$$\begin{aligned}&\int\frac{d\tau_1}{\langle\tau_1-(\tau-\tau_2+d_\epsilon(\xi_1+\xi_2))\rangle^{2b}\langle\tau_1+d_\epsilon(\xi_1+\xi)\rangle^{2b}}\\&\lesssim\langle\tau_2-(\tau+d_\epsilon(\xi_1+\xi_2)+d_\epsilon(\xi_1+\xi))\rangle^{-2b}.\\&\int\frac{d\tau_2}{\langle\tau_2-(\tau+d_\epsilon(\xi_1+\xi_2)+d_\epsilon(\xi_1+\xi))\rangle^{2b}\langle\tau_2-d_\epsilon(\xi_2)\rangle^{2b}}\\&\lesssim\langle\tau+d_\epsilon(\xi_1+\xi_2)+d_\epsilon(\xi_1+\xi)-d_\epsilon(\xi_2)\rangle^{-2b}.\end{aligned}$$

Since $\langle\xi-A\rangle\langle\xi-B\rangle\gtrsim\langle A-B\rangle$, we have $\langle\xi_1+\xi_2\rangle^{2s}\langle\xi_2\rangle^{2s}\langle\xi_1+\xi\rangle^{2s}\gtrsim\langle\xi\rangle^{2s}$.

Thus, the problem reduces to showing

$$\sup_{\xi,\tau} \frac{\langle \xi \rangle^{2a}}{\langle \tau - d_\epsilon(\xi)\rangle^{2\gamma}} \int \frac{\langle \epsilon\xi_1\rangle^{-4} d\xi_1 d\xi_2}{\langle \tau + d_\epsilon(\xi_1+\xi_2) + d_\epsilon(\xi_1+\xi) - d_\epsilon(\xi_2)\rangle^{2b}} < \infty.$$

The expression $\tau + d_\epsilon(\xi_1+\xi_2) + d_\epsilon(\xi_1+\xi) - d_\epsilon(\xi_1)$ is a cubic polynomial in ξ_2 with an inflection point at $\xi_2 = -\frac{\xi_1}{2}$, and therefore after changing the variable $\xi_2 \mapsto \xi_2 - \frac{\xi_1}{2}$, the integral becomes

$$\int \frac{\langle \epsilon\xi_1\rangle^{-4} d\xi_1 d\xi_2}{\langle 4\epsilon^2\xi_1\xi_2^3 + \xi_1(\epsilon^2\xi_1^2+2)\xi_2 + (1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2 + \tau\rangle^{2b}} < \infty.$$

In doing ξ_2-integral, if $\xi_1 < 0$, then via another change of variable $\xi_2 \mapsto -\xi_2$, the integral is invariant when

$$4\epsilon^2\xi_1\xi_2^3 + \xi_1(\epsilon^2\xi_1^2+2)\xi_2 + (1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2 + \tau$$

is replaced with

$$4\epsilon^2|\xi_1|\xi_2^3 + |\xi_1|(\epsilon^2\xi_1^2+2)\xi_2 + (1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2 + \tau.$$

Similarly the expression above can be replaced with

$$4\epsilon^2|\xi_1|\xi_2^3 + |\xi_1|(\epsilon^2\xi_1^2+2)\xi_2 + |(1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2 + \tau|,$$

leaving the integral invariant.

Another change of variable $\xi_2 \mapsto \frac{\xi_2}{|\xi_1|^{1/3}}$, followed by $\xi_1 \mapsto -\xi_1$, eliminates the ξ_1-dependence in the leading coefficient of this cubic polynomial, and our task simplifies to showing the following estimate:

$$\sup_{\xi,\tau} \frac{\langle \xi \rangle^{2a}}{\langle \tau - d_\epsilon(\xi)\rangle^{2\gamma}} \cdot \int \frac{\langle \epsilon\xi_1\rangle^{-4}|\xi_1|^{-1/3} d\xi_1 d\xi_2}{\langle 4\epsilon^2\xi_2^3 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)\xi_2 + |(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2 + \tau|\rangle^{2b}} < \infty. \tag{3.5}$$

It is easy to see that $\sup\limits_{\xi\in\mathbb{R}}$ can be reduced to $\sup\limits_{\xi>0}$, which we assume henceforth; if $\xi < 0$, let $\xi' = -\xi$ and do a change of variable $\xi_1 \mapsto -\xi_1$ in the integral. However, we must consider $\tau > 0, \tau < 0$ separately.

Case I. $\tau < 0$.

Since $\langle \xi \rangle^{2a} \langle \tau - d_\epsilon(\xi) \rangle^{-2\gamma} \le \langle \xi \rangle^{2a} \langle d_\epsilon(\xi) \rangle^{-2\gamma} \lesssim_\epsilon 1$, it suffices to show

$$\sup_{\xi,\tau} \iint \frac{|\xi_1|^{-1/3} d\xi_1 d\xi_2}{\langle 4\epsilon^2 \xi_2^3 + |\xi_1|^{2/3}(\epsilon^2 \xi_1^2 + 2)\xi_2 + |(1+\epsilon^2(\xi_1 - \xi)^2)(\xi_1 - \xi)^2 + \tau| \rangle^{2b}} < \infty. \quad (3.6)$$

We do the ξ_2-integral in three disjoint regions: $(-\infty, r) \cup (r, 0) \cup (0, \infty)$.

(i) Consider the Taylor expansion of $|P(\xi_2)|$ on $(-\infty, r)$ at $\xi_2 = r$.

$$\begin{aligned} |P(\xi_2)| &= -(12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2 \xi_1^2 + 2))(\xi_2 - r) - 12\epsilon^2 r(\xi_2 - r)^2 - 4\epsilon^2(\xi_2 - r)^3 \\ &\ge \max\Big(- |\xi_1|^{2/3}(\epsilon^2 \xi_1^2 + 2)(\xi_2 - r), -4\epsilon^2(\xi_2 - r)^3 \Big) \ge 0. \end{aligned}$$

Integrating these lower bounds, we obtain

$$\begin{aligned} |\xi_1|^{-1/3} \int_{-\infty}^{r} \frac{d\xi_2}{\langle p(\xi_2) \rangle^{2b}} &\le |\xi_1|^{-1/3} \int_{-\infty}^{r} \frac{d\xi_2}{\langle |\xi_1|^{2/3}(\epsilon^2 \xi_1^2 + 2)(\xi_2 - r) \rangle^{2b}} \\ &= |\xi_1|^{-1/3} \int_{0}^{\infty} \frac{d\xi_2}{\langle |\xi_1|^{2/3}(\epsilon^2 \xi_1^2 + 2)\xi_2 \rangle^{2b}} \\ &= \frac{|\xi_1|^{-1/3}}{|\xi_1|^{2/3}(\epsilon^2 \xi_1^2 + 2)} \int_{0}^{\infty} \frac{d\xi_2}{\langle \xi_2 \rangle^{2b}} \simeq_b \frac{1}{|\xi_1|(\epsilon^2 \xi_1^2 + 2)}, \end{aligned}$$

and

$$\begin{aligned} |\xi_1|^{-1/3} \int_{-\infty}^{r} \frac{d\xi_2}{\langle p(\xi_2) \rangle^{2b}} &\le |\xi_1|^{-1/3} \int_{-\infty}^{r} \frac{d\xi_2}{\langle 4\epsilon^2(\xi_2 - r)^3 \rangle^{2b}} \\ &= |\xi_1|^{-1/3} \int_{0}^{\infty} \frac{d\xi_2}{\langle 4\epsilon^2 \xi_2^3 \rangle^{2b}} \simeq_{b,\epsilon} |\xi_1|^{-1/3}. \end{aligned}$$

Hence,

$$|\xi_1|^{-1/3} \int_{-\infty}^{r} \frac{d\xi_2}{\langle p(\xi_2)\rangle^{2b}} \lesssim_{b,\epsilon} \min\left(\frac{1}{|\xi_1|(\epsilon^2\xi_1^2+2)}, |\xi_1|^{-1/3}\right) \tag{3.7}$$

and the desired result follows by integrating with respect to ξ_1.

(ii) Let $\xi_2 \in (r, 0)$. Since r is a root of $P(\xi_2)$,

$$\begin{aligned} |P(\xi_2)| = P(\xi_2) &= (\xi_2 - r)(4\epsilon^2\xi_2^2 + 4\epsilon^2 r\xi_2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2) + 4\epsilon^2 r^2) \\ &\geq (\xi_2 - r)(|\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)) \geq 0. \end{aligned}$$

Using this lower bound,

$$\begin{aligned} |\xi_1|^{-1/3} \int_{r}^{0} \frac{d\xi_2}{\langle p(\xi_2)\rangle^{2b}} &\leq |\xi_1|^{-1/3} \int_{r}^{0} \frac{d\xi_2}{\langle (\xi_2 - r)(|\xi_1|^{2/3}(\epsilon^2\xi_1^2+2))\rangle^{2b}} \\ &= |\xi_1|^{-1/3} \int_{0}^{|r|} \frac{d\xi_2}{\langle \xi_2 \cdot (|\xi_1|^{2/3}(\epsilon^2\xi_1^2+2))\rangle^{2b}} \\ &\leq \frac{|\xi_1|^{-1/3}}{|\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)} \int_{0}^{\infty} \frac{d\xi_2}{\langle \xi_2\rangle^{2b}} \simeq_b \frac{1}{|\xi_1|(\epsilon^2\xi_1^2+2)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} P(\xi_2) &= (12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2))(\xi_2 - r) + 12\epsilon^2 r(\xi_2 - r)^2 + 4\epsilon^2(\xi_2 - r)^3 \\ &\geq 4\epsilon^2(\xi_2 - r)^3 \geq 0, \end{aligned}$$

where the inequality holds since $(12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2))(\xi_2 - r) + 12\epsilon^2 r(\xi_2 - r)^2 \geq 0$

on $\xi_2 \in (r, 0)$. Then,

$$|\xi_1|^{-1/3} \int_r^0 \frac{d\xi_2}{\langle p(\xi_2)\rangle^{2b}} \leq |\xi_1|^{-1/3} \int_r^0 \frac{d\xi_2}{\langle 4\epsilon^2(\xi_2 - r)^3\rangle^{2b}}$$
$$= |\xi_1|^{-1/3} \int_0^{|r|} \frac{d\xi_2}{\langle 4\epsilon^2\xi_2^3\rangle^{2b}} \leq |\xi_1|^{-1/3} \int_0^{\infty} \frac{d\xi_2}{\langle 4\epsilon^2\xi_2^3\rangle^{2b}} \simeq_{b,\epsilon} |\xi_1|^{-1/3}.$$

Hence,

$$|\xi_1|^{-1/3} \int_r^0 \frac{d\xi_2}{\langle p(\xi_2)\rangle^{2b}} \lesssim_{b,\epsilon} \min\Big(\frac{1}{|\xi_1|(\epsilon^2\xi_1^2+2)}, |\xi_1|^{-1/3}\Big) \tag{3.8}$$

and the desired result follows by integrating with respect to ξ_1.

(iii) Similarly on $\xi_2 \in (0, \infty)$,

$$|\xi_1|^{-1/3} \int_0^\infty \frac{d\xi_2}{\langle p(\xi_2)\rangle^{2b}} \lesssim_{b,\epsilon} \min\Big(\frac{1}{|\xi_1|(\epsilon^2\xi_1^2+2)}, |\xi_1|^{-1/3}\Big), \tag{3.9}$$

where we use the two lower bounds of $|P(\xi_2)| = P(\xi_2) \geq \max\Big(4\epsilon^2\xi_2^3, |\xi_1|^{2/3}(\epsilon^2\xi_1^2 + 2)\xi_2\Big)$ to argue as before. This concludes the proof for case I.

Case II. $\tau \in [0, \frac{d_\epsilon(\xi)}{2}] \cup [2d_\epsilon(\xi), \infty)$.

If $\tau \in [0, \frac{d_\epsilon(\xi)}{2}]$, then $d_\epsilon(\xi) - \tau \geq \frac{d_\epsilon(\xi)}{2} \geq 0$, and therefore $\langle \tau - d_\epsilon(\xi)\rangle^{-2\gamma} \leq \langle \frac{d_\epsilon(\xi)}{2}\rangle^{-2\gamma}$. If $\tau \in [2d_\epsilon(\xi), \infty)$, then $\tau - d_\epsilon(\xi) \geq d_\epsilon(\xi)$, and therefore $\langle \tau - d_\epsilon(\xi)\rangle^{-2\gamma} \leq \langle d_\epsilon(\xi)\rangle^{-2\gamma}$. In either cases, the analysis reduces to showing

$$\sup_{\xi,\tau} \iint \frac{|\xi_1|^{-1/3} d\xi_1 d\xi_2}{\langle 4\epsilon^2\xi_2^3 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)\xi_2 + (1+\epsilon^2(\xi_1-\xi))(\xi_1-\xi)^2 + \tau\rangle^{2b}} < \infty, \tag{3.10}$$

which can be done as in case I.

Case III. $\tau \in (\frac{d_\epsilon(\xi)}{2}, 2d_\epsilon(\xi))$.

On this region, $\langle \tau - d_\epsilon(\xi)\rangle^{-2\gamma} \leq 1$, and therefore, we need to extract an algebraic decay in ξ from the double integral. As before, we derive good lower bounds on $|p(\xi_2)|$ on three disjoint regions, $(-\infty, r) \cup (r, 0) \cup (0, \infty)$ and bound the integral separately. Moreover we can assume $\xi > 1$ without loss of generality since $\left\{(\xi, \tau) : \xi \in [0,1], \tau \in [\frac{d_\epsilon(\xi)}{2}, 2d_\epsilon(\xi)]\right\}$ is a compact subset of $\mathbb{R}^2$, and therefore, extreme value theorem applies on our double integral.

(i) On $\xi_2 \in (-\infty, r)$,

$$\begin{aligned}|P(\xi_2)| = -(12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2 + 2))(\xi_2 - r) - 12\epsilon^2 r(\xi_2 - r)^2 - 4\epsilon^2(\xi_2 - r)^3 \\ \geq -(12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2 + 2))(\xi_2 - r) \geq 0\end{aligned}$$

$$\begin{aligned}\Rightarrow |\xi_1|^{-1/3}\int_{-\infty}^{r} \frac{d\xi_2}{\langle P(\xi_2)\rangle^{2b}} &\leq |\xi_1|^{-1/3}\int_0^\infty \frac{d\xi_2}{\langle(12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2))\xi_2\rangle^{2b}} \\ &\simeq_b \frac{|\xi_1|^{-1/3}}{12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)}.\end{aligned}$$

Consider a change of variable $z = \xi_1^{\frac{8}{3}} + c_\epsilon \xi^{\frac{8}{3}}$ where $c_\epsilon > 0$ is to be determined and $p \in [1, 3)$. Then,

$$\begin{aligned}&\int_0^\infty \frac{\langle \epsilon\xi_1\rangle^{-4}|\xi_1|^{-1/3}d\xi_1}{12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)} \\ &= \int_0^\xi \frac{\langle \epsilon\xi_1\rangle^{-4}|\xi_1|^{-1/3}d\xi_1}{12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)} + \int_\xi^\infty \frac{\langle \epsilon\xi_1\rangle^{-4}|\xi_1|^{-1/3}d\xi_1}{12\epsilon^2 r^2 + |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)} \\ &\lesssim_\epsilon |\xi|^{-\frac{8}{3}} \|\langle\epsilon\xi_1\rangle^{-4}\|_{L^{p'}_{(0,\xi)}} \|\xi_1^{-1/3}\|_{L^p_{(0,\xi)}} + \langle\epsilon\xi\rangle^{-4}\xi^{-1/3}\int_\xi^\infty \frac{d\xi_1}{\xi_1^{8/3} + c_\epsilon\xi^{8/3}} \\ &\lesssim_p \langle\xi\rangle^{-(3-\frac{1}{p})} + \langle\epsilon\xi\rangle^{-4}\xi^{-1/3}\int_{c_\epsilon\xi^{8/3}}^\infty \frac{dz}{z(z - c_\epsilon\xi^{8/3})^{\frac{5}{8}}} \lesssim \langle\xi\rangle^{-(3-\frac{1}{p})},\end{aligned}$$

where we use lemma 3.2.3 as a lower bound on $|r|$.

(ii) On $\xi_2 \in (r, 0)$, we have

$$
\begin{aligned}
|P(\xi_2)| &= P(\xi_2) \\
&\geq \frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|r|}\xi_2 + (1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau \geq 0
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
&|\xi_1|^{-1/3}\int\limits_r^0 \frac{d\xi_2}{\langle \frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|r|}\xi_2 + (1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau\rangle^{2b}} \\
&\qquad \lesssim_b \frac{|\xi_1|^{-1/3}|r(\xi_1)|}{\langle (1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2 + \frac{d_\epsilon(\xi)}{2}\rangle}.
\end{aligned}
$$

We change variable $z = \xi_1^4 + \frac{\xi^4}{2}$ and integrate with respect to ξ_1 as in the previous case (i), and use the upper bound in lemma 3.2.3.

$$
\begin{aligned}
&\int\limits_{-\infty}^{\infty}\int\limits_r^0 \frac{\langle \epsilon\xi_1\rangle^{-4}|\xi_1|^{-1/3}d\xi_2 d\xi_1}{\langle p(\xi_2)\rangle^{2b}} \\
&\lesssim \int\limits_0^{\xi} \frac{\langle \epsilon\xi_1\rangle^{-4}|\xi_1|^{-1/3}d_\epsilon(\xi)^{1/3}d\xi_1}{\langle \frac{d_\epsilon(\xi)}{2}\rangle} + \int\limits_{\xi}^{\infty} \frac{\langle \epsilon\xi_1\rangle^{-3}d\xi_1}{\langle (1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2 + \frac{d_\epsilon(\xi)}{2}\rangle} \\
&\lesssim_b \langle \xi\rangle^{-(3-\frac{1}{p})} + \frac{\langle \epsilon\xi\rangle^{-3}}{\epsilon^2}\int\limits_0^{\infty} \frac{d\xi_1}{\xi_1^4 + \frac{\xi^4}{2}} \lesssim \langle \xi\rangle^{-(3-\frac{1}{p})}.
\end{aligned}
$$

(iii) On $\xi_2 \in (0, \infty)$, we note

$$
|P(\xi_2)| = P(\xi_2) \geq 4\epsilon^2\xi_2^3 + \frac{d_\epsilon(\xi)}{2} \geq 0.
$$

Consider the following change of variable: $z = 4\epsilon^2\xi_2^3 + \frac{d_\epsilon(\xi)}{2}$. Then,

$$|\xi_1|^{-1/3}\int_0^\infty \frac{d\xi_2}{\langle P(\xi_2)\rangle^{2b}} \leq |\xi_1|^{-1/3}\int_0^\infty \frac{d\xi_2}{\langle 4\epsilon^2\xi_2^3 + \frac{d_\epsilon(\xi)}{2}\rangle^{2b}}$$
$$\simeq_\epsilon |\xi_1|^{-1/3}\int_{\frac{d_\epsilon(\xi)}{2}}^\infty \frac{dz}{\langle z\rangle^{2b}(z-\frac{d_\epsilon(\xi)}{2})^{2/3}} \lesssim_b |\xi_1|^{-\frac{1}{3}}|d_\epsilon(\xi)|^{-\frac{2}{3}},$$

where in the last inequality, we note that $\langle z\rangle^{-2b} \lesssim |z|^{-1}$ on the region of integration. On the other hand, we use $4\epsilon^2\xi_2^3 + (1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2 + \frac{d_\epsilon(\xi)}{2} \geq 0$ as another lower bound to derive a similar estimate for $|\xi_1| > 2\xi$ on which $|\xi_1 - \xi| \geq \frac{|\xi_1|}{2}$.

$$|\xi_1|^{-1/3}\int_0^\infty \frac{d\xi_2}{\langle P(\xi_2)\rangle^{2b}} \lesssim_{b,\epsilon} \frac{|\xi_1|^{-1/3}}{\langle (1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2 + \frac{d_\epsilon(\xi)}{2}\rangle^{2/3}} \leq \frac{|\xi_1|^{-1/3}}{\langle \frac{\epsilon^2\xi_1^4}{16} + \frac{d_\epsilon(\xi)}{2}\rangle^{2/3}}$$
$$\lesssim \frac{|\xi|^{-1/3}}{\left(\frac{\epsilon^2\xi_1^4}{16} + \frac{d_\epsilon(\xi)}{2}\right)^{2/3}},$$

and the remaining ξ_1 integral proceeds as before. □

proof of proposition 3.2.2. We prove by induction on $k \geq 0$. Let $I_k = [\frac{4}{3}k, \frac{4}{3}(k+1))$ and consider the following statement: for all $s \in I_k$, there exists C_s, a non-decreasing function, such that $\|u^{(\epsilon)}(t)\|_{H^s} \leq C_s(\|g\|_{H^s})\langle t\rangle^{\frac{1}{2}(3^{k+1}-1)}$ for all $t \in \mathbb{R}$; it suffices to assume $t \geq 0$ by time-reversal symmetry of solutions. Let $\alpha_k = \frac{1}{2}(3^{k+1}-1)$ for $k \geq 0$; note that $\alpha_k = 3\alpha_{k-1} + 1$.

Let $k = 0$ and fix $b \in (\frac{1}{2}, 1-\gamma), \gamma \in [\frac{1}{3}, \frac{1}{2})$ once and for all. By the local theory in $L^2(\mathbb{R})$ with $\delta \simeq \|g\|_{L^2}^{-\rho}$ for some $\rho = \rho(b,\gamma) > 0$,

$$\|u^{(\epsilon)}(t)\|_{H^s} \leq \|g\|_{H^s} + C\|u^{(\epsilon)}\|^3_{X^{0,b}_{(\epsilon),\delta}} \leq \|g\|_{H^s} + C'\|g\|^3_{L^2}, t \in (0,\delta].$$

Time evolving $u^{(\epsilon)}$ iteratively for $j = 1, 2, ...$ and $t \in ((j-1)\delta, j\delta]$,

$$\|u^{(\epsilon)}(t)\|_{H^s} \leq \|g\|_{H^s} + C'j\|g\|^3_{L^2} < \|g\|_{H^s} + C'(1+\frac{t}{\delta})\|g\|^3_{L^2}$$
$$\leq \|g\|_{H^s} + C'(1+\frac{t}{\delta})\|g\|^3_{H^s} \leq \|g\|_{H^s} + C'\|g\|^3_{H^s} + C''\|g\|^{3+\rho}_{H^s}t.$$

Noting that $\langle t\rangle \simeq 1+|t|$ and that C', C'' are universal constants that only depend on the given parameters, there exists a non-decreasing function C_s such that

$$\|g\|_{H^s} + C'\|g\|_{H^s}^3 + C''\|g\|_{H^s}^{3+\rho} t \le C_s(\|g\|_{H^s})\langle t\rangle.$$

For example, one can explicitly check that $C_s(\zeta) = C''\zeta^{3+\rho} + C'\zeta^3 + \zeta$ would do.

Now, suppose $k \ge 1$ and that the inductive hypothesis holds for $j = 0, 1, ..., k-1$, and let $s \in I_k$. Fix $a \in (0, \frac{4}{3})$ such that $s - ja \in I_{k-j}$ for $j = 1, 2, ..., k$ and $s-(k+1)a \le 0$. We again use the L^2 local theory on $[0,\delta]$ where $\delta = \tilde{c}\|g\|_{L^2}^{-\rho}$ for some $\tilde{c} > 0$. Iteratively applying smoothing estimate for $N(u^{(\epsilon)})$ and using triangle inequaltiy $(a+b)^3 \lesssim a^3 + b^3$ for $a, b \ge 0$, we obtain

$$\begin{aligned}
t \in (0,\delta] : \|u^{(\epsilon)}(t)\|_{H^s} &\le \|g\|_{H^s} + C(\|g\|_{H^{s-a}}^3 + \cdots + \|g\|_{H^{s-ka}}^{3^k}) + C\|g\|_{L^2}^{3^{k+1}} \\
t \in (\delta, 2\delta] : \|u^{(\epsilon)}(t)\|_{H^s} &\le \|g\|_{H^s} + C(\|g\|_{H^{s-a}}^3 + \cdots + \|g\|_{H^{s-ka}}^{3^k}) + 2C\|g\|_{L^2}^{3^{k+1}} \\
&\quad + C\Big(C_{s-a}(\|g\|_{H^{s-a}})^3\langle\delta\rangle^{3\alpha_{k-1}} + \cdots + C_{s-ka}(\|g\|_{H^{s-ka}})^{3^k}\langle\delta\rangle^{3^k\alpha_0}\Big) \\
&\le \|g\|_{H^s} + C(\|g\|_{H^{s-a}}^3 + \cdots + \|g\|_{H^{s-ka}}^{3^k}) + 2C\|g\|_{L^2}^{3^{k+1}} \\
&\quad + C\Big(C_{s-a}(\|g\|_{H^s})^3 + \cdots + C_{s-ka}(\|g\|_{H^s})^{3^k}\Big)\langle\delta\rangle^{3\alpha_{k-1}} \\
t \in ((j-1)\delta, j\delta] : \|u^{(\epsilon)}(t)\|_{H^s} &\le \|g\|_{H^s} + C(\|g\|_{H^{s-a}}^3 + \cdots + \|g\|_{H^{s-ka}}^{3^k}) + jC\|g\|_{L^2}^{3^{k+1}} \\
&\quad + (j-1)C\Big(C_{s-a}(\|g\|_{H^s})^3 + \cdots + C_{s-ka}(\|g\|_{H^s})^{3^k}\Big)\langle (j-1)\delta\rangle^{3\alpha_{k-1}}.
\end{aligned}$$

Hence for all $t \ge 0$,

$$\begin{aligned}
\|u^{(\epsilon)}(t)\|_{H^s} &< \|g\|_{H^s} + C(\|g\|_{H^s}^3 + \cdots + \|g\|_{H^s}^{3^k}) + C(1 + \frac{\|g\|_{H^s}^\sigma}{\tilde{c}} t)\|g\|_{H^s}^{3^{k+1}} \\
&\quad + C\frac{\|g\|_{H^s}^\sigma}{\tilde{c}}\Big(C_{s-a}(\|g\|_{H^s})^3 + \cdots + C_{s-ka}(\|g\|_{H^s})^{3^k}\Big)\langle t\rangle^{3\alpha_{k-1}+1},
\end{aligned}$$

and therefore, there exists a non-decreasing function C_s such that

$$\|u^{(\epsilon)}(t)\|_{H^s} \le C_s(\|g\|_{H^s})\langle t\rangle^{\alpha_k}.$$

$\square$

3.3 NLS limit.

In this section, we study the $\epsilon \to 0$ problem. Heuristically the mNLS is a perturbation of NLS, and it is our goal to make this statement rigorous. We first study the convergence of linear evolution.

Proposition 3.3.1. *Let $\epsilon > 0, s \in \mathbb{R}, t \in \mathbb{R}\setminus\{0\}$. Consider $U_\epsilon(t)$ and $U(t)$ as unitary operators on $H^s(\mathbb{R}^d)$. Then,*

1. *$U_\epsilon(t) \nrightarrow U(t)$ in norm operator topology as $\epsilon \to 0$.*
2. *$U_\epsilon(t) \xrightarrow[\epsilon\to 0]{} U(t)$ in strong operator topology. More generally if $u_0^{(\epsilon)}, u_0 \in H^s(\mathbb{R}^d)$ and $u_0^{(\epsilon)} \xrightarrow[\epsilon\to 0]{} u_0$, then $U_\epsilon(t)u_0^{(\epsilon)} \xrightarrow[\epsilon\to 0]{} U(t)u_0$ in $C([0,T], H^s(\mathbb{R}^d))$ for every $0 < T < \infty$.*
3. *For all $0 < T < \infty$ and $u \in C([0,T], H^s(\mathbb{R}^d))$, we have $U_\epsilon(t)u(t) \xrightarrow[\epsilon\to 0]{H^s} U(t)u(t)$ uniformly in $t \in [0,T]$. In particular, for every initial datum $u_0 \in H^s(\mathbb{R}^d)$, we have $U_\epsilon(t)u_0 \xrightarrow[\epsilon\to 0]{} U(t)u_0$ in $C([0,T], H^s(\mathbb{R}^d))$.*
4. *There exists some $u_0 \in H^s(\mathbb{R}^d)$ such that $U_\epsilon(t)u_0 \nrightarrow U(t)u_0$ in $C([0,\infty), H^s(\mathbb{R}^d))$ as $\epsilon \to 0$.*

proof of proposition 3.3.1. We use the characterization of Fourier multiplier on $L^2(\mathbb{R}^d)$ where the operator norm of a Fourier multiplier equals the L^∞ norm of the corresponding symbol. Since $H^s(\mathbb{R}^d)$ is unitarily isomorphic to $L^2(\mathbb{R}^d)$ via $f \mapsto \langle\xi\rangle^s \hat{f}$, the unitary action $U_\epsilon(t)$ on $H^s(\mathbb{R}^d)$ is unitarily isomorphic to a multiplication operator on $L^2(\mathbb{R}^d)$ via $F(\xi) \mapsto e^{-it(|\xi|^2+\epsilon^2|\xi|^4)}F(\xi)$. Hence to show $U_\epsilon(t) \nrightarrow U_0(t)$ in norm topology, it suffices to show $\|e^{-it|\xi|^2}(e^{-i\epsilon^2 t|\xi|^4} - 1)\|_{L^\infty_\xi} \nrightarrow 0$ as $\epsilon \to 0$. Indeed $\|e^{-it|\xi|^2}(e^{-i\epsilon^2 t|\xi|^4} - 1)\|_{L^\infty_\xi} = 2$ for all $\epsilon > 0$ and $t \in \mathbb{R}\setminus\{0\}$. This proves the first claim.

Noting that

$$\|U_\epsilon(t)u_0^{(\epsilon)} - U(t)u_0\|_{H^s} \leq \|u_0^{(\epsilon)} - u_0\|_{H^s} + \|(U_\epsilon(t) - U(t))u_0\|_{H^s},$$

by the triangle inequality and unitarity, it suffices to show that the latter tends to zero uniformly in $t \in [0,T]$. Since $|e^{-i\epsilon^2 t|\xi|^4} - 1|^2 = 2(1 - \cos(\epsilon^2 t|\xi|^4))$, we will show $\lim\limits_{\epsilon\to 0+} I = 0$ where $I = \int(1-\cos(\epsilon^2 t|\xi|^4))\langle\xi\rangle^{2s}|\widehat{u_0}|^2 d\xi$. We first observe that $1-\cos(\epsilon^2 t|\xi|^4)$ converges to zero as $\epsilon\to 0$ for a fixed t and ξ. For $n\in\mathbb{N}$, we observe that $\xi_n = \xi_n(\epsilon,t) = (\cos^{-1}(\frac{n-1}{n}))^{1/4}\epsilon^{-1/2}t^{-1/4}$ solves $1-\cos(\epsilon^2 t\xi_n^4) = \frac{1}{n}$ and $0\le 1-\cos(\epsilon^2 t|\xi|^4)\le\frac{1}{n}$ whenever $|\xi|\le\xi_n(\epsilon,t)$. Estimating I in two different regions, we obtain

$$I = \int\limits_{|\xi|\le\xi_n(\epsilon,t)} + \int\limits_{|\xi|>\xi_n(\epsilon,t)} \le \frac{\|u_0\|_{H^s}^2}{n} + 2\int\limits_{|\xi|>\xi_n(\epsilon,T)}\langle\xi\rangle^{2s}|\hat{u_0}|^2 d\xi.$$

$$\Rightarrow \overline{\lim_{\epsilon\to 0}}\, I \le \frac{\|u_0\|_{H^s}^2}{n}, \forall n\in\mathbb{N}.$$

which yields the desired result as $n\to\infty$. This proves the second claim.

Let $I = \int(1-\cos(\epsilon^2 t|\xi|^4))\langle\xi\rangle^{2s}|\widehat{u(t)}|^2 d\xi$. On $t\in[0,T]$, we have $0\le 1-\cos(\epsilon^2 t|\xi|^4)\le\frac{1}{n}$ whenever $|\xi|\le\xi_n(\epsilon,T)$. Hence,

$$I = \int\limits_{|\xi|\le\xi_n(\epsilon,T)} + \int\limits_{|\xi|>\xi_n(\epsilon,T)} \le \frac{\|u\|_{C_T^0 H_x^s}^2}{n} + 2\int\limits_{|\xi|>\xi_n(\epsilon,T)}\langle\xi\rangle^{2s}|\widehat{u(t)}|^2 d\xi.$$

We define $F_\epsilon(t) = \int_{|\xi|>\xi_n(\epsilon,T)}\langle\xi\rangle^{2s}|\widehat{u(t)}|^2 d\xi$ and show $\lim\limits_{\epsilon\to 0}\sup\limits_{t\in[0,T]} F_\epsilon(t) = 0$ by an Arzelà-Ascoli argument. Since $u\in C([0,T],H^s(\mathbb{R}))$, it is straightforward to see $F_\epsilon\in C([0,T],\mathbb{R})$ with a pointwise bound; in fact, $F_\epsilon(t)\to 0$ as $\epsilon\to 0$ pointwise. For $t_1,t_2\in[0,T]$,

$$\begin{aligned}|F_\epsilon(t_1)-F_\epsilon(t_2)| &= \int\limits_{|\xi|>\xi_n(\epsilon,T)}\langle\xi\rangle^{2s}\Big(|\widehat{u(t_1)}|^2-|\widehat{u(t_2)}|^2\Big)d\xi\\ &\le \Bigg(\Bigg(\int\limits_{|\xi|>\xi_n(\epsilon,T)}\langle\xi\rangle^{2s}|\widehat{u(t_1)}|^2 d\xi\Bigg)^{\frac{1}{2}} + \Bigg(\int\limits_{|\xi|>\xi_n(\epsilon,T)}\langle\xi\rangle^{2s}|\widehat{u(t_2)}|^2 d\xi\Bigg)^{\frac{1}{2}}\Bigg)\\ &\cdot\|u(t_1)-u(t_2)\|_{H^s}\le 2\|u\|_{C_T^0 H_x^s}\|u(t_1)-u(t_2)\|_{H^s},\end{aligned}$$

and hence uniform equicontinuity on $[0,T]$. This proves the third claim.

Lastly, define $u_0 = \langle\nabla\rangle^{-s}|\nabla|^{\frac{1-d}{2}}e^{-\frac{|x|^2}{2}}$. Then we claim

$$\sup_{t\in\mathbb{R}}\int_{-\infty}^{\infty}(1-\cos(\epsilon^2 t\xi^4))\langle\xi\rangle^{2s}|\widehat{u_0}|^2 d\xi$$

$$= \sup_{t\in[0,\infty)}\int_{-\infty}^{\infty}(1-\cos(\epsilon^2 t\xi^4))\langle\xi\rangle^{2s}|\widehat{u_0}|^2 d\xi = 2\pi,\ \forall\epsilon>0.$$

First, observe that

$$\langle\xi\rangle^s\widehat{u_0}(\xi) = \sqrt{2\pi}|\xi|^{\frac{1-d}{2}}e^{-\frac{|\xi|^2}{2}} \in L^2(\mathbb{R}^d).$$

By a direct computation,

$$\int_{-\infty}^{\infty}(1-\cos(\epsilon^2 t\xi^4))\langle\xi\rangle^{2s}|\widehat{u_0}|^2 d\xi = 2\pi|S^{d-1}|\int_0^{\infty}(1-\cos(\epsilon^2 t r^4))e^{-r^2}dr$$

$$=\frac{\pi^{3/2}}{4}|S^{d-1}|\left(\frac{\sqrt{2\pi}\left(\sin\left(\frac{1}{8}\left(\pi-\frac{1}{t\epsilon^2}\right)\right)J_{\frac{1}{4}}\left(\frac{1}{8t\epsilon^2}\right)-\cos\left(\frac{1}{8}\left(\frac{1}{t\epsilon^2}+\pi\right)\right)J_{-\frac{1}{4}}\left(\frac{1}{8t\epsilon^2}\right)\right)}{\sqrt{t}\epsilon}+4\right),$$

where $J_{\pm\frac{1}{4}}(x)$, the Bessel functions of the first kind, are the two linearly independent solutions to

$$x^2y'' + xy' + (x^2 - \frac{1}{16})y = 0;$$

here the $'$ is the derivative with respect to x. The right-hand side of our direct computation is known to be an increasing function in $t \in [0,\infty)$ whose limit as $t\to\infty$ is $\pi^{\frac{3}{2}}|S^{d-1}|$ for all $\epsilon > 0$. $\square$

To study the convergence of nonlinear evolution, we need to control the nonlinear term in the Duhamel formula, and it suffices, but is most likely not necessary, to assume that our solutions are continuous in space.

Lemma 3.3.1. *Let $s > \frac{1}{2}$ and $u^{(\epsilon)}$ be the global solution corresponding to the initial data $u_0^{(\epsilon)}$ where $\left\{u_0^{(\epsilon)}\right\} \subseteq B_{H^s(\mathbb{R})}(0,R)$ for some $R > 0$. Then there exists $C = C(R,s) > 0$ such that $\sup_{\epsilon>0}\|u^\epsilon(t)\|_{H^s} \leq Re^{Ct}$ for all $t \geq 0$.*

proof of lemma 3.3.1. When $s > \frac{1}{2}$, we can establish (unconditional) local wellposedness of mNLS in $C([0,T], H^s(\mathbb{R}))$ such that $\|u^{(\epsilon)}(t)\|_{H^s} \leq 2R$ for $t \in [0,T]$, by using that $H^s(\mathbb{R})$ is a Sobolev algebra, where $T \simeq R^{-2}$ where the implicit constant is independent of $\epsilon > 0$. Then by writing the solution in an integral form, and by applying Gronwall's inequality, one can deduce

$$\sup_{\epsilon>0} \|u^{\epsilon}(t)\|_{H^s} \leq Re^{Ct}, \forall t \in [0,T],$$

for some $C = C(R,s)$. In fact, we show that the inequality above holds for all $t \geq 0$. For a contradiction, suppose the desired inequality is false for some time element and define

$$\begin{aligned} T_1 &:= \sup \left\{ t \geq 0 : \sup_{\epsilon>0} \|u^{(\epsilon)}(\tau)\|_{H^s} \leq Re^{C\tau}, \forall \tau \in [0,t] \right\} \\ T_2 &:= \inf \left\{ t \geq 0 : \sup_{\epsilon>0} \|u^{(\epsilon)}(t)\|_{H^s} > Re^{Ct} \right\}. \end{aligned}$$

Then $T_0 := T_1 = T_2 > 0$ where the strict inequality is by the previous local wellposedness argument. If T_0 were to satisfy the desired inequality, then we can run another local wellposedness argument, which would contradict the maximality of T_1. Hence there exists $\epsilon_0 > 0$ such that $\|u^{(\epsilon_0)}(T_0)\|_{H^s} > Re^{CT_0}$. However by continuity, $\|u^{(\epsilon_0)}(T_0)\|_{H^s} = \lim_{t \to T_0-} \|u^{(\epsilon_0)}(t)\|_{H^s} \leq Re^{CT_0}$, a contradiction. □

proof of theorem 3.1.2. The second part of the statement is proposition 3.3.1. As for the first, the strategy is to write the solutions in the integral form and show that the difference goes to zero as $\epsilon \to 0$ by applying Gronwall's inequality.

Let $\delta > 0$. By proposition 3.3.1, there exists $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$, then

$$\sup_{t \in [0,T]} \|U_\epsilon(t) u_0^{(\epsilon)} - U(t) u_0\|_{H^s} < \delta.$$

To proceed with the nonlinear estimates, note that

$$\int_0^t U_\epsilon(t-\tau)\Big(J_\epsilon(|u^{(\epsilon)}|^2)u^{(\epsilon)}\Big)(\tau) - U(t-\tau)\Big(|u|^2u\Big)(\tau)d\tau = I_1 + I_2 + I_3 + I_4, \text{ where}$$

$$I_1 = \int_0^t U_\epsilon(t-\tau)\bigg(J_\epsilon\Big(|u^{(\epsilon)}|^2\Big)(u^{(\epsilon)} - u)\bigg)d\tau;\ I_2 = \int_0^t U_\epsilon(t-\tau)\bigg(J_\epsilon\Big(|u^{(\epsilon)}|^2 - |u|^2\Big)u\bigg)d\tau.$$

$$I_3 = \int_0^t U_\epsilon(t-\tau)\bigg(\Big(J_\epsilon\Big(|u|^2\Big) - |u|^2\Big)u\bigg)d\tau;\ I_4 = \int_0^t \Big(U_\epsilon(t-\tau) - U(t-\tau)\Big)(|u|^2u)d\tau.$$

We estimate one by one.

$$\|I_1\|_{H^s} \leq \int_0^t \|J_\epsilon\Big(|u^{(\epsilon)}|^2\Big)(u^{(\epsilon)} - u)\|_{H^s}d\tau \lesssim_s \Big(\sup_{\epsilon>0}\|u^{(\epsilon)}\|_{C_T^0H_x^s}\Big)^2 \int_0^t \|u^{(\epsilon)} - u\|_{H^s}d\tau,$$

and similarly

$$\|I_2\|_{H^s} \lesssim_s (\sup_{\epsilon>0}\|u^{(\epsilon)}\|_{C_T^0H_x^s} + \|u\|_{C_T^0H_x^s})\|u\|_{C_T^0H_x^s}\int_0^t \|u^{(\epsilon)} - u\|_{H^s}d\tau.$$

As for I_3, we have

$$\|I_3\|_{H^s} \lesssim_s \|u\|_{C_T^0H_x^s}\int_0^t \|(J_\epsilon - I)\Big(|u|^2\Big)\|_{H^s}d\tau.$$

By Arzelà-Ascoli argument, we show that the right-hand side converges to 0 uniformly on $t \in [0,T]$. Let $F_\epsilon(t) = \int_0^t \|(J_\epsilon - I)\Big(|u|^2\Big)\|_{H^s}d\tau$. By dominated convergence, $F_\epsilon(t) \xrightarrow[\epsilon\to 0]{} 0$ pointwise. Since $|F_\epsilon(t) - F_\epsilon(t')| \lesssim \|u\|^2_{C_T^0H_x^s}|t-t'|$, the family $\{F_\epsilon\} \subseteq C([0,T],\mathbb{R})$ is uniformly equicontinuous, which proves the claim. Then there exists $\epsilon_1 > 0$ such that if $\epsilon \in (0,\epsilon_1)$, then

$$\|u\|_{C_T^0H_x^s}\int_0^t \|(J_\epsilon - I)\Big(|u|^2\Big)\|_{H^s}d\tau < \delta,$$

uniformly on $t \in [0,T]$.

Lastly, by a change of variable, we have

$$\|I_4\|_{H^s} \le \int_0^t \Big\|\Big(U_\epsilon(t-\tau) - U(t-\tau)\Big)(|u|^2 u)\Big\|_{H^s} d\tau$$
$$= \int_0^t \Big\|\Big(U_\epsilon(\tau) - U(\tau)\Big)(|u(t-\tau)|^2 u(t-\tau))\Big\|_{H^s} d\tau.$$

We claim $\Big\|\Big(U_\epsilon(\tau) - U(\tau)\Big)(|u(t-\tau)|^2 u(t-\tau))\Big\|_{H^s} \xrightarrow[\epsilon\to 0]{} 0$ uniformly for $(t,\tau) \in \Omega_T = \{(t',\tau') \in \mathbb{R}^2 : 0 \le t' \le T, 0 \le \tau' \le t'\}$. Define $w(\tau) := |u(t-\tau)|^2 u(t-\tau)$ for $(t,\tau) \in \Omega_T$. With $\xi_n(\epsilon,\tau)$ defined as before,

$$\Big\|\Big(U_\epsilon(\tau) - U(\tau)\Big)w(\tau)\Big\|_{H^s}^2 \le \frac{2}{n} \int_{|\xi| \le \xi_n(\epsilon,\tau)} \langle\xi\rangle^{2s} |\widehat{w(\tau)}|^2 d\xi + 4 \int_{|\xi| > \xi_n(\epsilon,\tau)} \langle\xi\rangle^{2s} |\widehat{w(\tau)}|^2 d\xi$$
$$\le \frac{C_s \|u\|_{C_T^0 H_x^s}^6}{n} + 4 \int_{|\xi| > \xi_n(\epsilon,T)} \langle\xi\rangle^{2s} |\widehat{w(\tau)}|^2 d\xi.$$

Choose N sufficiently big such that $\frac{C_s \|u\|_{C_T^0 H_x^s}^6}{n} < \frac{\delta}{2}$ for all $n \ge N$. It suffices to show $F_\epsilon(t,\tau) := \int_{|\xi| > \xi_N(\epsilon,T)} \langle\xi\rangle^{2s} |\widehat{w(\tau)}|^2 d\xi \xrightarrow[\epsilon\to 0]{} 0$ uniformly on Ω_T. Let $(t,\tau),(t',\tau') \in \Omega_T$.

Then

$$\begin{aligned}
&|F_\epsilon(t,\tau) - F_\epsilon(t',\tau')| \\
\leq& \Bigg(\Big(\int\limits_{|\xi|>\xi_N(\epsilon,T)} \langle\xi\rangle^{2s} |\widehat{w(\tau)}|^2 d\xi \Big)^{\frac{1}{2}} + \Big(\int\limits_{|\xi|>\xi_N(\epsilon,T)} \langle\xi\rangle^{2s} |\widehat{w(\tau')}|^2 d\xi \Big)^{\frac{1}{2}} \Bigg) \\
&\cdot \Big(\int\limits_{|\xi|>\xi_N(\epsilon,T)} \langle\xi\rangle^{2s} |\widehat{w(\tau)} - \widehat{w(\tau')}|^2 d\xi \Big)^{\frac{1}{2}} \\
&\lesssim_s \|u\|^3_{C^0_T H^s_x} \Big(\int\limits_{|\xi|>\xi_N(\epsilon,T)} \langle\xi\rangle^{2s} |\widehat{w(\tau)} - \widehat{w(\tau')}|^2 d\xi \Big)^{\frac{1}{2}} \leq \|u\|^3_{C^0_T H^s_x} \|w(\tau) - w(\tau')\|_{H^s} \\
&\leq \|u\|^3_{C^0_T H^s_x} \Bigg(\bigg\| \Big(|u(t-\tau)|^2 + |u(t'-\tau')|^2 \Big) \Big(u(t-\tau) - u(t'-\tau') \Big) \bigg\|_{H^s} \\
&+ \Big\| u(t-\tau) u(t'-\tau') \cdot \overline{u(t-\tau) - u(t'-\tau')} \Big\|_{H^s} \Bigg) \\
&\lesssim_s \|u\|^5_{C^0_T H^s_x} \|u(t-\tau) - u(t'-\tau')\|_{H^s}.
\end{aligned}$$

This estimate combined with the fact that $F_\epsilon(t,\tau) \underset{\epsilon\to 0}{\longrightarrow} 0$ pointwise on Ω_T by the dominated convergence theorem implies the desired result by an Arzelà-Ascoli argument.

Hence there exists $\epsilon_2 > 0$ such that if $\epsilon \in (0,\epsilon_2)$, then

$$\Big\| \Big(U_\epsilon(\tau) - U(\tau) \Big) (|u(t-\tau)|^2 u(t-\tau)) \Big\| < \delta, \forall (t,\tau) \in \Omega_T,$$

and therefore

$$\int\limits_0^t \Big\| \Big(U_\epsilon(\tau) - U(\tau) \Big) (|u(t-\tau)|^2 u(t-\tau)) \Big\|_{H^s} d\tau < \delta t.$$

By restricting $\epsilon \in (0, \min(\epsilon_0,\epsilon_1,\epsilon_2))$, we obtain

$$\|u^{(\epsilon)}(t) - u(t)\|_{H^s} \lesssim_{R,s,T} \delta\langle t\rangle + \int\limits_0^t \|u^{(\epsilon)}(\tau) - u(\tau)\|_{H^s} d\tau, \forall t \in [0,T],$$

from which Gronwall's inequality yields the following:

$$\|u^{(\epsilon)}(t) - u(t)\|_{H^s} \leq C\delta\langle t\rangle e^{Ct}, \forall t \in [0,T],$$

where $C = C(R,s,T)$. Since $\delta > 0$ is arbitrary, the proof is complete. $\square$

From the following lemma follows the failure of uniform continuity of the data-to-solution map in negative Sobolev spaces:

Lemma 3.3.2. *(Tao, 2006, Exercise 3.5) Let $s \in (-\infty, 0)$. For every $0 < \delta \ll \epsilon < 1$, there exists $f_1, f_2 \in \mathscr{S}(\mathbb{R})$ with their H^s-norms of $O(\epsilon)$ and the separation of $O(\delta)$ such that there exists $T_\epsilon = O(\epsilon)$ for which $\|u[f_1](T_\epsilon) - u[f_2](T_\epsilon)\|_{H^s} \simeq \epsilon$.*

proof of theorem 3.1.3. We study the time-evolution for the difference $v := u^{(\epsilon)} - u$, where $u^{(\epsilon)}(0) = u(0) = u_0$, via various Sobolev embeddings and Gronwall's inequality to obtain an upper bound on $\|v(t)\|_{H^s}$. Then v satisfies

$$\begin{cases} i\partial_t v = (\epsilon^2\partial_{xx} - \partial_{xx})v + N(v) + F(x,t),\ (x,t) \in \mathbb{R}\times\mathbb{R} \\ v(0) = 0. \end{cases}$$

where $F(x,t) = \epsilon^2\partial_{xx}^2 u + (I - J_\epsilon)(|u^{(\epsilon)}|^2)u^{(\epsilon)}$ and $N(v) = -(|u^{(\epsilon)}|^2 + |u|^2)v + u^{(\epsilon)}u\overline{v}$. Observe that the linear contribution of the solution is always zero since $v(0) = 0$. Hence any contribution to the solution directly comes from F and N at least for a short time.

Writing the solution in the Duhamel form and using the unitarity of $U_\epsilon(t)$ in Sobolev spaces, we obtain

$$\|v(t)\|_{H^s} = \left\| \int_0^t U_\epsilon(t-\tau)\Big(N(v) + F\Big)(\tau)d\tau \right\|_{H^s} \leq \int_0^t \|N(v)\|_{H^s}d\tau \int_0^t \|F(\tau)\|_{H^s}d\tau.$$

where $t \in [0,T]$ for some T to be determined later.

We first estimate $\|F(\tau)\|_{H^s}$. The complete integrability of cubic NLS on $\mathbb{R}$ gives $\|\partial_{xx}^2 u\|_{H^s} \leq C(\|u_0\|_{H^4})$. On the other hand, note that $(I - J_\epsilon)(|u^{(\epsilon)}|^2) = \epsilon^2\partial_{xx}J_\epsilon(|u^{(\epsilon)}|^2)$. By Theorem 3.1.2, we obtain $\|u^{(\epsilon)}(\tau)\|_{H^j} \leq C(\|u_0\|_{H^j})$ for $j = 1,2$

on $\tau \in [0,T]$, if ϵ is sufficiently small depending on T and u_0. Hence

$$\begin{aligned}\|(I-J_\epsilon)(|u^{(\epsilon)}|^2)u^{(\epsilon)}\|_{H^s} &\le \|(I-J_\epsilon)(|u^{(\epsilon)}|^2)u^{(\epsilon)}\|_{L^2} \le \|(I-J_\epsilon)(|u^{(\epsilon)}|^2)\|_{L^2}\|u^{(\epsilon)}\|_{L^\infty}\\ &\le \epsilon^2\|\partial_{xx}(|u^{(\epsilon)}|^2)\|_{L^2}\|u^{(\epsilon)}\|_{L^\infty}\\ &\lesssim \epsilon^2\|u^{(\epsilon)}\|_{H^1}\|u^{(\epsilon)}\|_{H^2}^2 \le \epsilon^2 C(\|u_0\|_{H^2}).\end{aligned}$$

Hence the Duhamel contribution by F is at most $C(\|u_0\|_{H^4})\epsilon^2 t$ for $t \in [0,T]$ and $0 < \epsilon < \epsilon_0$, for some $\epsilon_0 = \epsilon_0(T, u_0) > 0$.

To estimate $\|N(v)(\tau)\|_{H^s}$, let $s' = \lceil -s \rceil$. Then

$$\||u|^2 v\|_{H^s} \lesssim \||u|^2\|_{H^{s'}}\|v\|_{H^s} \lesssim \|u\|_{H^{s'}}^2\|v\|_{H^s} \le C(\|u_0\|_{H^{s'}})\|v\|_{H^s},$$

where the first inequality is by (Behzadan and Holst, 2015, Theorem 8.1). Similarly by Theorem 3.1.2 and the complete integrability of cubic NLS, we obtain

$$\|N(v)(t)\|_{H^s} \le C(\|u_0\|_{H^{s'}})\|v(t)\|_{H^s},$$

for $t \in [0,T]$ and for sufficiently small ϵ. This yields

$$\|v(t)\|_{H^s} \le \epsilon^2 C(\|u_0\|_{H^2})t + \int_0^t C(\|u_0\|_{H^{s'}})\|v(\tau)\|_{H^s} d\tau,$$

which by Gronwall's inequality implies

$$\|v(t)\|_{H^s} \le \epsilon^2 \frac{C(\|u_0\|_{H^2})}{C(\|u_0\|_{H^{s'}})}\Big(e^{C(\|u_0\|_{H^{s'}})t} - 1\Big), \tag{3.11}$$

for all $t \in [0,T]$ and $\epsilon \in (0, \epsilon_0)$. Pick $\epsilon' > 0$ small enough so that $\epsilon' \ll R$ and $T_{\epsilon'} \ll T$ where $T_{\epsilon'}$ is as in lemma 3.3.2. Then for every $0 < \delta \ll \epsilon'$, there exists a pair of classical solutions (to the cubic NLS flow) u_1, u_2 such that $\|f_1 - f_2\|_{H^s} \lesssim \delta$, where $f_i = u_i(0), i = 1, 2$, and $\|u_1(T_{\epsilon'}) - u_2(T_{\epsilon'})\|_{H^s} \simeq \epsilon'$. From the Gronwall's inequality, there exists $\epsilon_1 = \epsilon_1(T, f_1, f_2) > 0$ such that if $\epsilon \in (0, \epsilon_1)$, we obtain

$$\|u^{(\epsilon)}[f_1](T_{\epsilon'}) - u^{(\epsilon)}[f_2](T_{\epsilon'})\|_{H^s} \gtrsim \frac{\epsilon'}{2},$$

which proves the desired claim. $\square$

3.4 Soliton Solutions in High Dimensions.

We consider solutions of the form $u^{(\epsilon)}(x,t) = e^{i\tau t} Q_{\epsilon,\tau}(x)$ where u satisfies (3.1), which after substitution yields

$$\epsilon^2 \Delta^2 Q_{\epsilon,\tau} - \Delta Q_{\epsilon,\tau} - J_\epsilon(|Q_{\epsilon,\tau}|^2) Q_{\epsilon,\tau} + \tau Q_{\epsilon,\tau} = 0, \tag{3.12}$$

where we consider $\tau > 0$ and assume $Q_{\epsilon,\tau}$ is real-valued; we drop the subscript notation whenever ϵ, τ are fixed. Whereas Fang-Segata-Wu use the method of constrained minimization to obtain ground state solutions in $d = 1, 2, 3$ (see (Fang et al., 2018) and references therein), we use the mountain-pass theorem to obtain ground state solutions in $1 \leq d \leq 9$. Our method directly comes from that in (Zhao et al., 2015) where they studied the soliton solutions to a Schrödinger-type equation generated by the second harmonic generation, a nonlinear optical process that has applications in laser physics. Due to a sufficient amount of overlap with the work of Zhao-Zhao-Shi, we give a sketch of our proof, highlighting the key difference in technical details that rises from a different choice in the function space due to the singular perturbation from the biharmonic operator.

We approach the analysis of (3.12) from variational point of view. For real-valued $u \in H^2(\mathbb{R}^d)$, $v \in H^1(\mathbb{R}^d)$, define the action functional:

$$I(u,v) = \int\limits_{\mathbb{R}^d} \frac{\epsilon^2 (\Delta u)^2}{2} + \frac{|\nabla u|^2}{2} + \frac{\tau u^2}{2} + \frac{\epsilon^2 |\nabla v|^2}{4} + \frac{v^2}{4} - \frac{u^2 v}{2} dx$$

where for convenience, we denote the integrand as $\mathscr{L}$ for Lagrangian. Then formally, we obtain the following Euler-Lagrange equation:

$$\begin{cases} \epsilon^2 \Delta^2 u - \Delta u + \tau u - uv = 0 \\ -\epsilon^2 \Delta v + v - u^2 = 0, \end{cases} \tag{3.13}$$

where we require (u, v) to vanish at the infinity. We are interested in the existence of strictly positive solution that obtains the minimum value of the action.

Proposition 3.4.1. *For $1 \leq d \leq 9$. there exists a pair $(\overline{u}, \overline{v}) \in H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$ of smooth, strictly positive solution to (3.13) that minimises the action. There exists no non-trivial smooth solution when $d \geq 12$.*

From the proposition above, $\overline{u}$ is our desired solution for (3.12). Since the action functional I is sufficiently smooth, in fact $I \in C^2(\mathscr{H}, \mathbb{R})$ where $\mathscr{H} = H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$, every weak solution to (3.13) equipped with the boundary condition $\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = 0$ rises from the *critical points* of the action, or the collection of $x \in \mathscr{H}$ such that $I'(x) = 0$.[2] Moreover since the linear part of the Lagrangian has a moderate growth rate, in fact quadratic in u, v and their higher derivatives, whereas the nonlinear part is smooth in (u, v), every weak solution is a classical solution. Hence our goal is to find a *ground state* $(\overline{u}, \overline{v}) \in \mathscr{H} \setminus \{0\}$ such that $I'(\overline{u}, \overline{v}) = 0$ and $I(\overline{u}, \overline{v}) = \inf \{I(u, v) : I'(u, v) = 0, (u, v) \in \mathscr{H} \setminus \{0\}\}$. We first check that I is well-defined.

Lemma 3.4.1. *Let $1 \leq d \leq 10$. For all $u, v \in H^2(\mathbb{R}^d)$, $w \in H^1(\mathbb{R}^d)$, we have $\|uvw\|_{L^1} \lesssim_d \|u\|_{H^2} \|v\|_{H^2} \|w\|_{H^1}$. This estimate fails for $d \geq 11$.*

Since the estimates in the proof below are used throughout, we include them explicitly.

proof of lemma 3.4.1. For $d = 1, 2$, a straightforward application of Sobolev embed-

[2] By $\oplus$, we mean the Hilbert space direct sum where $\langle (u, v), (\phi, \psi) \rangle_{\mathscr{H}} = \langle u, \phi \rangle_{H^2} + \langle v, \psi \rangle_{H^1}$. We shall drop the subscript notation whenever the context is clear. Having equipped $\mathscr{H}$ with an inner product, we can discuss how regular the action is. Recall that $I \in C^1(\mathscr{H}, \mathbb{R})$ if for every $x \in \mathscr{H}$, there exists unique $I'(x) \in \mathscr{H}^*$ such that $I(x + h) = I(x) + \langle I'(x), h \rangle + o(h)$ as $h \xrightarrow{\mathscr{H}} 0$ where $\langle \cdot, \cdot \rangle$ is the Hilbert space dual pairing, and the map $x \mapsto I'(x)$ is continuous under the Hilbert space topology. Further recall that $I \in C^2(\mathscr{H}, \mathbb{R})$ if $I \in C^1(\mathscr{H}, \mathbb{R})$ and for every $x \in \mathscr{H}$, there exists unique $I''(x) \in \mathscr{B}(\mathscr{H}, \mathscr{H}^*)$ such that $I'(x + h) = I'(x) + I''(x)h + o(h)$ as $h \xrightarrow[\mathscr{H}]{} 0$, and the map $x \mapsto I''(x)$ continuous where $\mathscr{B}(\mathscr{H}, \mathscr{H}^*)$ is endowed with the norm operator topology.

ding yields

$$\|uvw\|_{L^1} \leq \|u\|_{L^3}\|v\|_{L^3}\|w\|_{L^3} \lesssim \|u\|_{H^2}\|v\|_{H^2}\|w\|_{H^1}.$$

Likewise for $d = 3, \ldots, 10$,

$$\|uvw\|_{L^1} \leq \|u\|_{L^{\frac{4d}{d+2}}}\|v\|_{L^{\frac{4d}{d+2}}}\|w\|_{L^{\frac{2d}{d-2}}} \lesssim \|u\|_{H^2}\|v\|_{H^2}\|w\|_{H^1}.$$

However the nonlinear estimate $\|uvw\|_{L^1} \lesssim \|u\|_{H^2}\|v\|_{H^2}\|w\|_{H^1}$ fails for $d \geq 11$. To see this, let $\hat{\phi} \in C_c^\infty(\mathbb{R}^d)$ be a smooth radial bump with the Fourier support $|\xi| \in [0,1]$. Let $\hat{u}(\xi) = \hat{\phi}(\xi) - \hat{\phi}(2\xi)$. For $k \in \mathbb{N}$, define $u_k(x) = \mathcal{F}^{-1}[\hat{u}(\frac{\cdot}{2^k})](x) = 2^{dk}u(2^k x)$. An explicit computation yields

$$\|u_k\|_{L^3}^3 \simeq 2^{2dk};\ \|u_k\|_{H^s} \simeq 2^{(s+\frac{d}{2})k},$$

where $s \in \mathbb{R}$ and the implicit constant only depends on ϕ. If the nonlinear estimate were to hold, then

$$2^{2dk} \lesssim 2^{2(2+\frac{d}{2})k}2^{(1+\frac{d}{2})k} = 2^{(5+\frac{3d}{2})k} \Rightarrow 1 \lesssim 2^{(5-\frac{d}{2})k} \Rightarrow d \leq 10.$$

$\square$

The version of mountain-pass theorem that we use (Willem, 1997, Theorem 1.15) assumes that the action is C^2, which can be checked by the following direct computation:

Lemma 3.4.2. *Let $(u,v), (\phi,\psi), h = (h_1,h_2) \in \mathscr{H}$. Then the first two derivatives are as follows:*

$$\langle I'(u,v),(\phi,\psi)\rangle = \int \epsilon^2 \Delta u \Delta\phi + \nabla u \cdot \nabla\phi + \tau u\phi + \frac{\epsilon^2}{2}\nabla v \cdot \nabla\psi + \frac{v\psi}{2} - \frac{u^2\psi}{2} - uv\phi.$$

$$\langle I''(u,v)h,(\phi,\psi)\rangle = \langle I'(h_1,h_2),(\phi,\psi)\rangle - \int (uh_2 + h_1 v)\phi + uh_1\psi.$$

The non-existence of non-trivial solutions in $d \geq 12$ immediately follows from the following *Pohozaev's identity*, which can be proved by multiplying the first and second equation of (14) by $x \cdot \nabla u$ and $x \cdot \nabla v$, respectively, and integrating by parts.

$$\begin{aligned} -2\epsilon^2(d-4)\int(\Delta u)^2 - 2(d-2)\int|\nabla u|^2 - 2\tau d\int u^2 \\ -\epsilon^2(d-2)\int|\nabla v|^2 - d\int v^2 + 2d\int u^2 v = 0. \end{aligned} \tag{3.14}$$

If $(u,v) \in \mathscr{H}$ is a critical point, then $\langle I'(u,v),(u,v)\rangle = 0$, and writing this explicitly,

$$\int \epsilon^2(\Delta u)^2 + |\nabla u|^2 + \tau u^2 + \frac{\epsilon^2}{2}|\nabla v|^2 + \frac{v^2}{2} - \frac{3}{2}u^2 v = 0. \tag{3.15}$$

Combining the two equations above, we obtain

$$(8-\frac{2d}{3})\epsilon^2\int(\Delta u)^2 + (4-\frac{2d}{3})\int|\nabla u|^2 + (2-\frac{d}{3})\epsilon^2\int|\nabla v|^2 = \frac{2\tau d}{3}\int u^2 + \frac{d}{3}\int v^2, \tag{3.16}$$

from which we deduce that for $d \geq 12$, only trivial solution $(0,0)$ exists, and this proves the second part of proposition 3.4.1. Note that if $\epsilon = 0$, then non-trivial solutions do not exist for $d \geq 6$.

For $d = 11$, the action is unbounded and for $d = 10$, we happen to be at the borderline case of the concentration-compactness lemma, a crucial tool in our argument, and therefore we focus on $1 \leq d \leq 9$.

Lemma 3.4.3. *(Lions, 1984, Lemma 1.1) Let $p \in (1,\infty]$ with $p \neq \frac{2d}{d+2}$ if $p < d$. Let $p^* = \frac{dp}{d-p}$ if $p < d$ and $p^* = \infty$ if $p \geq d$. Suppose $\{u_n\} \subseteq L^2(\mathbb{R}^d)$ is bounded and $\{\nabla u_n\} \subseteq L^p(\mathbb{R}^d)$ is bounded. If $\lim\limits_{n\to\infty} \sup_{y\in\mathbb{R}^d}\int_{B(y,1)}|u_n|^2 = 0$, then $u_n \xrightarrow[n\to\infty]{} 0$ in $L^\alpha(\mathbb{R}^d)$ for all $\alpha \in (2,p^*)$.*

Before we apply the mountain-pass argument on I, we recall some useful notions. For $c \in \mathbb{R}$, a sequence $\{x_n\} \subseteq \mathscr{H}$ is called a *Palais-Smale sequence at c*, or $(PS)_c$ sequence for short, if $I(x_n) \xrightarrow[n\to\infty]{} c$ and $I'(x_n) \xrightarrow[n\to\infty]{\mathscr{H}^*} 0$. In practice, we generate a sequence of approximate solutions that is of $(PS)_c$ where c is defined such that we can minimise the action using the properties of *Nehari manifold*:

$$N := \{x \in \mathscr{H} \setminus \{0\} : \langle I'(x), x\rangle = 0\}.$$

Lemma 3.4.4. *If $x \in N$, then $I(x) = \max_{t\geq 0} I(tx)$.*
Moreover let $c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$ where

$$\Gamma = \{\gamma \in C([0,1], \mathscr{H}) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Then we have the following minimax identity:

$$c = \inf\{I(u,v) : (u,v) \in N\} = \inf\{I(u,v) : I'(u,v) = 0, (u,v) \in \mathscr{H} \setminus \{0\}\}.$$

Remark 3.4.1. *The analogue of the result above is proved in (Zhao et al., 2015). However it is straightforward to check that essentially the same proof works for our Lagrangian that has an additional Laplacian term.*

A non-trivial critical point is necessarily an element in the Nehari manifold though the converse need not be true. The following lemma is likely to be well-known, but we provide a proof for the sake of completion.

Lemma 3.4.5. *If $x \in N$ and $I(x) = c$, then $I'(x) = 0$.*

proof of lemma 3.4.5. Assume $I'(x) \neq 0$. By continuity of I', there exists a small neighbourhood $S \subseteq \mathscr{H}$ around x such that $\{I'(y) : y \in S\}$ is uniformly bounded away from $0 \in \mathscr{H}^*$. By the following deformation lemma ((Willem, 1997, Lemma 2.3)), there exists $\eta \in C([0,1] \times \mathscr{H}, \mathscr{H})$ that lowers the sublevel set of I at the value c; more precisely, we will use properties (i), (v) and (vi) of (Willem, 1997, Lemma 2.3).

On the other hand, by Lemma 3.4.1, we obtain $I(u,v) \gtrsim \|(u,v)\|^2_{\mathscr{H}}$, and hence I has a strict local minimum at the origin, which implies $c > 0$. As in (Zhao et al., 2015, Lemma 2.4), one can show that $f(t) := I(tx)$ is a C^2-function that obtains the global maximum at $t = 1$, $\lim_{t\to\infty} f(t) = -\infty$ and $f''(t) < 0$ for all $t \in (0,\infty)$. We consider $g(t) := I(\eta(1,tx))$ as a function of $t \in [0,\infty)$. By (v), $g(t) \leq f(t)$. Near the origin, by (i) we have $g(t) = f(t)$ and therefore is strictly increasing. Moreover for all $t > T$ for $T > 0$ sufficiently large, $g(t) = f(t)$ and therefore monotonically decreases to $-\infty$. Around $t = 1$, $g(t) < c$ by (vi). Note that $\gamma(t) \coloneqq \eta(1, tx)$, or more precisely an appropriate re-parameterisation of γ, contributes to the mountain-pass value, and therefore

$$c \leq \max_{t \geq 0} g(t) < c.$$

$\square$

proof of proposition 3.4.1. We first claim that I has at least one non-trivial critical point $(\overline{u}, \overline{v}) \in \mathscr{H} \setminus \{0\}$ such that $I(\overline{u}, \overline{v}) \leq c$. Invoking the definition of c defined in Lemma 3.4.4, we apply the mountain-pass theorem to obtain a $(PS)_c$ sequence (u_n, v_n), which is bounded in $\mathscr{H}$. By compactness, there exists $\delta \geq 0$ and a subsequence of (u_n, v_n) (without re-labelling the index) such that

$$\delta = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \int\limits_{B_1(y)} u_n^2 + v_n^2.$$

If $\delta > 0$, then our argument proceeds verbatim as in (Zhao et al., 2015, Theorem 2.2), thereby proving our claim; the sequence (u_n, v_n) weakly converges to some non-trivial critical point $(\overline{u}, \overline{v})$ up to translation and by Fatou's lemma, one can show $I(\overline{u}, \overline{v}) \leq c$. Then by Lemma 3.4.4, we have $I(\overline{u}, \overline{v}) = \inf \{I(u, v) : I'(u, v) = 0, (u, v) \in \mathscr{H} \setminus \{0\}\}$, a ground state solution. Going back to the equation (3.13), $\overline{v} > 0$ by strong maximum principle. If $|\overline{u}| \in H^2(\mathbb{R}^d)$, then it is straightforward to check $I(\overline{u}, \overline{v}) = I(|\overline{u}|, \overline{v})$ and $(|\overline{u}|, \overline{v}) \in N$, which by lemma 3.4.5 implies $I'(|\overline{u}|, \overline{v}) = 0$. We can choose $\overline{u} \geq 0$ without loss of generality, which then implies $\overline{u} > 0$ by another application of strong maximum principle. Hence it suffices to show $|\overline{u}| \in H^2(\mathbb{R}^d)$ and $\delta > 0$[3].

Recall that $\overline{u}$ is a classical solution that satisfies $A\overline{u} \coloneqq (\epsilon^2 \Delta^2 - \Delta + \tau)\overline{u} = J_\epsilon(\overline{u}^2)\overline{u}$ pointwise. Hence $|\overline{u}| \in H^1(\mathbb{R}^d)$ satisfies $A|\overline{u}| = J_\epsilon(\overline{u}^2)|\overline{u}|$ a.e., or more precisely, on an open subset $\left\{x \in \mathbb{R}^d : \overline{u} \neq 0\right\}$. Since A is a linear elliptic operator of order 4, if $J_\epsilon(\overline{u}^2)|\overline{u}| \in H^\sigma(\mathbb{R}^d)$ for some $\sigma \leq 1$, then $|\overline{u}| \in H^{\sigma+4}(\mathbb{R}^d)$. In fact, this process can be iterated infinitely many times to show $|\overline{u}| \in H^\infty(\mathbb{R}^d) \coloneqq \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^d)$[4].

[3] If $\epsilon = 0$, then the appropriate energy class is $H^1(\mathbb{R}^d)$, and it is generally true that $f \in H^1(\mathbb{R}^d)$ implies $|f| \in H^1(\mathbb{R}^d)$. This is not true when $\epsilon > 0$ and thus the appropriate energy class is $H^2(\mathbb{R}^d)$.

[4] Since $\|\overline{u}^2\|_{H^2} \lesssim \|\overline{u}\|_{H^2} \|\overline{u}\|_{L^\infty} < \infty$, we have $J_\epsilon(\overline{u}^2) \in H^4(\mathbb{R}^d)$ by Lemma 3.2.2. By keeping

If $\delta = 0$, then from

$$c = \lim_{n\to 0} I(u_n, v_n) = \lim_{n\to 0} \Big(I(u_n, v_n) - \frac{1}{2}\langle I'(u_n, v_n), (u_n, v_n)\rangle \Big) = \frac{1}{4}\lim_{n\to 0} \int u_n^2 v_n,$$

and from lemma 3.4.3, we conclude $c = 0$, a contradiction by the strict local minimality of I at the origin. To elaborate, we first consider the case $1 \le d \le 5$ where

$$\|u_n^2 v_n\|_{L^1} \le \|u_n\|_{L^3}^2 \|v_n\|_{L3} \xrightarrow[n\to\infty]{} 0,$$

where the last limit follows from lemma 3.4.3 where we use $(p, \alpha) = (2, 3)$. However for $6 \le d \le 9$, we need to use the fact that $\{u_n\}$ is bounded not only in $H^1(\mathbb{R}^d)$, but also in $H^2(\mathbb{R}^d)$ to rule out $\delta = 0$. In particular, consider the following:

$$\|u_n^2 v_n\|_{L^1} \le \|u_n\|^2_{L^{\frac{4d}{d+2}}} \|v_n\|_{L^{2^*}} \lesssim \|u_n\|^2_{L^{\frac{4d}{d+2}}} \|v_n\|_{H^1} \xrightarrow[n\to\infty]{} 0,$$

where the last limit follows from lemma 3.4.3, where we use $(p, \alpha) = (\frac{2d}{d-2}, \frac{4d}{d+2})$, since $2 < \frac{4d}{d+2} < \frac{2d}{d-4}$, and from an apriori uniform bound of $\{v_n\}$ in $H^1(\mathbb{R}^d)$. Unfortunately $d = 10$ is the borderline case where the concentration compactness lemma does not apply. $\square$

3.5 Appendix

proof of lemma 3.2.3. The first statement is a hyperbolic trigonometric representation of a cubic root for a unique real root, which can be verified by a direct substitution.

For the second statement, since $|r(-\xi_1)| \ge |r(\xi_1)|$ for all $\xi_1 \ge 0$, it suffices to show $|r(\xi_1)| \gtrsim_\epsilon |\xi|^{4/3}$ for $\xi_1 \ge 0$. Observe that $|r|$ is a decreasing function on $\xi_1 \in (0, \xi)$ since $P(0) = \Big(1+\epsilon^2(\xi_1-\xi)^2\Big)(\xi_1-\xi)^2+\tau$ is decreasing and $\partial_{\xi_2}P(0) = |\xi_1|^{2/3}(\epsilon^2\xi_1^2+2)$ is increasing on $\xi_1 \in (0, \xi)$; sketch a graph to see this. Hence for $\xi_1 \in (0, \xi]$,

$$|r(\xi_1)| \gtrsim_\epsilon |\xi|^{1/3}\langle \epsilon\xi\rangle \sinh\left(\frac{1}{3}\sinh^{-1}\left(3\sqrt{3}\epsilon\frac{\frac{d_\epsilon(\xi)}{2}}{|\xi|(\epsilon^2\xi^2+2)^{3/2}}\right)\right) \gtrsim_\epsilon |\xi|^{4/3}.$$

track of multiplication properties of Sobolev spaces, we can show $J_\epsilon(\overline{u}^2)|\overline{u}| \in H^{-1}(\mathbb{R}^d)$, which implies $|\overline{u}| \in H^3(\mathbb{R}^d)$; see (Behzadan and Holst, 2015, Theorem 8.1). Another iteration of elliptic regularity argument is sufficient to conclude $|\overline{u}| \in H^{\frac{9}{2}+}(\mathbb{R}^d)$, a Sobolev algebra for $1 \le d \le 9$. We iteratively use this bootstrapping argument to conclude $|\overline{u}| \in H^\infty(\mathbb{R}^d)$.

For $\xi_1 \in (\xi, 2\xi]$,

$$\begin{aligned}|r(\xi_1)| &\simeq_\epsilon |\xi_1|^{1/3}\langle\epsilon\xi_1\rangle \sinh\left(\frac{1}{3}\sinh^{-1}\left(3\sqrt{3}\epsilon\frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}}\right)\right)\\ &\geq |\xi|^{1/3}\langle\epsilon\xi\rangle \sinh\left(\frac{1}{3}\sinh^{-1}\left(3\sqrt{3}\epsilon\frac{\frac{d_\epsilon(\xi)}{2}}{2|\xi|(4\epsilon^2\xi^2+2)^{3/2}}\right)\right) \gtrsim_\epsilon |\xi|^{4/3},\end{aligned}$$

where the second inequality holds since for $\xi > 1$,

$$\frac{\frac{d_\epsilon(\xi)}{2}}{2|\xi|(4\epsilon^2\xi^2+2)^{3/2}} \gtrsim \frac{d_\epsilon(\xi)}{\xi^4} \gtrsim_\epsilon 1.$$

For $\xi_1 \in (2\xi, \infty)$, we make use of $\frac{\xi_1-\xi}{\xi_1} \geq \frac{1}{2}$ to show that the argument inside $\sinh^{-1}$ is bounded below by a positive constant as follows:

$$\begin{aligned}\frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}} &\geq \frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}}\\ &\gtrsim_\epsilon \frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2}{\xi_1^4} \gtrsim_\epsilon 1,\end{aligned}$$

and this proves our claim as in the previous case for $\xi_1 \in (\xi, 2\xi)$.

To show the third statement, we recall that $|r|$ is decreasing for $\xi_1 \in [0, \xi)$, and therefore for such ξ_1

$$|r(\xi_1)| \leq |r(0)| \simeq_\epsilon (d_\epsilon(\xi)+\tau)^{1/3} \lesssim d_\epsilon(\xi)^{1/3}$$

Furthermore for $\xi_1 \in [0, \frac{\xi}{2})$, we claim $|r(-\xi_1)| \lesssim_\epsilon |r(\xi_1)|$. Consequently,

$$|r(\xi_1)| \lesssim_\epsilon d_\epsilon(\xi)^{1/3}$$

for $|\xi_1| \in [0, \frac{\xi}{2})$.

It suffices to show that there exists a constant $C > 0$ such that the following holds:

$$\frac{\sinh\left(\frac{1}{3}\sinh^{-1}\left(3\sqrt{3}\epsilon\frac{(1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2+\tau}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}}\right)\right)}{\sinh\left(\frac{1}{3}\sinh^{-1}\left(3\sqrt{3}\epsilon\frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}}\right)\right)} \leq C$$

Let X, Y be the arguments inside the numerator and denominator of the hyperbolic sine, respectively. Then,

$$\frac{\sinh(X)}{\sinh(Y)} = \frac{e^X - e^{-X}}{e^Y - e^{-Y}} \leq \frac{e^X}{e^Y - e^{-Y}} \lesssim_\epsilon \frac{e^X}{e^Y}$$

where the last inequality follows since Y is bounded below by a positive constant since

$$\frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}} \geq \frac{\frac{d_\epsilon(\xi)}{2}}{\frac{|\xi|}{2}(\frac{\epsilon^2\xi^2}{4}+2)^{3/2}} \gtrsim_\epsilon 1.$$

Then using the following identity

$$\sinh^{-1}(t) = \ln(t+\sqrt{1+t^2}), \forall t \in \mathbb{R},$$

and letting α_1, α_2 be the arguments in the numerator and denominator of $\sinh^{-1}$, respectively,

$$\left(\frac{e^X}{e^Y}\right)^3 = \frac{\alpha_1+\sqrt{1+\alpha_1^2}}{\alpha_2+\sqrt{1+\alpha_2^2}} \leq \frac{\alpha_1+\sqrt{1+\alpha_1^2}}{\alpha_2} \lesssim_\epsilon \frac{\alpha_1}{\alpha_2},$$

since α_1 is bounded below by a positive constant (similar to $Y \gtrsim_\epsilon 1$). Note that our hypothesis on ξ_1 implies

$$-3 \leq \frac{\xi_1+\xi}{\xi_1-\xi} \leq -1.$$

Then,

$$\begin{aligned}\frac{\alpha_1}{\alpha_2} = \frac{(1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2+\tau}{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau} &= \frac{(1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2}{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau} \\ &\quad + \frac{\tau}{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau} \\ &\leq \frac{(1+\epsilon^2(\xi_1+\xi)^2)(\xi_1+\xi)^2}{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2}+1 \\ &\leq 9\Big(1+\Big(\frac{\xi_1+\xi}{\xi_1-\xi}\Big)^2\Big)+1 \leq 91,\end{aligned}$$

as desired.

For $\xi_1 > \frac{\xi}{2}$, we claim $\frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+\tau}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}} \lesssim \frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2+d_\epsilon(\xi)}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}} \lesssim_\epsilon 1$, which yields the claim. Note that our hypothesis on ξ_1 implies

$$-1 \leq \frac{\xi_1 - \xi}{\xi_1} \leq 1,$$

from which

$$\begin{aligned}\frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2 + d_\epsilon(\xi)}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}} &= \frac{(1+\epsilon^2(\xi_1-\xi)^2)(\xi_1-\xi)^2}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}} + \frac{d_\epsilon(\xi)}{|\xi_1|(\epsilon^2\xi_1^2+2)^{3/2}} \\ &\lesssim \frac{\Big(1+\epsilon^2(\xi_1-\xi)^2\Big)(\xi_1-\xi)^2}{\epsilon^3\xi_1^4} + c_\epsilon \lesssim \frac{1}{\epsilon^3\xi^2} + \frac{1}{\epsilon} + c_\epsilon \\ &\lesssim_\epsilon 1,\end{aligned}$$

as desired. Arguing as above, one can show $|r(-\xi_1)| \lesssim_\epsilon |\xi_1|^{1/3}\langle\epsilon\xi_1\rangle$ for $\xi_1 > \frac{\xi}{2}$ since for such ξ_1, $1 \leq \frac{\xi_1+\xi}{\xi_1} \leq 3$. $\square$

Chapter 4

Modified Strichartz Estimates and the Periodic Nonlinear Schrödinger Equations

4.1 Introduction.

This paper is concerned with the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u - \epsilon^2 \Delta^2 u = N(u),\ (x,t) \in \mathbb{T} \times \mathbb{R} := \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}. \\ u(0) = u_0 \in H^s(\mathbb{T}), \end{cases} \tag{4.1}$$

where for $\epsilon > 0$ and $\mu = \pm 1$, the nonlinearity takes the form

$$N_1(u) = \mu J_\epsilon(|u|^2)u := \mu(I - \epsilon^2\Delta)^{-1}(|u|^2)u, \text{ or } N_2(u) = \mu|u|^2 u, \text{ or } N_3(u) = \mu|u|^4 u.$$

Let $U_\epsilon(t) = e^{it(-\epsilon^2\Delta^2+\Delta)}$ be defined via functional calculus or as the Fourier multiplier $k \mapsto e^{-it(\epsilon^2|k|^4+|k|^2)}$, where we use the Fourier transform convention

$$\hat{u}(k) = \int_0^{2\pi} u(x)e^{-ikx}dx;\ \check{v}(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} v(k)e^{ikx}.$$

The desired existence result is obtained via a fixed point argument that considers the integral form of (4.1) (Duhamel formula):

$$u(t) = U_\epsilon(t)u_0 - i\int_0^t U_\epsilon(t-\tau)[N(u)(\tau)]d\tau. \tag{4.2}$$

We are motivated by a specific form of the nonlinearity, namely $N_1(u)$ with $\mu = -1$, where the model above forms the adiabatic limit of the quantum Zakharov system. This system models the nonlinear interaction between Langmuir waves and ion-acoustic waves in plasma physics. The case $\epsilon > 0$ accounts for a more-detailed quantum effect caused by, for instance, the presence of a dense and cold plasma. For more physical background on the quantum Zakharov system, see (Garcia et al., 2005; Haas and Shukla, 2009; Haas, 2011). For more details on the adiabatic limiting transition of the Zakharov system to the NLS, see (Ozawa and Tsutsumi, 1992; Schochet and Weinstein, 1986). Finally, for more details on the mathematical analysis of the quantum Zakharov system on $\mathbb{R}^d$, see (Chen et al., 2017; Fang et al., 2016; Fang et al., 2019; Guo et al., 2013a; Jiang et al., 2014). We also remark that (4.1) is a Hamiltonian PDE admitting conservation of L^2 norm. For the particular nonlinearity stated above, we have

$$ENERGY[u(t)] = \frac{\epsilon^2}{2}\|\partial_{xx}u\|_{L^2}^2 + \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{4}\|J_\epsilon(|u|^2)|u|^2\|_{L^1}.$$

We studied low-regularity well-posedness of the model above on $\mathbb{R}$ in (Choi, 2019). Here we handle the more difficult (and less dispersive) version of the problem, on the torus with periodic boundary condition. It is a standard fact that dispersive estimates are weaker on a bounded domain than on the full space, and hence to obtain the well-posedness theory of (4.1), we need a priori estimates that do not depend on the standard Strichartz estimate for $\mathbb{R}$. To achieve this goal, we use the method of Fourier restriction norms, where we consider solutions of the form $u \in C([0,T], H^s(\mathbb{T})) \cap X_T^{s,b}$ for appropriate real parameters s, b. It has been shown (in abundance) that solutions that are (space-time) square-integrable with respect to the *elliptic derivative* $\langle \nabla \rangle^s$ and the *dispersive derivative* $\langle i\partial_t - L \rangle^b$ exhibit useful a priori estimates that often yield low-regularity well-posedness results. Heuristically the $X^{s,b}$

space takes into account the geometry of a dispersion relation in the Fourier space, which often yields smoothing estimates in the L^2-sense. However the uniqueness of a solution is technically *not* obtained in $C([0,T], H^s(\mathbb{T}))$, and therefore we do not obtain *unconditional uniqueness* directly from such an $X^{s,b}$ method[1].

Our main tool is a continuous embedding of $X^{s,b}$ into the Strichartz space $L^p(\mathbb{T} \times \mathbb{R})$ for $p = 4, 6$, cases where Fourier-analytic tools are available; here we denote $\|G\|_{L^p} := \|G\|_{l^p_k L^p_\tau}$ whenever $G = G(k, \tau)$. We are motivated by the following result of (Oh and Tzvetkov, 2017):

$$\|u\|_{L^4(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{5}{16}}},$$

where the linear operator is ∂_{xx}^2, which correspondingly yields a quartic monomial dispersion relation. Unlike (Oh and Tzvetkov, 2017), where solutions exhibit the scaling symmetry $u_\lambda(x,t) := \lambda^{-2} u(\frac{x}{\lambda}, \frac{t}{\lambda^4})$ with $u_0(x) = \lambda^{-2} u_0(\frac{x}{\lambda})$, our model has no such scaling symmetry. For similar estimates associated to the linear operators of Schrödinger and KdV equations, see (Bourgain, 1993a; Tao, 2006; Erdoğan and Tzirakis, 2016).

Theorem 4.1.1. *For $w(k) = \epsilon^2 k^4 + k^2$ with $\epsilon > 0$, we have $\|u\|_{L^4(\mathbb{T}\times\mathbb{R})} \lesssim \epsilon^{-\frac{1}{8}} \|u\|_{X^{0,\frac{5}{16}}}$ for every $u \in C_c^\infty(\mathbb{T} \times \mathbb{R})$. Moreover we have $\|u\|_{L^6(\mathbb{T}\times\mathbb{R})} \lesssim \epsilon^{-\frac{1}{6}} \|u\|_{X^{0,\frac{5}{12}}}$ for every $u \in C_c^\infty(\mathbb{T} \times \mathbb{R})$.*

Corollary 4.1.1. *The Cauchy problem (4.1), with nonlinearities N_i for $i = 1, 2, 3$, is globally well-posed in $H^s(\mathbb{T})$ for every $s \geq 0$ and $\epsilon > 0$. The data-to-solution map is locally Lipschitz continuous.*

---

[1] Unconditional uniqueness of (1) with $N_i(u)$, $i = 1, 2, 3$ for $s > \frac{1}{2}$ follows immediately from the Sobolev algebra property of $H^s(\mathbb{T})$. For $s \geq \frac{1}{6}$ and $N_i(u)$, $i = 1, 2$, one can imitate the proof of (Guo et al., 2013b), i.e., by applying normal form reductions infinitely many times, to obtain unconditional uniqueness. However we would like to ask whether the presence of fourth-order dispersion makes it possible to perform differentiation by parts *finitely* many times, instead of *infinitely* many times, and still obtain unconditional uniqueness. We note that for KdV, such finite iteration (in fact, twice) yields unconditional uniqueness in $L^2(\mathbb{T})$ (Babin et al., 2011).

Remark 4.1.1. *The result above can be generalized to higher order derivatives. For instance if* $w(k) = k^\delta$ *with* $\delta \in \{2, 3, 4, \dots\}$, *we have* $\|u\|_{L^4(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{\delta+1}{4\delta}}}$ *for every* $u \in C_c^\infty(\mathbb{T} \times \mathbb{R})$.

It is expected that the inequality of remark 4.1.1, for every integer $\delta \geq 2$, holds not only for monomials, but also for every polynomial w of degree δ with real coefficients. Since the proof of theorem 4.1.1 and remark 4.1.1 heavily depends on the algebraic structure of a given monomial, it seems unwieldy to produce a direct proof for the case of an arbitrary polynomial in the spirit of remark 4.1.1. Furthermore we would like to ask whether such embedding continues to hold when a polynomial dispersion relation is replaced by a continuous function that behaves asymptotically as a polynomial as $|k| \to \infty$.

For $b > \frac{1}{2}$, $s \in \mathbb{R}$ and some spacetime Banach space Y, we remark that embeddings of the type $X^{s,b} \hookrightarrow Y$ are closely related to the Strichartz estimates of the form $\|e^{-itL} f\|_Y \lesssim \|f\|_{H^s}$; see (Tao, 2006, Lemma 2.9). Here we are mainly interested in refining the b-index to $b < \frac{1}{2}$ so that some time element of the form T^β for $\beta > 0$ is extracted when performing a fixed point argument.

When $\epsilon = 0$, however, we remark that the embedding $X^{0,b} \hookrightarrow L^6(\mathbb{T} \times \mathbb{R})$ for $b < \frac{1}{2}$ fails to hold. If it were to hold, then local well-posedness for $N(u) = \pm|u|^4 u$ can be obtained via Picard iteration (for example, see the proof of Corollary 4.1.1), and this implies that the solution map (from H_x^s to $C_t^0 H_x^s$) is analytic; see (Bejenaru and Tao, 2006, Theorem 3). However Kishimoto showed that such solution map defined on $L^2(\mathbb{T})$, if it exists at all, fails to be C^5 for quintic (mass-critical) NLS; see (Kishimoto, 2014, Corollary 1.3). Our method yields a sequence of solutions to (4.1), say $\{u^\epsilon\}_{\epsilon>0}$, that corresponds to the quintic nonlinearity with $u^\epsilon(0) = u_0 \in L^2(\mathbb{T})$. For $T > 0$, $p \in (1, \infty)$, since $\{u^\epsilon\}_{\epsilon>0} \subseteq C([0,T], L^2(\mathbb{T})) \hookrightarrow L^p([0,T], L^2(\mathbb{T}))$ with $\|u^\epsilon\|_{L^p([0,T],L^2(\mathbb{T}))} = T^{\frac{1}{p}} \|u_0\|_{L^2}$, due to the (mass) conservation law and the reflexivity of

$L^p([0,T],L^2(\mathbb{T}))$, we can extract a weakly convergent subsequence in $L^p([0,T],L^2(\mathbb{T}))$. However without any extra regularity on the solutions, it is insufficient to show that the limit defines a strong solution. Hence our method of adding a small fourth-order dispersion seems unlikely to yield positive results for the mass-critical periodic NLS.

On the other hand, we show that (4.1) is mildly ill-posed in negative Sobolev spaces, in the spirit of Burq, Gérard, and Tzvetkov (Burq et al., 2002).

Proposition 4.1.1. *If $s<0$ and $d\in\mathbb{N}$, then the data-to-solution map of (4.1), if it exists, fails to be uniformly continuous on closed balls of $H^s(\mathbb{T}^d)$ to $C([0,T],H^s(\mathbb{T}^d))$ for any $T>0$.*

The failure of uniform well-posedness itself does not disprove well-posedness in the Hadamard sense where only the continuity of data-to-solution is required. However it implies that a fixed point argument via Picard iteration cannot be applied in negative Sobolev spaces. One of the earliest works in the ill-posedness of dispersive models comes from that of Kenig, Ponce, and Vega (Kenig et al., 2001) where they exploited the Galilean invariance and the structure of ground state solutions of the focusing cubic NLS on $\mathbb{R}$. For more analysis on NLS with gauge-invariant power-type nonlinerities on $\mathbb{R}^d$, see Christ, Colliander, Tao (Christ et al., 2003b). For an analysis of norm inflation of NLS with more general nonlinearities on various domains, both compact and non-compact, see Kishimoto (Kishimoto, 2018).

Lastly we consider the regularity of solutions with respect to the parameter ϵ. We fix a nonlinearity motivated from the quantum Zakharov system. We are also interested in the NLS limit as $\epsilon\to 0$ and the case $\epsilon\to\infty$; note that $J_\epsilon \xrightarrow[\epsilon\to\infty]{} \mathbb{P}$ strongly in $L^2(\mathbb{T})$ where $\mathbb{P}f=\frac{1}{2\pi}\int f$ is the projector at the zeroth frequency while $J_\epsilon \xrightarrow[\epsilon\to\infty]{} 0$ strongly in $L^2(\mathbb{R}^d)$. For $\epsilon=0$, we denote u^0 by the well-posed solution to the focusing cubic NLS on $\mathbb{T}$ (Bourgain, 1993a).

This short article is organised as follows: section 2 contains proofs of modified

Strichartz estimates adapted to the Fourier restriction space. We remark that the estimates proved are sharp. Section 3 discusses the failure of uniform well-posedness of (4.1) in negative Sobolev spaces. Lastly section 4 discusses the NLS limit as $\epsilon \to 0$ for $N(u) = -J_\epsilon(|u|^2)u$. Here we complexify ϵ to emphasize the qualitative differences in the behavior of solutions depending on the perturbation parameter. In particular, we show the existence of $u_0 \in H^s(\mathbb{T})$ such that there exists no analytic extension of the solution map (whose initial condition is u_0) in the neighbourhood of $\epsilon = 0$ in $\mathbb{C}$.

4.2 Modified Strichartz Estimates.

proof of theorem 4.1.1 (L^4-estimate). We closely follow the proof of (Tao, 2006, Proposition 2.13). For $m \in \mathbb{N} \cup \{0\}$, define the following dyadic projector $\chi_m(\tau, k) := \chi_{2^m \leq \langle \tau + w(k) \rangle < 2^{m+1}}$ and $\widehat{u_{2^m}}(\tau, k) := \hat{u}(\tau, k)\chi_m(\tau, k)$. Then

$$\|u\|_{L^4}^2 = \|u^2\|_{L^2} \lesssim \sum_{m,n \geq 0} \|u_{2^m} u_{2^{m+n}}\|_{L^2} = \sum_{m,n \geq 0} \|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2}.$$

Since $\chi_m^2 = \chi_m$, we have

$$\begin{aligned} &\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}(\tau, k) \\ =& \sum_{k_1 \in \mathbb{Z}} \int \widehat{u_{2^m}}(\tau_1, k_1) \widehat{u_{2^{m+n}}}(\tau - \tau_1, k - k_1) \chi_m(\tau_1, k_1) \chi_{m+n}(\tau - \tau_1, k - k_1) d\tau_1. \end{aligned}$$

By applying the Cauchy-Schwarz inequality in τ_1, k_1, Hölder's inequality in $l_k^2 L_\tau^\infty$, and Young's inequality for convolution,

$$\begin{aligned} \|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2} &\leq \|\chi_m * \chi_{m+n}\|_{l_k^\infty L_\tau^\infty}^{1/2} \left\| |\widehat{u_{2^m}}|^2 * |\widehat{u_{2^{m+n}}}|^2 \right\|_{l_k^1 L_\tau^1}^{1/2} \\ &\leq \|\chi_m * \chi_{m+n}\|_{l_k^\infty L_\tau^\infty}^{1/2} \|u_{2^m}\|_{L^2} \|u_{2^{m+n}}\|_{L^2}. \end{aligned}$$

It remains to extract a sufficient decay (in m, n) from $\|\chi_m * \chi_{m+n}\|_{L^\infty}^{1/2}$. Since

$$\chi_m * \chi_{m+n}(\tau, k) = \sum_{k_1} \int \chi_m(\tau_1, k_1) \chi_{m+n}(\tau - \tau_1, k - k_1) d\tau_1,$$

each non-zero integral, for a fixed k_1, is at most $O(2^m)$. It remains to count how many k_1s give rise to non-zero integrals. For a fixed τ, k, if the integral is non-zero, then

$$\tau_1 + w(k_1) = O(2^m),\ \tau - \tau_1 + w(k - k_1) = O(2^{m+n}),$$

and therefore

$$\tau + w(k_1) + w(k - k_1) = O(2^{m+n}). \tag{4.3}$$

Viewing w as a function defined on $\mathbb{R}$, i.e., $w(x) = \epsilon^2 x^4 + x^2$, it can be shown algebraically that $x \mapsto w(x) + w(k - x) = w(x) + w(x - k)$ has a global minimum at $x = \frac{k}{2}$. Changing variable $k_1' = k_1 - \frac{k}{2}$ so that in the new coordinate, $x \mapsto w(x) + w(k - x)$ is centered at the origin, the LHS of (4.3) becomes

$$\begin{aligned}&\tau + 2\epsilon^2 k_1^4 + (2 + 3\epsilon^2 k^2)k_1^2 + \frac{k^2(4 + \epsilon^2 k^2)}{8} \\ =&\tau + 2\epsilon^2\Big(k_1^2 + \frac{1}{2}(\epsilon^{-2} + \frac{3k^2}{2})\Big)^2 - \Big(\epsilon^2 k^4 + k^2 + \frac{1}{2\epsilon^2}\Big),\end{aligned}$$

where we re-labelled k_1' into k_1. This implies that k_1s are constrained in finitely many intervals of length $O(\epsilon^{-\frac{1}{2}} 2^{\frac{m+n}{4}})$, and therefore

$$\|\chi_m * \chi_{m+n}\|_{L^\infty}^{1/2} \lesssim \epsilon^{-\frac{1}{4}} 2^{\frac{5m}{8} + \frac{n}{8}},$$

from which

$$\begin{aligned}\|u\|_{L^4}^2 &\lesssim \epsilon^{-\frac{1}{4}} \sum_{m,n\geq 0} 2^{\frac{5m}{8}+\frac{n}{8}} \|u_{2^m}\|_{L^2} \|u_{2^{m+n}}\|_{L^2} \\ = \epsilon^{-\frac{1}{4}} \sum_{n\geq 0} 2^{-\frac{3n}{16}} &\sum_{m\geq 0} 2^{\frac{5m}{16}} \|u_{2^m}\|_{L^2} 2^{\frac{5}{16}(m+n)} \|u_{2^{m+n}}\|_{L^2} \lesssim \epsilon^{-\frac{1}{4}} \|u\|^2_{X^{0,\frac{5}{16}}},\end{aligned} \tag{4.4}$$

where we use Cauchy-Schwarz inequality at the last step and

$$\|u\|_{X^{0,b}}^2 \simeq \sum_{m\geq 0} 2^{2bm} \|u_{2^m}\|_{L^2}^2,$$

by the Plancherel's Theorem. $\square$

proof of remark 4.1.1. As in the previous proof, we estimate $\|\chi_m * \chi_{m+n}\|_{L^\infty_{k,\tau}}^{1/2}$ in terms of dyadic powers in m and n such that the sum $\sum_{m,n\geq 0} \|\chi_m * \chi_{m+n}\|^{1/2}_{L^\infty_{k,\tau}} \|u_{2^m}\|_{L^2} \|u_{2^{m+n}}\|_{L^2}$ converges. As before, our problem reduces to estimating the cardinality of the following set:

$$E(\tau, k) = \left\{k_1 \in \mathbb{Z} : \tau + w(k_1) + w(k - k_1) = O(2^{m+n})\right\}.$$

We first assume $w(k) = k^\delta$ where $\delta \geq 2$ is even. Considering w as a function on $\mathbb{R}$ as before, $x \mapsto w(x) + w(k - x)$ is convex with a global minimum at $x = \frac{k}{2}$. Changing variable $x \mapsto x + \frac{k}{2}$, we have $|E(\tau, k)| = |\tilde{E}(\tau, k)|$ where

$$\tilde{E}(\tau, k) = \left\{k_1 \in \mathbb{Z} : \tau + w(k_1 + \frac{k}{2}) + w(\frac{k}{2} - k_1) = O(2^{m+n})\right\}.$$

For a fixed $k \in \mathbb{Z}$, since $\tilde{w}(k_1) \coloneqq w(k_1 + \frac{k}{2}) + w(\frac{k}{2} - k_1)$ is even (in k_1) and convex with a global minimum of $2 \cdot (\frac{k}{2})^\delta$, and the τ-term corresponds to translating $\tilde{w}(k_1)$, we have

$$\begin{aligned} \sup_{\tau\in\mathbb{R}} |\tilde{E}(\tau, k)| &\lesssim \left|\left\{k_1 : \tilde{w}(k_1) - 2 \cdot \left(\frac{k}{2}\right)^\delta = O(2^{m+n})\right\}\right| \\ &\leq \left|\left\{k_1 : k_1^\delta = O(2^{m+n})\right\}\right| \lesssim 2^{\frac{m+n}{\delta}}, \end{aligned}$$

where the second inequality holds since it can be shown by a direct computation that $\tilde{w}(k_1) - 2 \cdot \left(\frac{k}{2}\right)^\delta \geq k_1^\delta$ for all $k_1, k \in \mathbb{Z}$. Hence $\|\chi_m * \chi_{m+n}\|^{1/2}_{L^\infty_{k,\tau}} \lesssim 2^{\frac{(\delta+1)m}{2\delta} + \frac{n}{2\delta}}$, and the argument proceeds as in (4.4).

Now assume δ is odd. We closely follow the argument in (Erdoğan and Tzirakis, 2016, Theorem 3.18). Convexity of dispersion relation for even powers played a crucial role in the previous counting argument. For an odd-power dispersion relation, we consider low and high frequencies separately. Note that for $k = O(1)$, we have $w(k) = O(1)$, i.e., dispersion relation plays no crucial role for low frequencies. Hence as in (Erdoğan and Tzirakis, 2016), we have $\|\widehat{u_{2^m}} * \widehat{u_{2^{m+n}}}\|_{L^2_\tau l^2_{|k|\leq 2^a}} \lesssim 2^{\frac{a+m}{2}} \|u_{2^m}\|_{L^2} \|u_{2^{m+n}}\|_{L^2}$ for $a \geq 0$.

It remains to estimate $\|\chi_m * \chi_{m+n}\|^{1/2}_{L^\infty_\tau l^\infty_{|k|>2^a}}$ where, as before, each non-zero integral has $O(2^m)$ contribution. After changing variable $k_1 \mapsto k_1 + \frac{k}{2}$, $w(k_1) + w(k - k_1)$

becomes $w(k_1+\frac{k}{2})-w(k_1-\frac{k}{2})$, which by a direct computation amounts to:

$$\begin{aligned}\tilde{w}(k_1) &= (k_1+\frac{k}{2})^\delta-(k_1-\frac{k}{2})^\delta = 2\sum_{j \text{ even},\, 0\le j\le\delta}\binom{\delta}{j}k_1^j(\frac{k}{2})^{\delta-j}\\ &= 2k\sum_{j \text{ even},\, 0\le j\le\delta}\binom{\delta}{j}k_1^j\frac{k^{\delta-j-1}}{2^{\delta-j}}.\end{aligned}$$

Since $\tilde{w}$ is even and convex with a global minimum $2\cdot\left(\frac{k}{2}\right)^\delta$, we have

$$\begin{aligned}\sup_{\tau\in\mathbb{R}}|\tilde{E}(\tau,k)| &\lesssim \left|\left\{k_1 : \tilde{w}(k_1)-2\cdot\left(\frac{k}{2}\right)^\delta = O(2^{m+n})\right\}\right|\\ &= \left|\left\{k_1 : 2k\sum_{j \text{ even},\, 2\le j\le\delta}\binom{\delta}{j}k_1^j\frac{k^{\delta-j-1}}{2^{\delta-j}} = O(2^{m+n})\right\}\right|\\ &\lesssim \left|\left\{k_1 : \sum_{j \text{ even},\, 2\le j\le\delta}\binom{\delta}{j}k_1^j\frac{k^{\delta-j-1}}{2^{\delta-j}} = O(2^{m+n-a})\right\}\right|\\ &\le \left|\left\{k_1 : \frac{\delta}{2}k_1^{\delta-1} = O(2^{m+n-a})\right\}\right|,\end{aligned}$$

which implies $\|\chi_m*\chi_{m+n}\|^{1/2}_{L^\infty_\tau l^\infty_{|k|>2^a}} \lesssim 2^{\frac{\delta m+n-a}{2(\delta-1)}}$. Setting $a=\frac{m+n}{\delta}$ to equate the bounds from the low and high frequencies, we obtain

$$\|u_{2^m}u_{2^{m+n}}\|_{L^2_{x,t}} \lesssim 2^{\frac{(\delta+1)m}{2\delta}+\frac{n}{2\delta}}\|u_{2^m}\|_{L^2}\|u_{2^{m+n}}\|_{L^2},$$

from which the argument proceeds as in (4.4). □

proof of theorem 4.1.1 (L^6-estimate). Let $w(x)=\epsilon^2x^4+x^2$. We need to derive trilinear estimates on the dyadic components of u as the following computation suggests:

$$\begin{aligned}&\|u\|^3_{L^6_{x,t}} = \|u^3\|_{L^2_{x,t}} \lesssim \sum_{m,n,l\ge0}\|u_{2^m}u_{2^{m+n}}u_{2^{m+n+l}}\|_{L^2_{x,t}} = \sum_{m,n,l\ge0}\|\widehat{u_{2^m}}*\widehat{u_{2^{m+n}}}*\widehat{u_{2^{m+n+l}}}\|_{L^2}\\ &= \sum_{m,n,l\ge0}\left\|\sum_{k_1,k_2\in\mathbb{Z}}\iint_{\tau_1,\tau_2}\widehat{u_{2^m}}(\tau_1,k_1)\widehat{u_{2^{m+n}}}(\tau_2,k_2)\widehat{u_{2^{m+n+l}}}(\tau-\tau_1-\tau_2,k-k_1-k_2)\right\|_{L^2} \qquad (4.5)\\ &\le \sum_{m,n,l\ge0}\|\chi_m*\chi_{m+n}*\chi_{m+n+l}\|^{1/2}_{L^\infty_{k,\tau}}\|u_{2^m}\|_{L^2_{x,t}}\|u_{2^{m+n}}\|_{L^2_{x,t}}\|u_{2^{m+n+l}}\|_{L^2_{x,t}}.\end{aligned}$$

We claim $\|\chi_m * \chi_{m+n} * \chi_{m+n+l}\|^{1/2}_{L^\infty_{k,\tau}} \lesssim \epsilon^{-\frac{1}{2}} 2^{\frac{5m+3n+l}{4}}$.

Using the support conditions of dyadic projectors, we have

$$\begin{aligned}\tau_1 + w(k_1) = O(2^m);\ \tau_2 + w(k_2) = O(2^{m+n})\\ \tau - \tau_1 - \tau_2 + w(k - k_1 - k_2) = O(2^{m+n+l}),\end{aligned}$$

and combining the two,

$$\tau + w(k_1) + w(k_2) + w(k - k_1 - k_2) = O(2^{m+n+l}).$$

Define

$$E(\tau, k) = \left\{(k_1, k_2) \in \mathbb{Z}^2 : \tau + w(k_1) + w(k_2) + w(k - k_1 - k_2) = O(2^{m+n+l})\right\}.$$

By considering $(k_1, k_2) \mapsto \tilde{w}(k_1, k_2) \coloneqq w(k_1) + w(k_2) + w(k - k_1 - k_2)$ as a map defined on $\mathbb{R}^2$ with a fixed $k \in \mathbb{Z}$, it can be shown by a direct computation that $\tilde{w}$ has a global minimum at $(\frac{k}{3}, \frac{k}{3})$. We change variables $(k_1, k_2) \mapsto (k_1 + \frac{k}{3}, k_2 + \frac{k}{3})$, after which $\tilde{w}$ in the new variables, without re-labelling, can be understood as a polynomial of degree 4 in two variables centered at the origin. We transform $\tilde{w}$ further by considering another change of variable, $k_1 = r\cos(\theta)$, $k_2 = r\sin(\theta)$, and considering $\tilde{w}$ as a polynomial of one variable, namely r, for every fixed $\theta \in [0, 2\pi)$.[2] Then we have

$$\begin{aligned} v(r) \coloneqq \tilde{w}(r,\theta) = \frac{r^2}{12}\Big(3\epsilon^2\Big(9 - \cos(4\theta) + 8\sin(2\theta)\Big)r^2 \\ - 12\epsilon^2 k\Big(\cos(\theta) - \cos(3\theta) + \sin(\theta) + \sin(3\theta)\Big)r \\ + (8\epsilon^2k^2 + 12)(\sin(2\theta) + 2)\Big) + \frac{k^2}{27}(\epsilon^2k^2 + 9). \end{aligned} \tag{4.6}$$

We claim v is an increasing and convex function on $r > 0$ for every $\theta \in [0, 2\pi)$, $k \in \mathbb{Z}$. Since v has no r^1-term, it is clear that $v'(0) = 0$, and it is shown by a direct computation that $v''(r) \geq 0$ on $r \geq 0$. Indeed

$$\begin{aligned} v''(r) = 6\epsilon^2\Big(\sin(2\theta) + 2\Big)^2 r^2 + 6\epsilon^2 k\Big(\cos(3\theta) - (\sin(\theta) + \sin(3\theta) + \cos(\theta))\Big)r \\ + \Big(\frac{4}{3}\epsilon^2k^2 + 2\Big)\Big(\sin(2\theta) + 2\Big), \end{aligned}$$

[2] Here we slightly abuse notations and do not re-name $\tilde{w}$.

and we leave it as a calculus exercise to show the expression above is non-negative for every $\theta \in [0, 2\pi)$, $k \in \mathbb{Z}$. Hence for every $k \in \mathbb{Z}$, $|E(\tau, k)|$ is maximized at $\tau = -\frac{k^2}{27}(\epsilon^2 k^2 + 9)$ since the τ-term corresponds to translating $\tilde{w}$. To obtain a uniform estimate in k, we note that the following lower bound holds on $r \geq 0$ where the implicit constant is independent of θ, k:

$$\frac{1}{r^2}\Big(v(r) - \frac{k^2}{27}(\epsilon^2 k^2 + 9)\Big) \gtrsim \epsilon^2 r^2,$$

which can be shown similarly as $v''(r) \geq 0$. Then

$$\sup_{\tau \in \mathbb{R}} |E(\tau, k)| \lesssim \big|\{(k_1, k_2) \in \mathbb{Z}^2 : \epsilon^2 (k_1^2 + k_2^2)^2 = O(2^{m+n+l})\}\big| \lesssim \epsilon^{-1} 2^{\frac{m+n+l}{2}},$$

and together with the $O(2^{2m+n})$ contribution from each integral corresponding to $(k_1, k_2) \in E(\tau, k)$, we have

$$\|\chi_m * \chi_{m+n} * \chi_{m+n+l}\|_{L^\infty_{k,\tau}} \lesssim \epsilon^{-1} 2^{\frac{m+n+l}{2} + 2m+n} = \epsilon^{-1} 2^{\frac{5m+3n+l}{2}}.$$

The rest follows immediately as in (4.4). $\square$

We remark that the key idea behind our L^4, L^6-estimates is to keep track of the support of convolutions of dyadic projectors. Initially we wanted to take the approach of (Burq et al., 2007) and obtain a priori estimates using trilinear Strichartz estimates. However such approach would require that the following Strichartz estimate holds given an initial data in L^2:

$$\|U_\epsilon(t) f\|_{L^6_{x,t}(\mathbb{T}^2)} \lesssim_\epsilon \|f\|_{L^2(\mathbb{T})},$$

for $\epsilon > 0$; Bourgain showed that $\epsilon = 0$ case is false (Bourgain, 1993a). When one imitates his proof of L^6-estimate with a derivative loss for the Schrödinger operator, one again encounters a number-theoretic problem where one needs to count the number of lattice points (k_1, k_2) in $\mathbb{Z}^2$ that satisfies the following fourth-order resonance condition:

$$\begin{aligned} &r_{N,n,j} \\ &:= \Big|\Big\{|k_1|, |k_2| \leq N : \epsilon^2(k_1^4 + k_2^4 + (n-k_1-k_2)^4) + k_1^2 + k_2^2 + (n-k_1-k_2)^2 = j\Big\}\Big|, \end{aligned}$$

where $|\cdot|$ denotes the cardinality of a set and $N, j \in \mathbb{N}$, $n \in \mathbb{Z}$. As N grows, it seems not true that $r_{N,n,j}$ stays bounded. Our use of software not only suggests that $r_{N,n,j}$ grows as N grows, but also Bombieri and Pila (Bombieri et al., 1989, Theorem 1) proved:

$$|t\Gamma \cap \mathbb{Z}^2| \lesssim_{\epsilon'} t^{\epsilon'},$$

for all $\epsilon' > 0$ as $t \to \infty$ where Γ is the image of an analytic function defined on $\mathbb{T}$. Hence an application of Bourgain's counting method to our problem yields:

$$\|U_\epsilon(t)f\|_{L^6_{x,t}(\mathbb{T}^2)} \lesssim_{\epsilon,\epsilon'} \|f\|_{H^{\epsilon'}(\mathbb{T})},$$

which does not yield well-posedness when $s = 0$.

proof of Corollary 4.1.1. This proof follows by a standard argument using the properties of $X^{s,b}$ space, but we include it for completion. For the sake of concreteness, we fix $N(u) = N_1(u)$ with $\mu = -1$; from Theorem 1.1, we use the L^4-estimate for N_i, $i = 1, 2$ and the L^6-estimate for N_3. For $s \geq 0$, $\frac{1}{2} < b < \frac{11}{16}$ and $T \leq 1$, define the following contraction operator on $X^{s,b}_T$:

$$\Gamma u := U_\epsilon(t)u_0 + i\int_0^t U_\epsilon(t-\tau)\Big(J_\epsilon(|u|^2)\Big)(\tau)d\tau.$$

To indeed show that Γ is a contraction, we first need the following lemma.

Lemma 4.2.1. *For $s \geq 0$ and $\epsilon > 0$, we have*

$$\|J_\epsilon(f\overline{g})h\|_{X^{s,-\frac{5}{16}}} \lesssim_{s,\epsilon} \|f\|_{X^{s,\frac{5}{16}}}\|g\|_{X^{0,\frac{5}{16}}}\|h\|_{X^{0,\frac{5}{16}}} + \|f\|_{X^{0,\frac{5}{16}}}\|g\|_{X^{s,\frac{5}{16}}}\|h\|_{X^{0,\frac{5}{16}}} + \|f\|_{X^{0,\frac{5}{16}}}\|g\|_{X^{0,\frac{5}{16}}}\|h\|_{X^{s,\frac{5}{16}}}.$$

Then by the lemma above and lemma 1.2.1,

$$\begin{aligned}
\|\Gamma u\|_{X_T^{s,b}} &\leq \|U_\epsilon(t)u_0\|_{X_T^{s,b}} + \left\| \int_0^t U_\epsilon(t-\tau)\Big(J_\epsilon(|u|^2)\Big)(\tau)d\tau \right\|_{X_T^{s,b}} \\
&\lesssim \|u_0\|_{H^s} + \|J_\epsilon(|u|^2)u\|_{X_T^{s,b-1}} \lesssim \|u_0\|_{H^s} + T^{\frac{11}{16}-b}\|J_\epsilon(|u|^2)u\|_{X_T^{s,-\frac{5}{16}}} \qquad (4.7) \\
&\lesssim \|u_0\|_{H^s} + T^{(\frac{5}{4}-b)-}\|u\|^2_{X_T^{0,b}}\|u\|_{X_T^{s,b}} \lesssim \|u_0\|_{H^s} + T^{(\frac{5}{4}-b)-}\|u\|^3_{X_T^{s,b}}.
\end{aligned}$$

Hence Γ is well-defined on some closed ball of $X_T^{s,b}$ of radius $C\|u_0\|_{H^s}$ where C is sufficiently large. The difference $\|\Gamma u - \Gamma v\|_{X_T^{s,b}}$ is estimated similarly where if $T \leq \min(1, c\|u_0\|_{H^s}^{-\frac{2}{\beta}})$ for $\beta = (\frac{5}{4} - b)-$ and some constant $c > 0$, then Γ defines a contraction on the closed ball. Uniqueness easily extends to the full $X_T^{s,b}$ and the local Lipschitz regularity of the data-to-solution map follows from a standard fixed point argument.

To extend the solution $u \in C([0,T], H^s(\mathbb{T})) \cap X_T^{s,b}$ globally in time, we apply the well-posedness result in L^2, whose solutions are global-in-time thanks to the L^2-norm conservation, to the estimate 4.7 to obtain $T_0 = T_0(\|u_0\|_{L^2}) > 0$ such that $\|u\|_{X_{T_0}^{s,b}} \lesssim \|u_0\|_{H^s}$. Since $b > \frac{1}{2}$, we have some constant $C > 0$ such that $\|u(t)\|_{H^s} \leq C\|u_0\|_{H^s}$ on $t \in [0, T_0]$, and iterating this procedure infinitely many times, we obtain $\|u(t)\|_{H^s} \leq C^{|t|}\|u_0\|_{H^s}$ for all $t \in \mathbb{R}$. □

proof of lemma 4.2.1. By duality if $\overline{v} \in (X^{s,-\frac{5}{16}})^*$,

$$\begin{aligned}
|\langle J_\epsilon(f\overline{g})h, v\rangle|_{L^2_{x,t}} &= |\langle\langle\nabla\rangle^s(J_\epsilon(f\overline{g})h), \langle\nabla\rangle^{-s}v\rangle_{L^2}| \leq \|\langle\nabla\rangle^s(J_\epsilon(f\overline{g})h)\|_{L^{4/3}}\|\langle\nabla\rangle^{-s}v\|_{L^4} \\
&\lesssim \|\langle\nabla\rangle^s(J_\epsilon(f\overline{g})h)\|_{L^{4/3}}\|v\|_{X^{-s,\frac{5}{16}}},
\end{aligned}$$

where the last inequality is by Theorem 4.1.1. Repeatedly applying the Leibniz rule for Sobolev space on $\|\langle\nabla\rangle^s(J_\epsilon(f\overline{g})h)\|_{L^{4/3}}$ (see lemma 1.2.3), we obtain the desired estimate. □

Remark 4.2.1. *As in (Oh and Tzvetkov, 2017, Footnote 9), we use projectors in the space-time Fourier space to show that the estimates proved above are sharp. Without loss of generality, let $w(k) = k^\delta$ for $\delta \in \{2, 3, \dots\}$, since any polynomials of degree δ are asymptotically equivalent to k^δ modulo the leading coefficient. Define $\widehat{u_N}(k,\tau) := \chi_N(k)\chi_{N^\delta}(\tau)$ where $\chi_N(k)$ is a characteristic function on $k \in [-N, N]$ and similarly for $\chi_{N^\delta}(\tau)$. A direct computation reveals*

$$\|u_N\|_{L^4_{x,t}} \simeq N^{\frac{3}{4}(1+\delta)},\ \|u_N\|_{L^6_{x,t}} \simeq N^{\frac{5}{6}(1+\delta)},\ \|u_N\|_{X^{0,b}} \simeq N^{\frac{1+(2b+1)\delta}{2}},$$

and this shows sharpness. In fact by considering $p = 2q$ for $q \in \{2, 3, \dots\}$, we derive $\|u_N\|_{L^{2q}_{x,t}} \simeq N^{\frac{2q-1}{2q}(1+\delta)}$. Hence for $X^{0,b} \hookrightarrow L^{2q}(\mathbb{T} \times \mathbb{R})$ to hold, it is necessary that

$$b \geq \frac{(q-1)(1+\delta)}{2q\delta}. \tag{4.8}$$

From the proof of Corollary 4.1.1, we see that local well-posedness follows from the embedding of the form $X^{0,b} \hookrightarrow L^p(\mathbb{T} \times \mathbb{R})$ where $b < \frac{1}{2}$. Combining with (4.8), we obtain $q - 1 < \delta$, and hence $q_{max} = \delta$, or equivalently $p_{max} = 2\delta$ if we are interested in the integer powers. We observe that the following dispersive model with $\delta \in \{2, 3, \dots\}$,

$$\begin{cases} i\partial_t u \pm \partial_x^\delta u = \pm|u|^{p-1}u,\ (x,t) \in \mathbb{T} \times \mathbb{R} := \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}. \\ u(0) = u_0 \in L^2(\mathbb{T}), \end{cases}$$

is L^2-critical when $p = 2\delta + 1$. Hence optimistically, this method of modified Strichartz estimates should yield global well-posedness of such dispersive models for every odd-power subcritical nonlinearities.

4.3 Ill-Posedness.

For concreteness, assume $N(u) = -J_\epsilon(|u|^2)u$. Essentially the same argument works for $\mu|u|^{p-1}u$ for $p > 1$.

proof of proposition 4.1.1. Let $u_{n,k}(x, 0) := k\langle n\rangle^{-s}e^{in\cdot x}$ where $k > 0$, $n \in \mathbb{Z}^d$, $x \in \mathbb{T}^d$ and $\cdot$ is the usual dot product. Let $\{k_n\}$ be a positive sequence that converges to k.

By a direct computation,

$$\|u_{n,k}(x,0)\|_{H^s_x} \simeq k; \; \|u_{n,k}(x,0) - u_{n,k_n}(x,0)\|_{H^s_x} \simeq |k_n - k| \xrightarrow[n\to\infty]{} 0.$$

By another direct computation, one can verify that the following is a classical solution to (4.1) with an initial data $u_{n,k}(x,0)$:

$$u_{n,k}(x,t) = k\langle n\rangle^{-s} e^{-it(\epsilon^2|n|^4+|n|^2-k^2\langle n\rangle^{-2s})+in\cdot x}.$$

Given any $T > 0$ and $t \in [0,T]$, we have

$$\begin{aligned} \|u_{n,k_n}(x,t) - u_{n,k}(x,t)\|_{H^s_x} &\simeq_s |k_n e^{itk_n^2\langle n\rangle^{-2s}} - ke^{itk^2\langle n\rangle^{-2s}}| \\ &\geq |k| \cdot |e^{it\langle n\rangle^{-2s}(k_n^2-k^2)} - 1| - |k_n - k|. \end{aligned} \tag{4.9}$$

By defining $k_n := \left(k^2 + t^{-1}\langle n\rangle^{2s}\pi\right)^{1/2}$, which is shown to converge to k since $s < 0$, we see that the RHS of (4.9) is bounded below by k for all but finitely many n. □

Remark 4.3.1. *We note that the compactness of* $\mathbb{T}$ *leads to an (uniform) ill-posedness proof that is more straightforward than that on* $\mathbb{R}$ *by considering the time-evolution of pure frequencies* $\{e^{inx}\}_{n\in\mathbb{Z}}$*, which of course fail to be square-integrable on* $\mathbb{R}$*.*

4.4 Regularity in $\epsilon \in \mathbb{C}$.

In this section, we complexify the quantum parameter ϵ and study the regularity of solutions (in ϵ) with a fixed nonlinearity $N(u) = -J_\epsilon(|u|^2)u$. Since the equation is invariant under $\epsilon \to -\epsilon$, it suffices to consider the closed half-plane $\{Im(\epsilon) \geq 0\}$. We expect a purely dispersive behaviour when $\epsilon \in \mathbb{R} \cup \left(i\mathbb{R} \setminus \left\{\frac{i}{n} : n \in \mathbb{Z} \setminus \{0\}\right\}\right)$ and hence we refer to this domain as the *dispersive regime*. Denoting $\epsilon^2 = \alpha + i\beta$ where $\alpha, \beta \in \mathbb{R}$, we expect the equation to be dissipative when $\beta < 0$, i.e., the (open) second and fourth quadrant. When $\beta > 0$, we expect solutions to exhibit blow-ups in some sense, which we show in the next proposition. Here we use the semi-group $S_\epsilon(t) = e^{-it(\epsilon^2\partial^2_{xx} - \partial_{xx})}$ for $t \geq 0$ since solutions exhibit the time-reversal symmetry only in the dispersive regime. Our analysis shows that the addition of

$\epsilon^2\partial_{xx}$ singularly perturbs the well-known focusing cubic NLS, i.e., the $\epsilon = 0$ case. Our results are for classical solutions, but we believe that analogous results should hold for less regular solutions.

We immediately note that $N(u)$ is not well-defined for $\epsilon = \frac{i}{n}$ where $n \in \mathbb{Z}\setminus\{0\}$, and therefore we exclude such ϵ in our analysis. When $\epsilon \in \mathbb{R}$, it is clear that $\|J_\epsilon f\|_{H^s} \leq \|f\|_{H^s}$ for all $s \in \mathbb{R}$. However this is no longer true when we complexify ϵ. In fact,

$$\|J_\epsilon f\|_{H^s} \leq \gamma(\epsilon)\|f\|_{H^s} := \left\|\frac{1}{1+\epsilon^2 n^2}\right\|_{l^\infty_n} \|f\|_{H^s}, \tag{4.10}$$

where $\gamma(\epsilon)$ denotes the best constant for J_ϵ. Unlike the $\epsilon \in \mathbb{R}$ case, it is possible that $\gamma(\epsilon)$ is arbitrarily large depending on $\epsilon \in \mathbb{C}$. However we leave it as a simple exercise that $\gamma(\epsilon) \leq 1$ if $|\epsilon| \geq \sqrt{2}$. We also note that $J_\epsilon \xrightarrow[\epsilon\to\infty]{} \mathbb{P}$ in uniform operator topology on $H^s(\mathbb{T})$ where the ϵ-limit is in the sense of extended complex plane.

We let $D := \mathbb{R} \cup \left(i\mathbb{R} \setminus \{\frac{i}{n} : n \in \mathbb{Z}\setminus\{0\}\}\right) \cup \{Re(\epsilon)\cdot Im(\epsilon) < 0\}$, i.e., the union of dispersive and dissipative regime. For $T > 0$, we denote $\Phi : D \times H^s(\mathbb{T}) \to C([0,T], H^s(\mathbb{T}))$ to be the solution map given by $(\epsilon, u_0) \mapsto u^\epsilon$ such that $u^\epsilon(x,0) = u_0(x)$. If the context is clear, we abuse our notation and denote $\Phi : D \times H^s(\mathbb{T}) \to C([0,\infty), H^s(\mathbb{T}))$ or consider Φ as a map defined only on D, provided that u_0 is fixed. The following proposition shows that Φ is well-defined. Our results that pertain to the interval $[0,T]$ also hold for time elements in any compact subset of $\mathbb{R}$.

Proposition 4.4.1. *Let $s > \frac{1}{2}$.*

1. *The Cauchy problem (4.1) is globally well-posed in $H^s(\mathbb{T})$ for $\epsilon \in D$ for every $u_0 \in H^s(\mathbb{T})$. On the other hand, the data-to-solution map, if it exists, exhibits norm inflation at the origin for $\epsilon \in \{Re(\epsilon)\cdot Im(\epsilon) > 0\}$.*

2. *For any $T > 0$, $\Phi : D \times H^s(\mathbb{T}) \to C([0,T], H^s(\mathbb{T}))$ is continuous, but not uniformly.*

3. *There exists $u_0 \in H^s(\mathbb{T})$ such that for any $\epsilon \in \mathbb{R} \cup \left(i\mathbb{R} \setminus \{\frac{i}{n} : n \in \mathbb{Z}\setminus\{0\}\}\right)$*

and any sequence $\{\epsilon_j\} \subseteq D$ *such that* $\epsilon_j \xrightarrow[j\to\infty]{} \epsilon$*, the convergence* $\Phi(\epsilon_j, u_0) \xrightarrow[j\to\infty]{} \Phi(\epsilon, u_0)$ *in* $C([0,\infty), H^s(\mathbb{T}))$ *fails. In particular,* Φ *is nowhere continuous on the dispersive regime for a certain class of initial data, if* $T = \infty$*.*

Remark 4.4.1. *As for the second statement of Proposition 4.4.1, it is expected that for each* $u_0 \in H^s(\mathbb{T})$, $\Phi : \Omega := \{\epsilon \in \mathbb{C} : Re(\epsilon) \cdot Im(\epsilon) < 0\} \to C([0,\infty), H^s(\mathbb{T}))$ *is analytic in the Banach space sense. However we prove a weaker result corresponding to the linearized version of our nonlinear equation. We show that if* $\Phi_L : \Omega \to C([\delta,\infty), H^s(\mathbb{T}))$ *is given by* $\epsilon \mapsto S_\epsilon(t)u_0$ *where* $s \in \mathbb{R}\,, \delta > 0$*, then* Φ_L *is analytic.*

Remark 4.4.2. *For the third statement of Proposition 4.4.1, we expect such nowhere continuity of the solution map not just for some special class of initial data but also for every nonzero* $u_0 \in H^s(\mathbb{T})$*. Physically we expect the (purely dispersive) solutions to undergo phase decoherence without damping, which cannot be controlled as the terminal time* $T \to \infty$*.*

Remark 4.4.3. *If* v *is the solution of the following linear equation:*

$$\begin{cases} i\partial_t v + \Delta v - \epsilon^2 \Delta^2 v = -\frac{\|u_0\|_{L^2}^2}{2\pi} v, \ (x,t) \in \mathbb{T} \times \mathbb{R} := \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}. \\ v(0) = u_0 \in H^s(\mathbb{T}), \end{cases}$$

and we have $\{u_0^\epsilon\}_{\epsilon \in D} \subseteq H^s(\mathbb{T})$ *such that* $u_0^\epsilon \xrightarrow[\epsilon\to\infty]{H^s} u_0$*, then for every* $T > 0$*,*

$$\|\Phi(\epsilon, u_0^\epsilon) - v\|_{C([0,T],H^s(\mathbb{T}))} \xrightarrow[\epsilon\to\infty]{} 0.$$

The proof is similar to that of Proposition 4.4.1 second statement.

proof of remark 4.4.1. Let $u_0 = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} c_n e^{inx}$ where $\|\langle n\rangle^s c_n\|_{l^2} < \infty$. Setting $\alpha = Re(\epsilon^2)$, $\beta = Im(\epsilon^2)$, we have

$$S_\epsilon(t)u_0 = \frac{1}{2\pi} \sum_n e^{-i\epsilon^2 n^4 t - in^2 t} c_n e^{inx} = \frac{1}{2\pi} \sum_n e^{\beta n^4 t} e^{-i(\alpha n^4 + n^2)t} c_n e^{inx}.$$

From the first equality, one could take the ϵ-derivative and directly show the analyticity of Φ_L. We show weak analyticity instead since they are equivalent. Let $\phi \in C([\delta,\infty), H^s(\mathbb{T}))'$. For every $f \in C_0([\delta,\infty), \mathbb{C})$, a complex-valued continuous function that vanishes at the infinity, define $\phi_n(f) = \phi(f \mapsto fe^{inx})$. Then by the Riesz-Markov-Kakutani representation theorem, there exists a (σ-finite) complex Radon measure

μ_n on $[\delta, \infty)$ with a finite total variation given by $\|\mu_n\| = \|\phi_n\|$ where the norm on ϕ_n denotes the operator norm on $C_0([\delta, \infty), \mathbb{C})'$. From the Fourier series representation, the zeroth mode is constant, and therefore it suffices to show that

$$\begin{aligned}\epsilon \mapsto \phi(\sum_{n\neq 0} e^{-i\epsilon^2 n^4 t - in^2 t} c_n e^{inx}) = \sum_{n\neq 0} \phi(e^{-i\epsilon^2 n^4 t - in^2 t} c_n e^{inx}) &= \sum_{n\neq 0} c_n \int e^{-i\epsilon^2 n^4 t - in^2 t} d\mu_n \\ &= \sum_{n\neq 0} c_n \int e^{\beta n^4 t} e^{-i(\alpha n^4 + n^2)t} d\mu_n\end{aligned}$$

is analytic in $\epsilon \in \Omega$ where we define $F_n(\epsilon) = \int e^{-i\epsilon^2 n^4 t - in^2 t} d\mu_n$.

First note that $(\epsilon, t) \mapsto e^{-i\epsilon^2 n^4 t - in^2 t}$ is continuous in $\Omega \times [\delta, \infty)$ and that for each $t \in [\delta, \infty)$, the integrand is complex analytic in $\epsilon \in \Omega$. Since $\|\mu_n\| < \infty$, the dominated convergence theorem yields that F_n is continuous on Ω for all $n \in \mathbb{Z} \setminus \{0\}$.

In addition to continuity, complex-analyticity of F_n is a straightforward application of Morera's theorem. In fact, let Ω' be an open set that is compactly contained in Ω, from which we have $\beta \leq \beta_0 < 0$, and let T be an oriented triangle compactly contained in Ω'. Since $|e^{\beta n^4 t} e^{-i(\alpha n^4 + n^2)t}| \leq e^{\beta_0 n^4 \delta}$, we have

$$\left| \int_T \int e^{-i\epsilon^2 n^4 t - in^2 t} d\mu_n d\epsilon \right| \leq e^{\beta_0 n^4 \delta} \|\mu_n\| (\text{Length of } T),$$

and by Fubini,

$$\int_T F_n(\epsilon) d\epsilon = \int \int_T e^{-i\epsilon^2 n^4 t - in^2 t} d\epsilon d\mu_n = 0.$$

To show that the desired series is analytic, it suffices to show the series converges uniformly in every compact subset of Ω. Denoting Ω' as before and observing that $\|\phi_n\| \leq c\langle n\rangle^s \|\phi\|$ for some $c > 0$, we have

$$\begin{aligned}|F_n(\epsilon)| &\leq e^{\beta_0 n^4 \delta} \|\mu_n\| \lesssim \langle n\rangle^s e^{\beta_0 n^4 \delta} \\ \sum_{n\neq 0} |c_n||F_n(\epsilon)| &\lesssim \delta^{-\frac{1}{8}} \|u_0\|_{H^s}.\end{aligned}$$

and the conclusion follows from the Weierstrass' M-test. We remark that the proof above does not yield a uniform estimate as $\delta \to 0$. $\square$

proof of proposition 4.4.1 (first statement). It has been shown that $\Phi(\epsilon, u_0)$ for $\epsilon \in \mathbb{R}$ is well-defined. We first consider $\epsilon \in i\mathbb{R}$. It suffices to show that J_ϵ defines a bounded operator on $H^s(\mathbb{T})$ if and only if $\epsilon \neq \frac{i}{n}$ for $n \in \mathbb{Z}\setminus\{0\}$, which holds since the spectrum of $\Delta_{\mathbb{T}}$ is $\{n^2 : n = 0, 1, 2, \dots\}$. Since $e^{it(\partial_{xx} - \epsilon^2 \partial_{xx}^2)}$ is unitary on $H^s(\mathbb{T})$, which are algebras for $s > \frac{1}{2}$, the proof of global well-posedness for Φ on $i\mathbb{R} \setminus \left\{\frac{i}{n} : n \in \mathbb{Z} \setminus \{0\}\right\}$ follows from a standard argument invoking the Duhamel formula and the Gronwall's inequality.

For $\epsilon \in \{Re(\epsilon) \cdot Im(\epsilon) < 0\}$, observe that $\|S_\epsilon(t) f\|_{H^s} \leq \|f\|_{H^s}$. Then the Duhamel integral formula allows us to set up a fixed point argument as follows:

$$\begin{aligned}
&\left\| S_\epsilon(t) u_0 + i \int_0^t S_\epsilon(t-\tau)\Big(J_\epsilon(|u|^2)u\Big)(\tau)d\tau \right\|_{H_x^s} \\
\leq &\|u_0\|_{H^s} + \int_0^t \|S_\epsilon(t-\tau)\Big(J_\epsilon(|u|^2)u\Big)(\tau)\|_{H_x^s} d\tau \lesssim_\epsilon \|u_0\|_{H^s} + t(\sup_{\tau \in [0,t]} \|u(\tau)\|_{H_x^s})^3,
\end{aligned}$$

We extend this local-in-time solution globally by iterating the local result with the Duhamel's formula as follows:

$$\|u(t)\|_{H_x^s} \lesssim_\epsilon \|u_0\|_{H^s} + \int_0^t \|u(\tau)\|_{H_x^s} d\tau,$$

from which we use the Gronwall's inequality to deduce $\|u(t)\|_{H_x^s} \leq Ce^{\tilde{c}t}$ for all $t \geq 0$ where the constants depend only on s and $\|u_0\|_{H^s}$. We omit standard details.

On the other hand, if $\beta > 0$, we show that the solutions exhibit norm inflation at the origin, that is, for every $\delta > 0$, there exists $0 < T < \delta$ and $\phi \in H^\infty(\mathbb{T})$ such that $\|\phi\|_{H^s} < \delta$ and $\|u[\phi](T)\|_{H^s} > \delta^{-1}$, if the solution exists on $[0, T]$. We again exploit the compactness of our domain and study the flow of pure frequencies. Let k_n be a non-negative real sequence that converges to 0 sub-exponentially. For $k \geq 0$, let $u_{n,k}(x, 0) = k\langle n\rangle^{-s} e^{inx}$, and by a direct computation, we have the classical solution:

$$u_{n,k}(x,t) = k\langle n\rangle^{-s} e^{\beta t n^4} e^{-it(\alpha n^4 + n^2 - k^2\langle n\rangle^{-2s})} e^{inx}.$$

Given $\delta > 0$, we choose $k = 0$ and $T = \frac{\delta}{2}$, and by the following,

$$\|u_{n,k_n}(x,0)\|_{H^s} \simeq k_n \xrightarrow[n\to\infty]{} 0;\ \|u_{n,k_n}(x,T)\|_{H^s} \simeq k_n e^{\beta T n^4},$$

a norm inflation at the origin has been shown for a sufficiently large n. $\square$

proof of proposition 4.4.1 (second statement). We first study the continuity properties of our semi-group. Then we show that the difference of the solutions measured in the desired space-time Banach space can be made arbitrarily small by using the Duhamel integral formula.

Lemma 4.4.1. *Let $\epsilon, \epsilon' \in D$, $T > 0$ and u_0^ϵ, $u_0^{\epsilon'} \in H^s(\mathbb{T})$ such that $u_0^{\epsilon'} \xrightarrow[\epsilon'\to\epsilon]{H^s} u_0^\epsilon$. Then $S_{\epsilon'}(t)u_0^{\epsilon'} \xrightarrow[\epsilon'\to\epsilon]{} S_\epsilon(t)u_0^\epsilon$ in $C([0,T], H^s(\mathbb{T}))$. Furthermore if $u \in C([0,T], H^s(\mathbb{T}))$, then $S_{\epsilon'}(t)u(t) \xrightarrow[\epsilon'\to\epsilon]{} S_\epsilon(t)u(t)$ in $C([0,T], H^s(\mathbb{T}))$.*

proof of lemma 4.4.1. Since $\|S_{\epsilon'}(t)u_0^{\epsilon'} - S_\epsilon(t)u_0^\epsilon\|_{H^s} \le \|u_0^{\epsilon'} - u_0^\epsilon\|_{H^s} + \|(S_{\epsilon'}(t) - S_\epsilon(t))u_0^\epsilon\|_{H^s}$, it suffices to show

$$\sum_n |m(n)|^2 \langle n\rangle^{2s} |\widehat{u_0^\epsilon}(n)|^2 \xrightarrow[\epsilon'\to\epsilon]{} 0$$

uniformly in $t \in [0,T]$ where $m(n)$ is the symbol of $S_{\epsilon'}(t) - S_\epsilon(t)$. Let $\epsilon^2 = \alpha + i\beta$, ${\epsilon'}^2 = {\alpha'}^2 + i\beta'$ where $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$. By a direct computation, we have

$$\begin{aligned}
|m(n)|^2 &= e^{2\beta' t n^4} + e^{2\beta t n^4} - 2e^{(\beta'+\beta)tn^4} \cos\Big((\alpha' - \alpha)tn^4\Big) \\
&= (e^{\beta' t n^4} - e^{\beta t n^4})^2 + 2e^{(\beta'+\beta)tn^4}\Big(1 - \cos\Big((\alpha' - \alpha)tn^4\Big)\Big) \\
&\le min\Bigg(4, |\beta' - \beta|^2 T^2 n_0^8 + 2\Big(1 - \cos\Big((\alpha' - \alpha)tn^4\Big)\Big)\Bigg),
\end{aligned}$$

where in the last inequality, the former conservative bound is used to estimate the tail-behavior of the Fourier series whereas the latter bound is used to control the low frequencies where we assume $|n| \le n_0 \in \mathbb{N}$ for some n_0. Given $\tilde{\epsilon} > 0$, there exists n_0 such that $\sum_{|n|>n_0} \langle n\rangle^{2s} |\widehat{u_0^\epsilon}(n)|^2 < \tilde{\epsilon}$. Let $0 < \delta < 1$ such that $\delta < \frac{\tilde{\epsilon}}{\|u_0^\epsilon\|_{H^s}^2}$ where we assume $u_0^\epsilon \ne 0$. It is a straightforward calculation to check that there exists $R = R(T) > 0$

such that if $|\epsilon' - \epsilon| < R$, then $|\alpha' - \alpha| < \delta_\alpha$, $|\beta' - \beta| < \delta_\beta$ where

$$\delta_\alpha \lesssim \frac{\cos^{-1}(1 - \frac{\delta}{2})}{Tn_0^4},\ \delta_\beta \lesssim \frac{\delta^{1/2}}{Tn_0^4}.$$

Then for every $\epsilon' \in D$ such that $|\epsilon' - \epsilon| < R$, we have

$$\begin{aligned}\sum_n |m(n)|^2 \langle n\rangle^{2s} |\widehat{u_0^\epsilon}(n)|^2 &= \sum_{|n|\le n_0} |m(n)|^2 \langle n\rangle^{2s} |\widehat{u_0^\epsilon}(n)|^2 + \sum_{|n|>n_0} |m(n)|^2 \langle n\rangle^{2s} |\widehat{u_0^\epsilon}(n)|^2 \\ &\lesssim \delta \|u_0\|_{H^s}^2 + 4 \sum_{|n|>n_0} \langle n\rangle^{2s} |\widehat{u_0^\epsilon}(n)|^2 \lesssim \tilde{\epsilon}.\end{aligned}$$

On the other hand, if $u \in C([0,T], H^s(\mathbb{T}))$, to show $S_{\epsilon'}(t)u(t) \xrightarrow[\epsilon'\to\epsilon]{} S_\epsilon(t)u(t)$ in $C([0,T], H^s(\mathbb{T}))$, it suffices to show that $n_0 = n_0(T) \in \mathbb{N}$ can be chosen such that $\sum\limits_{|n|>n_0} \langle n\rangle^{2s} |\widehat{u(t)}(n)|^2 < \tilde{\epsilon}$ for every $t \in [0,T]$, from which the argument proceeds as before. Such uniform estimate (in t) follows from the Arzelà-Ascoli argument since

$$\begin{aligned}&\left| \sum_{|n|>n_0} \langle n\rangle^{2s} \Big(|\widehat{u(t)}(n)|^2 - |\widehat{u(t')}(n)|^2 \Big) \right| \\ \le &\Bigg(\Big(\sum_{|n|>n_0} \langle n\rangle^{2s} |\widehat{u(t)}(n)|^2 \Big)^{1/2} + \Big(\sum_{|n|>n_0} \langle n\rangle^{2s} |\widehat{u(t')}(n)|^2 \Big)^{1/2} \Bigg) \|u(t) - u(t')\|_{H^s} \\ &\le 2\|u\|_{C_T H_x^s} \|u(t) - u(t')\|_{H^s}.\end{aligned}$$

□

Lemma 4.4.2. *γ, defined as in (4.10), is continuous on D. Moreover there exists $r > 0$ such that if $\epsilon' \in B(\epsilon, r) \subseteq D$, an open ball in the subspace topology, and if for each ϵ', there exists $u_0^{\epsilon'} \in H^s(\mathbb{T})$ with $\|u_0^{\epsilon'}\|_{H^s} \le R$ for some $R > 0$, then*

$$\sup_{\epsilon' \in B(\epsilon, r)} \|u^{\epsilon'}(t)\|_{H^s} \le Re^{Ct},\ t \ge 0,$$

where C only depends on $s, \gamma(\epsilon)$ and R.

proof of lemma 4.4.2. Continuity follows from the definition since $|\frac{1}{1+\epsilon^2 n^2} - \frac{1}{1+\epsilon'^2 n^2}|$ can be made arbitrarily small uniformly in n. To show the uniform rate of growth, pick a $r > 0$ such that if $|\epsilon' - \epsilon| < r$, then $\gamma(\epsilon') \le 2\gamma(\epsilon)$. Noting that the semi-group is a contraction on $H^s(\mathbb{T})$ for all $\epsilon \in D$ and that the given initial data are uniformly bounded in $H^s(\mathbb{T})$, the desired estimate follows from applying the Gronwall's inequality to the Duhamel integral formula. $\square$

Now we sub-divide the difference of nonlinear Duhamel terms and estimate them one by one.

$$\int_0^t \Big(S_{\epsilon'}(t-\tau)\Big(J_{\epsilon'}(|u^{\epsilon'}|^2)u^{\epsilon'}\Big)(\tau) - S_\epsilon(t-\tau)\Big(J_\epsilon(|u^\epsilon|^2)u^\epsilon\Big)\Big)(\tau)d\tau = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \int_0^t S_{\epsilon'}(t-\tau)\Big(J_{\epsilon'}(|u^{\epsilon'}|^2)(u^{\epsilon'} - u^\epsilon)\Big)(\tau)d\tau \\
I_2 &= \int_0^t S_{\epsilon'}(t-\tau)\Big(J_{\epsilon'}(|u^{\epsilon'}|^2 - |u^\epsilon|^2)u^\epsilon\Big)(\tau)d\tau \\
I_3 &= \int_0^t S_{\epsilon'}(t-\tau)\Big((J_{\epsilon'} - J_\epsilon)(|u^\epsilon|^2)u^\epsilon\Big)(\tau)d\tau \\
I_4 &= \int_0^t (S_{\epsilon'}(t-\tau) - S_\epsilon(t-\tau))\Big(J_\epsilon(|u^\epsilon|^2)u^\epsilon\Big)(\tau)d\tau.
\end{aligned}$$

Taking $|\epsilon' - \epsilon|$ sufficiently small, we use the uniform estimate above to show

$$\|I_1\|_{H^s} \lesssim_s \gamma(\epsilon')\|u^{\epsilon'}\|^2_{C_T H^s_x} \int_0^t \|u^{\epsilon'} - u^\epsilon\|_{H^s} d\tau \lesssim \gamma(\epsilon)\|u^\epsilon_0\|^2_{H^s} e^{CT} \int_0^t \|u^{\epsilon'} - u^\epsilon\|_{H^s} d\tau$$

$$\begin{aligned}
\|I_2\|_{H^s} &\lesssim_s \gamma(\epsilon')(\|u^\epsilon\|_{C_T H^s_x} + \|u^{\epsilon'}\|_{C_T H^s_x})\|u^\epsilon\|_{C_T H^s_x} \int_0^t \|u^{\epsilon'} - u^\epsilon\|_{H^s} d\tau \\
&\lesssim \gamma(\epsilon)\|u^\epsilon_0\|^2_{H^s} e^{CT} \int_0^t \|u^{\epsilon'} - u^\epsilon\|_{H^s} d\tau.
\end{aligned}$$

Similarly another application of Lemma 4.4.2 yields

$$\|I_3\|_{H^s} \lesssim_s \|u_0^\epsilon\|_{H^s} e^{CT} \int_0^t \|(J_{\epsilon'} - J_\epsilon)(|u^\epsilon|^2)\|_{H^s} d\tau.$$

To show that the integral on the RHS tends to zero as $\epsilon' \to \epsilon$ at a rate that only depends on $T > 0$, note that

$$\int_{t'}^t \|(J_{\epsilon'} - J_\epsilon)(|u^\epsilon|^2)\|_{H^s} d\tau \lesssim \gamma(\epsilon)\Big(\sup_{\tau\in[0,T]} \|u^\epsilon\|_{H^s}\Big)^2 |t'-t| \lesssim \gamma(\epsilon)\|u_0^\epsilon\|_{H^s}^2 e^{CT}|t'-t|,$$

and by an Arzelà-Ascoli argument, the desired integral tends to zero as $\epsilon' \to 0$ uniformly in $t \in [0,T]$. We apply a similar compactness argument to control the final term, I_4.

$$\begin{aligned}\|I_4\|_{H^s} &\leq \int_0^t \Big\|\Big(S_{\epsilon'}(t-\tau) - S_\epsilon(t-\tau)\Big)(J_\epsilon(|u^\epsilon|^2)u^\epsilon)\Big\|_{H^s} d\tau \\ &= \int_0^t \Big\|\Big(S_{\epsilon'}(\tau) - S_\epsilon(\tau)\Big)w_\epsilon(\tau)\Big\|_{H^s} d\tau,\end{aligned}$$

where $w_\epsilon(\tau) = J_\epsilon(|u^\epsilon(t-\tau)|^2)u^\epsilon(t-\tau)$ for $(t,\tau) \in \Omega_T$ where

$$\Omega_T := \left\{(t',\tau') \in \mathbb{R}^2 : 0 \leq t' \leq T, 0 \leq \tau' \leq t'\right\}.$$

We claim $\Big\|\Big(S_{\epsilon'}(\tau) - S_\epsilon(\tau)\Big)w_\epsilon(\tau)\Big\|_{H^s} \xrightarrow[\epsilon'\to\epsilon]{} 0$ uniformly on Ω_T. As in the proof of Lemma 4.4.1,

$$\Big\|\Big(S_{\epsilon'}(\tau) - S_\epsilon(\tau)\Big)w_\epsilon(\tau)\Big\|_{H^s}^2 \leq \sum_{|n|\leq n_0} |m(n)|^2\langle n\rangle^{2s}|\widehat{w_\epsilon(\tau)}(n)|^2 + 4\sum_{|n|>n_0}\langle n\rangle^{2s}|\widehat{w_\epsilon(\tau)}(n)|^2.$$

By the dominated convergence theorem, $\sum_{|n|>n_0}\langle n\rangle^{2s}|\widehat{w_\epsilon(\tau)}(n)|^2 \xrightarrow[n_0\to\infty]{} 0$ for every $(t,\tau) \in \Omega_T$. Furthermore for $F_{n_0}(t,\tau) := \sum_{|n|>n_0}\langle n\rangle^{2s}|\widehat{w_\epsilon(\tau)}(n)|^2$, we have

$$|F_{n_0}(t,\tau) - F_{n_0}(t',\tau')| \lesssim_\epsilon \|u^\epsilon\|_{C_T H_x^s}^5 \|u^\epsilon(t-\tau) - u^\epsilon(t'-\tau')\|_{H^s},$$

and therefore the claim follows from another application of an Arzelà-Ascoli argument.

Given any $\tilde{\epsilon} > 0$,

$$u^{\epsilon'}(t) - u^{\epsilon}(t) = S_{\epsilon'}(t)u_0^{\epsilon'} - S_{\epsilon}(t)u_0' + i(I_1 + I_2 + I_3 + I_4)$$
$$\Rightarrow \|u^{\epsilon'}(t) - u^{\epsilon}(t)\|_{H^s} \leq \tilde{\epsilon}\langle t\rangle + \gamma(\epsilon)\|u_0^{\epsilon}\|_{H^s}^2 e^{CT}\int_0^t \|u^{\epsilon'}(\tau) - u^{\epsilon}(\tau)\|_{H^s} d\tau. \tag{4.11}$$

where the first term of the RHS of (4.11) comes from lemma 4.4.1, I_3 and I_4 whereas the second terms comes from I_1 and I_2, where $|\epsilon' - \epsilon|$ is sufficiently small depending only on $\gamma(\epsilon)$, $\|u_0^{\epsilon}\|_{H^s}$ and T. Finally the Gronwall's inequality yields

$$\sup_{t\in[0,T]} \|u^{\epsilon'}(t) - u^{\epsilon}(t)\|_{H^s} \leq \tilde{\epsilon}\langle T\rangle e^{\tilde{C}T},$$

and since $\tilde{\epsilon}$ is arbitrary, the proof for continuity is complete.

To show that uniform continuity fails, let $u_0(x) = \langle n\rangle^{-s}e^{inx}$ to which

$$\Phi(\epsilon, u_0)(x,t) = e^{-it(\epsilon^2 n^4 + n^2 - \langle n\rangle^{-2s})}\langle n\rangle^{-s}e^{inx}.$$

For $T > 0$, we observe

$$\|\Phi(\epsilon, u_0) - \Phi(\epsilon', u_0)\|_{C_T H_x^s}^2 = \sup_{t\in[0,T]} |e^{-it\epsilon^2} - e^{-it\epsilon'^2}|^2 \tag{4.12}$$

We can let $\epsilon, \epsilon' > 0$ such that $|\epsilon' - \epsilon| = O(1)$ and $\epsilon, \epsilon' \to \infty$. Then

$$= \sup_{t\in[0,T]} 2 - 2\cos(t(\epsilon^2 - \epsilon'^2)) = 2.$$

$\square$

proof of proposition 4.4.1 (third statement). Suppose $\Phi : D \to C([0,\infty), H^s(\mathbb{T}))$ were continuous at $\epsilon_0 \in D \setminus \{\epsilon \in \mathbb{C} : Re(\epsilon)\cdot Im(\epsilon) < 0\}$ and let $\epsilon_j \xrightarrow[j\to\infty]{} \epsilon_0$ where $\epsilon_j \in D$. Let $u_0(x) = \langle n\rangle^{-s}e^{inx}$ for some $n \in \mathbb{N}$ for which we have

$$u_j(x,t) := \langle n\rangle^{-s}e^{\beta_j t n^4}e^{-it(\alpha_j n^4 + n^2 - \langle n\rangle^{-2s})}e^{inx}$$

where $\alpha_j = Re(\epsilon_j^2)$, $\beta_j = Im(\epsilon_j^2)$ and $u(x,t) := \langle n\rangle^{-s}e^{\beta t n^4}e^{-it(\alpha n^4 + n^2 - \langle n\rangle^{-2s})}e^{inx}$ where

$\alpha = Re(\epsilon_0^2)$, $\beta = Im(\epsilon_0^2)$, where $\beta = 0$ by the assumption on ϵ_0. Re-scaling the time variable $\tau = tn^4$, we have

$$\begin{aligned}\|u_j(x,t) - u(x,t)\|_{H^s_x} &\simeq e^{\beta_j tn^4}|e^{-(\beta_j - i(\alpha_j-\alpha))tn^4} - 1| = e^{\beta_j\tau}|e^{-(\beta_j - i(\alpha_j-\alpha))\tau} - 1| \\ &= |e^{i(\alpha_j-\alpha)\tau} - e^{\beta_j\tau}|.\end{aligned}$$

If $\beta_j = 0$ for all but finitely many js, then for such js,

$$\sup_{\tau\in[0,\infty)} |e^{i(\alpha_j-\alpha)\tau} - 1| = 2,$$

if $\alpha_j \neq \alpha$, and hence we assume $\beta_j < 0$ for all j without loss of generality. Similarly we assume $|\alpha - \alpha_j| > 0$ for all j.

Suppose $|\alpha_j - \alpha| \leq |\beta_j|$ for all $j \geq j_0 \in \mathbb{N}$. Recall from Taylor expansion that there exists $0 < \delta_0 < 1$ and $c_1 > 0$ such that if $|z| \leq \delta_0$, then $|e^z - 1| \geq c_1|z|$. Let $0 < c < \frac{\delta_0}{\sqrt{2}} < 1$. For $\tau > \frac{c}{|\beta_j|}$, we have

$$\sup_{\tau\in[\frac{c}{|\beta_j|},\infty)} |e^{i(\alpha_j-\alpha)\tau} - e^{\beta_j\tau}| \geq 1 - e^{-c} > 0.$$

For $\tau \in [0, \frac{c}{|\beta_j|}]$, let $z = -(\beta_j - i(\alpha_j - \alpha))\tau$. Then

$$|z| = ((\alpha_j - \alpha)^2 + \beta_j^2)^{1/2}\tau \leq \Big(\frac{(\alpha_j-\alpha)^2}{|\beta_j|^2} + 1\Big)^{1/2} c \leq \sqrt{2}c < \delta_0,$$

and therefore

$$\begin{aligned} & e^{\beta_j\tau}|e^{-(\beta_j - i(\alpha_j-\alpha))\tau} - 1| \geq c_1 e^{\beta_j\tau}\tau((\alpha_j-\alpha)^2 + \beta_j^2)^{1/2} \\ \Rightarrow & \sup_{\tau\in[0,\frac{c}{|\beta_j|}]} e^{\beta_j\tau}|e^{-(\beta_j - i(\alpha_j-\alpha))\tau} - 1| \geq \sup_{\tau\in[0,\frac{c}{|\beta_j|}]} c_1 e^{\beta_j\tau}\tau((\alpha_j-\alpha)^2 + \beta_j^2)^{1/2} \\ & \qquad = c_1 c e^{-c}\frac{((\alpha_j-\alpha)^2 + \beta_j^2)^{1/2}}{|\beta_j|} \geq c_1 c e^{-c} > 0,\end{aligned}$$

from which we conclude, from the continuity hypothesis of Φ at ϵ_0, that there cannot exist $j_0 \in \mathbb{N}$ such that $|\alpha_j - \alpha| \leq |\beta_j|$ for all $j \geq j_0$.

Passing to a subsequence without re-labelling the index, suppose $|\alpha_j - \alpha| > |\beta_j|$

for all $j \in \mathbb{N}$. As before, if $\tau > \frac{\pi}{2|\beta_j|}$, we have

$$\sup_{\tau \in [\frac{\pi}{2|\beta_j|}, \infty)} |e^{i(\alpha_j - \alpha)\tau} - e^{\beta_j \tau}| \geq 1 - e^{-\frac{\pi}{2}} > 0.$$

On the other hand, we have

$$\sup_{\tau \in [0, \frac{\pi}{2|\beta_j|}]} |e^{i(\alpha_j - \alpha)\tau} - e^{\beta_j \tau}| \geq \sup_{\tau \in [0, \frac{\pi}{2|\alpha_j - \alpha|}]} |e^{i(\alpha_j - \alpha)\tau} - e^{\beta_j \tau}| \geq 1.$$

Hence it cannot be that $\lim_{j \to \infty} \|\Phi(\epsilon_0, u_0) - \Phi(\epsilon_j, u_0)\|_{C([0,\infty), H^s(\mathbb{T}))} = 0$. $\square$

Chapter 5

Periodic Quantum Zakharov Systems

5.1 Introduction.

We consider the well-posedness and the semi-classical limit of the compact one-dimensional quantum Zakharov system (QZS). Thus we assume the periodic boundary condition

$$\begin{cases} (i\partial_t + \alpha\partial_{xx} - \epsilon^2\partial_{xx}^2)u = un, \ (x,t) \in \mathbb{T} \times [0,T] \\ (\beta^{-2}\partial_{tt} - \partial_{xx} + \epsilon^2\partial_{xx}^2)n = \partial_{xx}(|u|^2), \\ (u(x,0), n(x,0), \partial_t n(x,0)) = (u_0, n_0, n_1) \in H^{s,l} := H^s(\mathbb{T}) \times H^l(\mathbb{T}) \times H^{l-2}(\mathbb{T}), \end{cases} \tag{5.1}$$

where u is complex-valued, n is real-valued, $\mathbb{T} = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$, $T > 0$ is the time-of-existence (to be determined), and $\alpha, \beta > 0$, $s, l \in \mathbb{R}$. When $\epsilon = 0$, QZS is well-known as the classical Zakharov system (ZS), a pair of non-linear PDE developed to model the interaction of Langmuir turbulence waves and ion-acoustic waves. Here $u(x,t)$ denotes the slowly-varying envelope of electric field, and $n(x,t)$ represents an ion-acoustic wave that models the density fluctuation of ions (Zakharov et al., 1972). A thrust of interest in rigorously studying the quantum effects unexplained by ZS came from the physics community (Garcia et al., 2005). There the quantum effect is characterized by a fourth-order perturbation with a quantum parameter $\epsilon > 0$ that is non-negligible when either the ion-plasma frequency is high or the electrons temperature is low; for more background in the physics of this model, see (Haas, 2011; Simpson et al., 2009).

Our goal is to understand the effect of quantum perturbations, represented by the biharmonic operator. We will do this in the context of well-posedness theory, thereby extending results of (Takaoka et al., 1999). We show that the biharmonic operator provides an extra degree of smoothing that nullifies the distinction between resonance ($\frac{\beta}{\alpha} \in \mathbb{Z}$) and non-resonance ($\frac{\beta}{\alpha} \notin \mathbb{Z}$), something which played a central role in (Takaoka et al., 1999). More precisely, we show that the regions of Sobolev exponent pairs $(s, l) \in \mathbb{R}^2$ yielding well-posedness for $\epsilon = 0$ (which depend on $\frac{\beta}{\alpha} \in \mathbb{Z}$ or $\frac{\beta}{\alpha} \notin \mathbb{Z}$), are no longer different when $\epsilon > 0$. We apply the Bourgain norm method to show that if ZS is well-posed in a certain Sobolev space of initial data, then so is QZS. Under the condition $s \geq 0$, we show that our application of Bourgain norm method yields a region of Sobolev exponents for the local well-posedness that is sharp up to the boundary. With the more precise statement given in Section 5.4, we state our main result. We define the region $\Omega_L \subseteq \mathbb{R}^2$ by

$$\Omega_L \coloneqq \{s \geq 0,\ -1 \leq l < 2s + 1,\ -2 < s - l \leq 2\}. \tag{5.2}$$

Theorem 5.1.1. *Let $\alpha, \beta, \epsilon > 0$ and $(s, l) \in \Omega_L$. Then for every $(u_0, n_0, n_1) \in H^{s,l}$, there exists $(u, n, \partial_t n) \in C([0, T], H^{s,l})$, a strong solution to (5.1), where $T = T(\|u_0\|_{H^s}, \|n_0\|_{H^l}, \|n_1\|_{H^{l-2}}) > 0$. The solution is unique in the modified Bourgain space $X \subsetneq C([0, T], H^{s,l})$ and the data-to-solution map is Lipschitz continuous from $H^{s,l}$ to X.*

In the appendix, we give examples of spacetime functions that illustrate the necessity of the condition

$$s \geq -1,\ -1 \leq l \leq 2s + 1,\ -2 \leq s - l \leq 2. \tag{5.3}$$

for control of the nonlinear terms in the appropriate Bourgain norms.

Although the QZS model is relatively new, the method of multilinear weighted estimates via Fourier transform and the Cauchy-Schwarz inequality has been used successfully by many. These include (but are not limited to) Bourgain, Kenig-Ponce-Vega, and Ginibre-Tsutsumi-Velo (Bourgain, 1993a; Kenig et al., 1996b; Kenig et al., 1996a; Ginibre et al., 1997), in applications to various dispersive equations such as KdV, nonlinear Schrödinger equation with various nonlinearities, and ZS on $\mathbb{R}^d$. Additionally, Tao (Tao, 2001) investigated an alternative approach based on orthogonality and dyadic decompositions.

Typically the task of proving boundedness for certain multilinear operators reduces to spacetime Lebesgue-type estimates in Fourier space, which can be a challenge on periodic spatial domains where satisfactory Strichartz estimates are not available. Despite this difficulty, see (Erdoğan and Tzirakis, 2013; Kishimoto, 2013; Bourgain, 1993b) for various applications of Bourgain norm methods to ZS on periodic domains. On $\mathbb{R}^d$, as opposed to the compact case, it is generally expected that there is a wider range of Sobolev exponents for a well-posedness theory, with the full range of Strichartz estimates at one's disposal; for more recent work on QZS on $\mathbb{R}$, see (Fang et al., 2017; Chen et al., 2017; Jiang et al., 2014).

The QZS defines a Hamiltonian PDE with an energy functional H defined on $H^{2,1}$; see Section 5.2 for an explicit representation of this. We show, via the conservation law and an energy method, that the local flow obtained from Theorem 5.1.1 is global whenever initial data are sufficiently regular, with finite energy.

Theorem 5.1.2. *If* $(u_0, n_0, n_1) \in \Omega_G \subseteq H^{s,l}$ *with*

$$\Omega_G := \{(s,l) \in \Omega_L 0 :\leq s - l \leq 2,\ s + l \geq 4\} \cup \{(2,1)\},$$

then the local solution obtained from Theorem 5.1.1 can be extended to a global solution.

Here the difficulty is proving persistence of regularity, given that any initial data with a finite energy has a global solution in $H^{2,1}$, for which we derive an explicit growth rate of Sobolev norms. While our energy method for QZS provides an exponential bound on growth in time, see (Erdoğan and Tzirakis, 2013) for results on polynomial growth rates for the classical ZS on $\mathbb{T}$.

We expect, however, that the above QZS local flow can be uniquely extended to a global flow, from scaling-invariance perspectives suggested in (Ginibre et al., 1997), and provide here a heuristic argument for this. Assuming for the moment that $\alpha = \beta = \epsilon = 1$, suppose the long-time behavior of the solution is governed by the simplified system

$$\begin{cases} (i\partial_t - \Delta^2)u = un, \\ (\partial_{tt} + \Delta^2)n = \Delta(|u|^2). \end{cases} \tag{5.4}$$

Assuming further that both $n(x,t), \partial_t n(x,t)$ have mean zero for all $t \in \mathbb{R}$, consider a change of variable $N_\pm = n \pm i\Delta^{-1}\partial_t n$, under which from the previous equation yields

$$\begin{cases} (i\partial_t - \Delta^2)u = u\frac{N_+ + N_-}{2}, \\ (i\partial_t \mp \Delta)N_\pm = \mp|u|^2. \end{cases} \tag{5.5}$$

If we add the assumption, as in (Ginibre et al., 1997), that the higher order biharmonic operator dominates the scaling property of $N_\pm$, we can neglect the $\mp\Delta$ in the $N_\pm$ equation. Then (5.5) is scale-invariant under

$$u_\lambda(x,t) = \lambda^4 u(\lambda x, \lambda^4 t);\ n_\lambda(x,t) = \lambda^4 n(\lambda x, \lambda^4 t), \tag{5.6}$$

and hence the pair of critical Sobolev exponents is

$$(s_c, l_c) = (\frac{d}{2} - 4, \frac{d}{2} - 4). \tag{5.7}$$

When $d = 1$, (s_c, l_c) is strictly below the region of well-posedness (see (5.2), (5.3)), and hence we expect any QZS local solution to be global. Moreover we remark that Theorem 5.1.2 includes the case $\{s - l = 0\}$, which is relevant from the scaling perspective; recall from (Ginibre et al., 1997) that for ZS, we have $(s_c, l_c) = (\frac{d}{2} - \frac{3}{2}, \frac{d}{2} - 2)$.

In the last part of the paper, we consider the semi-classical limit of QZS to ZS as $\epsilon \to 0$. Under the ϵ-perturbation, we expect the qualitative behavior of solutions to differ from that of the unperturbed system, and hence singular perturbation theory lies at the core of the analysis of QZS. As is well known, similar issues arise in the WKB method, multiscale analysis, and boundary layer theory; see (DeVille et al., 2008) for an application of singular perturbation theory to ODE in the context of renormalization group and normal form method. Here we extend the results of Guo-Zhang-Guo (Guo et al., 2013a) to show that the solutions behave continuously as $\epsilon \to 0$ on a compact time interval. Although their work is on $\mathbb{R}^d$ and for integer Sobolev exponents, an analogue of their argument works on $\mathbb{T}$ as well, and extends to non-integer exponents. On the other hand, we provide a simple example that illustrates that the biharmonic operator $\epsilon^2 \Delta^2$, for any $\epsilon > 0$, is a singular perturbation on an infinite time interval. Here we address a subtlety based on the fact that QZS generates a flow on $H^{s,l}$ whereas the classical ZS does so on $H_0^{s,l} := H^s(\mathbb{T}) \times H^l(\mathbb{T}) \times H^{l-1}(\mathbb{T})$. To overcome this apparent *discontinuity* of solution space, we need to uniformly bound the solution in various norms, with bounds independent of $\epsilon > 0$.

We briefly outline the organization of the paper. In Section 5.2, we introduce important notations and invoke the Lagrangian formulation of (5.1). In Section 5.3, we summarize a set of linear estimates that are used throughout the paper. In Section 5.4, nonlinear estimates are proved and applied to yield local well-posedness of (5.1); in particular, we prove the more precise statement of theorem 5.1.1. In Section 5.5, we extend local solutions to global solutions for a fixed $\epsilon > 0$ and consider the $\epsilon \to 0$ problem. Throughout the paper, $\alpha, \beta > 0$ are fixed and the adiabatic limit $\beta \to \infty$ is not considered.

5.2 Background.

Whenever we take the direct product of normed spaces, we will define the product norm to be the sum of the components, for instance, $\|(u_0, n_0, n_1)\|_{H^{s,l}} = \|u_0\|_{H^s} + \|n_0\|_{H^l} + \|n_1\|_{H^{l-2}}$.

As a consequence of the invariance of (5.1) under $u(x,t) \mapsto e^{i\theta}u(x,t)$ and time-translation, mass and energy are conserved:

$$\begin{aligned} M[u,n,\partial_t n](t) &= \|u\|_{L^2}^2 = \|u_0\|_{L^2}^2 \\ H[u,n,\partial_t n](t) &= \alpha\|\partial_x u\|_{L^2}^2 + \epsilon^2\|\partial_{xx}u\|_{L^2}^2 + \frac{1}{2}\Big(\|n\|_{L^2}^2 + \frac{1}{\beta^2}\|\partial_t n\|_{\dot{H}^{-1}}^2 + \epsilon^2\|\partial_x n\|_{L^2}^2\Big) \\ &\quad + \int n|u|^2. \end{aligned} \tag{5.8}$$

We can assume that n_0, n_1 have zero means. If $\int n_0, \int n_1 \neq 0$, then we can consider the change of variable

$$u(x,t) \mapsto e^{i\left(\frac{t^2}{4\pi}\int n_1 + \frac{t}{2\pi}\int n_0\right)}u(x,t);\; n(x,t) \mapsto n(x,t) - \frac{t}{2\pi}\int n_1 - \frac{1}{2\pi}\int n_0, \tag{5.9}$$

which can be directly checked to satisfy (5.1) with zero means in the new variables. By integrating the second equation of (5.1) over space, one obtains $\frac{d^2}{dt^2}\int_{\mathbb{T}} n = 0$, and therefore the mean zero condition on n_0, n_1 allows us to make sense of $\|\partial_t n\|_{\dot{H}^{-1}}$ in the energy functional. We will use this idea extensively to obtain global solutions.

One expects a Hamiltonian system to have its Lagrangian counterpart via Legendre transform. Define

$$\mathscr{L} = \frac{i}{2}(\overline{u}\partial_t u - u\partial_t\overline{u}) - \alpha\partial_x u\partial_x\overline{u} - (\partial_x\nu)u\overline{u} + \frac{1}{2\beta^2}(\partial_t\nu)^2$$
$$-\frac{1}{2}(\partial_x\nu)^2 - \epsilon^2\partial_{xx}u\partial_{xx}\overline{u} - \frac{\epsilon^2}{2}(\partial_{xx}\nu)^2, \tag{5.10}$$

where u is a complex field, $\overline{u}$, the conjugate field of u, and ν, a real field where we impose $n \coloneqq \partial_x\nu$. The action functional $\mathcal{S}$ corresponding to $\mathscr{L}$ is defined in the usual way as follows:

$$S = \int \mathscr{L}(u, \overline{u}, \nu, \partial_\mu u, \partial_\mu\overline{u}, \partial_\mu\nu)dxdt, \tag{5.11}$$

where ∂_μ denotes higher derivatives. To look for the critical points of $\mathcal{S}$, we impose

$$\delta\mathcal{S} = 0, \tag{5.12}$$

which amounts to solving the Euler-Lagrange equations corresponding to the given fields. One can check that the Euler-Lagrange equation for u yields the first equation of (5.1), and the spatial derivative of the Euler-Lagrange equation for ν yields the second equation of (5.1).

We wish to obtain a strong solution $(u, n, \partial_t n)$ to (5.1) and by this we mean $(u, n, \partial_t n) \in C([0,T], H^{s,l})$ for some $T > 0$ that satisfies the Duhamel's principle

$$u(t) = U_\epsilon(t)u_0 - i\int_0^t U_\epsilon(t-t')(un)(t')dt' \tag{5.13}$$

$$n(t) = \partial_t V_\epsilon(t)n_0 + V_\epsilon(t)n_1 + \beta^2\int_0^t V_\epsilon(t-t')\partial_{xx}(|u|^2)(t')dt',$$

where $U_\epsilon(t), V_\epsilon(t), \partial_t V_\epsilon(t)$ for $\epsilon \geq 0$ are defined via Fourier multipliers as

$$\begin{aligned}
\widehat{U_\epsilon(t)u}(k) &= e^{-it(\alpha k^2+\epsilon^2 k^4)}\widehat{u}(k),\\
\widehat{V_\epsilon(t)n_1}(k) &= \begin{cases} \frac{\sin(\beta|k|\langle\epsilon k\rangle t)}{\beta|k|\langle\epsilon k\rangle}\widehat{n}_1(k), & k \neq 0\\ t\cdot\widehat{n}_1(k), & k=0,\end{cases}\\
\widehat{\partial_t V_\epsilon(t)n_0}(k) &= \cos(\beta|k|\langle\epsilon k\rangle t)\widehat{n}_0(k). \qquad (5.14)
\end{aligned}$$

To obtain low-regularity well-posedness, we define the modified Bourgain norm adapted to the linear operators of interest. Take a complex-valued $f \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ and define

$$\|f\|_{X_S^{s,b}} = \|\langle k\rangle^s\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^b\widehat{f}(k,\tau)\|_{L^2_\tau l^2_k}, \qquad (5.15)$$

$$\|f\|_{X_W^{l,b}} = \|\langle k\rangle^l\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle^b\widehat{f}(k,\tau)\|_{L^2_\tau l^2_k}, \qquad (5.16)$$

from which we define $X_S^{s,b}$ and $X_W^{l,b}$ as the closure of $C_c^\infty(\mathbb{T}\times\mathbb{R})$ with respect to the norms introduced above, respectively. We refer to expressions such as $\langle\tau+\alpha k^2+\epsilon^2k^4\rangle$ and $\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle$ as dispersive weights. These are Bourgain spaces adapted to the given dispersion relations.

Although for $b > \frac{1}{2}$, we have

$$X_S^{s,b} \hookrightarrow C(\mathbb{R}, H^s), \qquad (5.17)$$

we are interested in the endpoint case $b = \frac{1}{2}$ where the continuous embedding into $C(\mathbb{R}, H^s)$ fails. Motivated by the Fourier inversion theorem, we augment the norm and consider

$$\begin{aligned}
\|f\|_{Y_S^s} &= \|f\|_{X_S^{s,\frac{1}{2}}} + \|\widehat{f}(k,\tau)\langle k\rangle^s\|_{l_k^2 L_\tau^1} \\
\|f\|_{Y_W^l} &= \|f\|_{X_W^{l,\frac{1}{2}}} + \|\widehat{f}(k,\tau)\langle k\rangle^s\|_{l_k^2 L_\tau^1},
\end{aligned} \tag{5.18}$$

from which we can recover the desired continuous embedding by the dominated convergence theorem, that is,

$$Y_S^s \hookrightarrow C(\mathbb{R}, H^s), \tag{5.19}$$

and similarly for Y_W^l. To control the Duhamel term coming from the nonlinearities, we consider the companion spaces to Y_S^s, Y_W^l:

$$\begin{aligned}
\|f\|_{Z_S^s} &= \|f\|_{X_S^{s,-\frac{1}{2}}} + \left\|\frac{\langle k\rangle^s}{\langle \tau + \alpha k^2 + \epsilon^2 k^4\rangle}\widehat{f}(k,\tau)\right\|_{l_k^2 L_\tau^1} \\
\|f\|_{Z_W^l} &= \|f\|_{X_W^{l,-\frac{1}{2}}} + \left\|\frac{\langle k\rangle^l}{\langle |\tau| - \beta|k|\langle \epsilon k\rangle\rangle}\widehat{f}(k,\tau)\right\|_{l_k^2 L_\tau^1}.
\end{aligned} \tag{5.20}$$

To obtain solutions for small time, we further define the time-restricted space for $T > 0$

$$\|f\|_{X_{S,T}^{s,b}} = \inf_{\tilde{f}=f,\, t\in[0,T]} \|\tilde{f}\|_{X_S^{s,b}}, \tag{5.21}$$

where such restriction for other Bourgain spaces can be defined analogously.

5.3 Linear estimates.

Here we assume $\alpha, \beta, \epsilon > 0$.

Lemma 5.3.1 (Homogeneous Estimates)**.** *For* $s, l \in \mathbb{R}$,

$$\|U_\epsilon(t)u_0\|_{H^s} = \|u_0\|_{H^s};\ \|\psi(t)U_\epsilon(t)u_0\|_{Y^s_S} \leq c_1(\psi)\|u_0\|_{H^s}, \forall \epsilon \geq 0.$$
$$\|\partial_t V_\epsilon(t)n_0\|_{H^l} \leq \|n_0\|_{H^l};\ \|\psi(t)\partial_t V_\epsilon(t)n_0\|_{Y^l_W} \leq c_2(\psi)\|n_0\|_{H^l}, \forall \epsilon \geq 0.$$
$$\|V_\epsilon(t)n_1\|_{H^l} \leq c\Big(t + \frac{1}{\beta\epsilon}\Big)\|n_1\|_{H^{l-2}};\ \|\psi(t)V_\epsilon(t)n_1\|_{Y^l_W} \leq c_3(\psi)\Big(1 + \frac{1}{\beta\epsilon}\Big)\|n_1\|_{H^{l-2}},$$
$$\|V_0(t)n_1\|_{H^l} \leq c\Big(t + \frac{1}{\beta}\Big)\|n_1\|_{H^{l-1}};\ \|\psi(t)V_0(t)n_1\|_{Y^l_W} \leq c_4(\psi)\Big(1 + \frac{1}{\beta}\Big)\|n_1\|_{H^{l-1}}.$$

Proof. The first line of inequalities follows from the unitarity of Schrödinger operator; see (Tao, 2006) and lemma 1.2.1. A similar argument can be used to show the other inequalities. □

Lemma 5.3.2 (Duhamel Estimates)**.** *For* $s, l \in \mathbb{R}$ *and* $\rho \in [0,1]$,

$$\left\|\psi(t)\int_0^t U_\epsilon(t-t')F(t')dt'\right\|_{Y^s_S} \leq c_1(\psi)\|F\|_{Z^s_S},$$
$$\left\|\psi(t)\int_0^t V_\epsilon(t-t')|\nabla|^{2-\rho}F(t')dt'\right\|_{Y^l_W} \leq c_2(\psi)c_2(\rho,\beta,\epsilon)\|F\|_{Z^l_W},$$
$$\left\|\psi(t)\int_0^t \partial_t V_\epsilon(t-t')D^{2-\rho}F(t')dt'\right\|_{Y^{l-2}_W} \leq c_3(\psi)\|F\|_{Z^l_W}.$$

Proof. For the first inequality, see lemma 1.2.1. The second and third are proved similarly where

$$c_2(\rho,\beta,\epsilon) = \begin{cases} \frac{\left(\frac{1-\rho}{\rho\epsilon^2}\right)^{\frac{1-\rho}{2}}}{\beta\rho^{-1/2}}, & \rho \in (0,1) \\ \frac{1}{\beta\epsilon}, & \rho = 0 \\ \frac{1}{\beta}, & \rho = 1. \end{cases}$$

□

The following estimates allow us to extract a (small) positive time factor, which is applied to obtain local well-posedness.

Lemma 5.3.3. *Let $T \leq 1$, $s, l \in \mathbb{R}$, and $-\frac{1}{2} < b \leq b' < \frac{1}{2}$. Then*

$$\|\psi(t/T)u\|_{X_S^{s,b}} \lesssim_{\psi,b,b'} T^{b'-b}\|u\|_{X_S^{s,b'}},$$
$$\|\psi(t/T)u\|_{X_W^{l,b}} \lesssim_{\psi,b,b'} T^{b'-b}\|u\|_{X_W^{l,b'}}.$$

Proof. The first inequality follows from lemma 1.2.1. A similar argument can be used to show the second inequality. □

5.4 Nonlinear estimates.

Here we fix $\alpha, \beta, \epsilon > 0$ and $T \in (0, 1]$.

Proposition 5.4.1. *For $0 < \rho \leq 1$, suppose $s \geq 0$, $-1 \leq l \leq 2s + 1 - \rho$, $-2 + \rho \leq s - l \leq 2$ and $b \in (\frac{1}{6}, \frac{1}{2}]$. Then there exists $C = C(\alpha, \beta, \epsilon, \rho, s, l, b) > 0$ such that*

$$\|un\|_{X_S^{s,-\frac{1}{2}}} \leq C(\|u\|_{X_S^{s,b}}\|n\|_{X_W^{l,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|n\|_{X_W^{l,b}}),$$
$$\|D^\rho(u\overline{v})\|_{X_W^{l,-\frac{1}{2}}} \leq C(\|u\|_{X_S^{s,b}}\|v\|_{X_S^{s,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|v\|_{X_S^{s,b}}).$$

Proposition 5.4.2. *Assume the hypotheses of proposition 5.4.1. Then,*

$$\left\|\frac{\langle k\rangle^s}{\langle \tau + \alpha k^2 + \epsilon^2 k^4\rangle}\widehat{un}(k,\tau)\right\|_{l_k^2 L_\tau^1} \lesssim \|u\|_{X_S^{s,b}}\|n\|_{X_W^{l,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|n\|_{X_W^{l,b}},$$
$$\left\|\frac{\langle k\rangle^l}{\langle |\tau| - \beta|k|\langle \epsilon k\rangle\rangle}\widehat{D^\rho(u\overline{v})}(k,\tau)\right\|_{l_k^2 L_\tau^1} \lesssim \|u\|_{X_S^{s,b}}\|v\|_{X_S^{s,\frac{1}{2}}} + \|u\|_{X_S^{s,\frac{1}{2}}}\|v\|_{X_S^{s,b}}.$$

Corollary 5.4.1. *Assume the same hypotheses as before. Then for some $\theta \in (0, \frac{1}{3})$,*

$$\|un\|_{Z_S^s} \lesssim T^\theta\|u\|_{Y_S^s}\|n\|_{Y_W^l},$$
$$\|D^\rho(|u|^2)\|_{Z_W^l} \lesssim T^\theta\|u\|_{Y_S^s}^2.$$

Remark 5.4.1. *Note that corollary 5.4.1 is an immediate consequence of proposition 5.4.1, proposition 5.4.2, and lemma 5.3.3. Here we will not be concerned with obtaining an optimal range for b or θ.*

Remark 5.4.2. *Note that the LHS in proposition 5.4.2 is controlled by the same term as in proposition 5.4.1. Though proposition 5.4.2 is proved in a similar way to* *proposition 5.4.1 via the duality trick, one needs to be wary of the L^1_τ-estimate; see (Takaoka et al., 1999) for more detail.*

Remark 5.4.3. *The method of direct estimation by the Cauchy-Schwarz inequality does not seem to work, at least directly, when $\rho = 0$. One can check that the τ_1-integral in (5.50) is not justified. In fact if $k = 0$, then $IV = \infty$ by a direct computation.*

As a corollary, we show

Theorem 5.4.1. *If $(s,l) \in \Omega_L$, then (5.1) is locally well-posed; that is, there exists $T = T(\|u_0\|_{H^s}, \|n_0\|_{H^l}, \|n_1\|_{H^{l-2}}) > 0$ and a unique $(u, n, \partial_t n) \in Y^s_{S,T} \times Y^l_{W,T} \times Y^{l-2}_{W,T}$ that satisfies (5.1). Further, if $T' \in (0,T)$, then there exists a neighborhood $B \subseteq H^{s,l}$ around (u_0, n_0, n_1) such that the data-to-solution map $(u_0, n_0, n_1) \mapsto (u, n, \partial_t n)$ is Lipschitz-continuous from B to $Y^s_{S,T} \times Y^l_{W,T} \times Y^{l-2}_{W,T}$.*

proof of theorem 5.4.1. Define $X = Y^s_{S,T} \times Y^l_{W,T} \times Y^{l-2}_{W,T}$ with

$$\|(u, n, \partial_t n)\| := \|u\|_{X^s_{S,T}} + \|n\|_{Y^l_{W,T}} + \|\partial_t n\|_{Y^{l-2}_{W,T}},$$

and $X(R) = \{(u, n, \partial_t n) \in X : \|(u, n, \partial_t n)\| \leq R\sigma\}$ for $R > 0$ to be determined and $\sigma = \|u_0\|_{H^s} + \|n_0\|_{H^l} + \|n_1\|_{H^{l-2}}$. Further, define a map

$$\Gamma(u, n, \partial_t n) = (\Gamma_1(u, n), \Gamma_2(u), \Gamma_3(u))$$

on $X(R)$ where

$$\Gamma_1(u,n)(t) = \psi(t)U(t)u_0 - i\psi(t)\int_0^t U(t-t')(un)(t')dt',$$

$$\Gamma_2(u)(t) = \psi(t)\partial_t V(t) n_0 + \psi(t)V(t)n_1 + \beta^2\psi(t)\int_0^t V(t-t')\partial_{xx}(|u|^2)(t')dt', \quad (5.22)$$

and $\Gamma_3(u)(t) = \partial_t \Gamma_2(u)(t)$ is defined in the sense of distribution. If $(s,l) \in \Omega_L$, pick $\rho > 0$ sufficiently small such that the hypotheses of proposition 5.4.1 are fulfilled. Then by corollary 5.4.1, there exists $\theta > 0$ such that

$$\|\Gamma(u, n, \partial_t n)\| \lesssim \sigma + T^\theta \|u\|_{Y^s_{S,T}} \|n\|_{Y^l_{W,T}} + T^\theta \|u\|^2_{Y^s_{S,T}} \leq \sigma + 2T^\theta R^2 \sigma^2. \quad (5.23)$$

If R is chosen sufficiently big (depending on the choice of given parameters), then by choosing $0 < T \lesssim \sigma^{-\theta}$, we conclude that Γ maps into $X(R)$. Similarly given $(u_1, n_1, \partial_t n_1), (u_2, n_2, \partial_t n_2) \in X$, we have

$$
\begin{aligned}
&\|\Gamma(u_1, n_1, \partial_t n_1) - \Gamma(u_2, n_2, \partial_t n_2)\| \\
&\lesssim T^\theta \Big(\|u_1\|_{Y^s_{S,T}} \|n_1 - n_2\|_{Y^l_{W,T}} \\
&+ \|n_2\|_{Y^l_{W,T}} \|u_1 - u_2\|_{Y^s_{S,T}} \Big) + T^\theta \Big(\|u_1\|_{Y^s_{S,T}} \|u_1 - u_2\|_{Y^s_{S,T}} + \|u_2\|_{Y^s_{S,T}} \|u_1 - u_2\|_{Y^s_{S,T}} \Big) \\
&\lesssim T^\theta R\sigma \|(u_1, n_1, \partial_t n_1) - (u_2, n_2, \partial_t n_2)\|, \qquad (5.24)
\end{aligned}
$$

and by choosing $T \lesssim \sigma^{-\theta}$ sufficiently small, Γ defines a contraction operator on $X(R)$, and hence there exists a unique $(u, n, \partial_t n) \in X \hookrightarrow C([0,T], H^{s,l})$ that satisfies (5.1). Local lipschitz-continuity of the data-to-solution map follows from the contraction mapping principle. □

Remark 5.4.4. *By construction, $\partial_t n$ is indeed the distributional derivative of n. By defining X to include $\partial_t n$, not just (u, n), we obtain the continuity of $\partial_t n$ in t as a result of the contraction argument.*

Now the goal is to show the boundedness of the multilinear operators corresponding to the nonlinear terms which, at a technical level, involves directly estimating a $L^\infty L^1$-norm of a function defined on the spacetime Fourier space in different regions depending on which dispersive weight is most dominant. Observing that

$$
\begin{aligned}
&(\tau + \alpha k^2 + \epsilon^2 k^4) - (\tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4) - (\tau_2 \pm \beta k_2 \langle \epsilon k_2 \rangle) \\
=&(k - k_1)\Big((k + k_1)(\alpha + \epsilon^2(k^2 + k_1^2)) \mp \beta \langle \epsilon(k - k_1) \rangle \Big), \qquad (5.25)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\max \Big(|\tau + \alpha k^2 + \epsilon^2 k^4|, |\tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4|, \big||\tau_2| - \beta |k_2| \langle \epsilon k_2 \rangle\big| \Big) \\
\geq &\frac{1}{3} |k - k_1| \, \big|(k + k_1)(\alpha + \epsilon^2(k^2 + k_1^2)) \mp \beta \langle \epsilon(k - k_1) \rangle\big| , \qquad (5.26)
\end{aligned}
$$

where the sign on the RHS of (5.26) depends on τ_2, k_2. Since this subtlety does not affect our subsequent analysis, we do not keep track of the sign. For notational convenience, we define $h(k, k_1) = (k + k_1)(\alpha + \epsilon^2(k^2 + k_1^2)) \mp \beta\langle\epsilon(k - k_1)\rangle$. We set $\epsilon = 1$ where it is assumed that implicit constants depend on $\epsilon > 0$.

Lemma 5.4.1. *For all $e_1 > \frac{1}{4}, e_2 > \frac{1}{3}$, and $k \neq 0$ for the second inequality,*

$$\sigma_1(k,\tau) := \sum_{k_1 \neq k; \pm} \frac{1}{\langle \epsilon^2 k_1^4 + \alpha k_1^2 + \tau \pm \beta(k - k_1)\langle\epsilon(k-k_1)\rangle\rangle^{e_1}} \leq c_1(\alpha, \beta, \epsilon, e_1) < \infty,$$

$$\sigma_2(k,\tau) := \sum_{k_1} \frac{1}{\langle k_1^3 - \frac{3k}{2}k_1^2 + \left(\frac{\alpha + 2\epsilon^2 k^2}{2\epsilon^2}\right)k_1 + \frac{\tau - \alpha k^2 - \epsilon^2 k^4}{4\epsilon^2 k}\rangle^{e_2}} \leq c_2(\alpha, \beta, \epsilon, e_2) < \infty.$$

Lemma 5.4.2. *There exist $C(\alpha, \beta, \epsilon), c(\alpha, \beta, \epsilon) > 0$ such that for all $(k, k_1) \in \mathbb{Z}^2$ that satisfies $\{|k| \geq C(\alpha, \beta, \epsilon)\} \cup \{|k_1| \geq C(\alpha, \beta, \epsilon)\}$, we have*

$$|h(k, k_1)| \geq c(\alpha, \beta, \epsilon)|k - k_1|.$$

Lemma 5.4.3. *There exists $C(\alpha, \beta, \epsilon) > 0$ such that if $\{0 \neq |k| \geq 2|k_1|\}$ $\cap \{|k| \geq C(\alpha, \beta, \epsilon)\}$, we have $|k - k_1||h(k, k_1)| \gtrsim |k|^4$. Similarly if $\left\{0 \neq \frac{|k_1|}{2} \geq |k|\right\}$ $\cap \{|k_1| \geq C(\alpha, \beta, \epsilon)\}$, then $|k - k_1||h(k, k_1)| \gtrsim |k_1|^4$.*

proof of lemma 5.4.1. The second inequality can be proven in a similar way as to (Erdoğan and Tzirakis, 2013, lemma 3(c)). For the first inequality, there exists $c > 0$ independent of k, k_1 such that

$$|(k - k_1)\langle\epsilon(k - k_1)\rangle - (k - k_1)|\epsilon(k - k_1)|| \leq c.$$

Hence the term $\langle\epsilon(k - k_1)\rangle$ in the summation can be replaced with $|\epsilon(k - k_1)|$. Then

$$\sigma_1(k,\tau) \leq \sum_{k_1 \neq k; \pm} \frac{1}{\langle \epsilon^2 k_1^4 + \alpha k_1^2 + \tau \pm \beta\epsilon(k - k_1)^2\rangle^{e_1}} \leq c',$$

where the constant is independent of k, τ by an argument similar to (Erdoğan and Tzirakis, 2013, lemma 3(c)). □

proof of lemma 5.4.2. Assume $k \neq k_1$. For a fixed $k \in \mathbb{Z}$, let $r_\mp(k) \in \mathbb{R}$ be the unique real-root of $h(k, \cdot)$ where $r_-(k)$ corresponds to the minus sign in $h(k, \cdot)$, and similarly for $r_+(k)$; we drop the $\mp$-subscript. Noting that h is symmetric in both arguments, it suffices to assume $|k| \geq C(\alpha, \beta, \epsilon)$ where

$$C(\alpha, \beta, \epsilon) := \max\Big(C_1(\alpha, \beta, \epsilon), \sqrt{\frac{3\sqrt{2}\beta}{\epsilon}}, \frac{1}{3\epsilon}\Big) \tag{5.27}$$

where for all $|k| \geq C_1(\alpha, \beta, \epsilon) > 0$, we have $\beta\langle \epsilon k\rangle < |k|(\alpha + \epsilon^2 k^2)$.

We first show that for k sufficiently big, $r(k) \notin \mathbb{Z}$. For $k \in \mathbb{Z}$, consider the graphs of $k_1 \mapsto (k + k_1)(\alpha + \epsilon^2(k^2 + k_1^2))$ and $k_1 \mapsto \pm\beta\langle\epsilon(k - k_1)\rangle$. If we require that the y-intercept of the cubic polynomial is greater (in magnitude) than that of the square-root term, i.e., $\beta\langle\epsilon k\rangle < |k|(\alpha + \epsilon^2 k^2)$, then $r(k) \in [-c_2 k, 0]$ for $k > 0$ and $r(k) \in [0, -c_2 k]$ for $k < 0$ where $c_2 = c_2(\alpha, \beta, \epsilon) > 0$.

Now we claim $\lim\limits_{|k|\to\infty} |r(k) + k| = 0$. From $h(k, r(k)) = 0$, we obtain

$$|r(k) + k| = \left|\frac{\beta\langle\epsilon(k - r(k))\rangle}{\alpha + \epsilon^2(k^2 + r(k)^2)}\right| \lesssim \frac{\beta\epsilon|k - r(k)|}{\alpha + \epsilon^2 k^2} \leq \frac{(1 + c_2)\beta\epsilon|k|}{\alpha + \epsilon^2 k^2} \xrightarrow[|k|\to\infty]{} 0. \tag{5.28}$$

Hence if $|k|$ is sufficiently big and $r(k) \in \mathbb{Z}$, then $r(k) = -k$, which cannot be since $|h(k, -k)| = \beta\langle 2\epsilon k\rangle \geq \beta$. For $k \in \mathbb{Z}$, to show $\inf\limits_{k_1\in\mathbb{Z}} |h(k, k_1)|$ is attained at $k_1 = -k$, note that from standard calculus,

$$\partial_{k_1} h(k, k_1) = 3\epsilon^2 k_1^2 + 2\epsilon^2 k k_1 + \alpha + \epsilon^2 k^2 \pm \frac{\beta\epsilon^2(k - k_1)}{\langle\epsilon(k - k_1)\rangle} \geq \alpha + \frac{2}{3}\epsilon^2 k^2 \pm \frac{\beta\epsilon^2(k - k_1)}{\langle\epsilon(k - k_1)\rangle}, \tag{5.29}$$

and since $\frac{\beta\epsilon^2|k-k_1|}{\langle\epsilon(k-k_1)\rangle} \leq \beta\epsilon$, we have $\partial_{k_1} h \geq \alpha$ by (5.27), and hence

$$\inf_{|k|\geq C(\alpha,\beta,\epsilon),\ (k,k_1)\in\mathbb{Z}^2} |h(k, k_1)| \geq \beta.$$

If $|k - k_1| \leq 3|k|$, then $\inf\limits_{|k|\geq C(\alpha,\beta,\epsilon),\ (k,k_1)\in\mathbb{Z}^2} |\frac{h(k,k_1)}{k-k_1}| \geq \frac{\beta}{3C(\alpha,\beta,\epsilon)}$. If $|k - k_1| \geq 3|k|$, then $|k_1| \geq 2|k|$, $|k + k_1| \geq \frac{|k_1|}{2}$, and $|k - k_1| \leq \frac{3|k_1|}{2}$. Furthermore

$$\left|\frac{h(k, k_1)}{k - k_1}\right| \geq \frac{1}{3}(\alpha + \epsilon^2(k^2 + k_1^2)) - \left|\beta\frac{\langle\epsilon(k - k_1)\rangle}{k - k_1}\right| \geq \frac{1}{3}(\alpha + \epsilon^2(k^2 + k_1^2)) - \sqrt{2}\beta\epsilon \geq \frac{\alpha}{3}, \tag{5.30}$$

where the last inequality is by (5.27). $\square$

proof of lemma 5.4.3. Since h is symmetric in k, k_1, it suffices to prove the first statement. For $|k| \geq 2|k_1|$, we have $|k \pm k_1| \geq \frac{|k|}{2}$. If we further assume $|k| \geq \frac{2}{\epsilon}$, we have $\epsilon|k-k_1| \geq \frac{\epsilon}{2}|k| \geq 1$, and therefore

$$\beta\langle\epsilon(k-k_1)\rangle \leq \sqrt{2}\beta\epsilon|k-k_1| \leq \sqrt{2}\beta\epsilon(|k|+|k_1|) \leq \frac{3}{\sqrt{2}}\beta\epsilon|k|. \tag{5.31}$$

By the triangle inequality, if $|k| \geq 10\max(\frac{1}{\epsilon}, \sqrt{\frac{\beta}{\epsilon}})$,

$$|k-k_1||h(k,k_1)| \geq \frac{|k|}{2}\Big(\frac{\epsilon^2}{2}|k|^3 - \frac{3}{\sqrt{2}}\beta\epsilon|k|\Big) \geq \frac{\epsilon^2}{8}|k|^4. \tag{5.32}$$

□

proof of proposition 5.4.1. Though the main idea of this proof follows closely that of (Takaoka et al., 1999), we include a full proof here to address any subtleties that rise from the fourth-order perturbation. To use the duality argument, let $w \in L^2_{k,\tau}$, $\|w\|_{L^2} = 1$ and $w \geq 0$. Since

$$\|un\|_{X_S^{s,-\frac{1}{2}}} = \left\|\frac{\langle k\rangle^s}{\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^{1/2}} \sum_{k_1+k_2=k} \int_{\tau_1+\tau_2=\tau} \hat{u}(\tau_1,k_1)\hat{n}(\tau_2,k_2)\right\|_{l^2_kL^2_\tau}, \tag{5.33}$$

it suffices to estimate

$$E := \sum_{k_1+k_2-k=0} \int_{\tau_1+\tau_2-\tau=0} \frac{\langle k\rangle^s\langle k_1\rangle^{-s}\langle k_2\rangle^{-l} f(\tau_1,k_1)g(\tau_2,k_2)w(\tau,k)}{\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^{1/2}\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle^{b_1}\langle|\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle\rangle^{b_2}}, \tag{5.34}$$

where

$$f(\tau,k) = |\hat{u}(\tau,k)|\langle k\rangle^s\langle\tau+\alpha k^2+\epsilon^2k^4\rangle^{b_1};\, g(\tau,k) = |\hat{n}(\tau,k)|\langle k\rangle^l\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle^{b_2}, \tag{5.35}$$

and $b_1, b_2 \leq \frac{1}{2}$. By direct computation, one can rule out $k_2 = 0$, and hence we assume the sum is over $k_2 \neq 0$, or equivalently, $k_1 \neq k$.

I. $\max\Big(|\tau+\alpha k^2+\epsilon^2k^4|, |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|, ||\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle|\Big) = |\tau+\alpha k^2+\epsilon^2k^4|$.

Let $b_1 = b_2 = b = \frac{1}{2}-$. Applying the Cauchy-Schwarz inequality in variables (k_1,τ_1) and (k_2,τ_2), followed by the Young's inequality, it suffices to show

$$\sup_{\tau,k} I := \sup_{\tau,k} \frac{\langle k\rangle^{2s}}{\langle \tau+\alpha k^2+\epsilon^2 k^4\rangle} \cdot \sum_{k_1\neq k} \int \frac{d\tau_1}{\langle k_1\rangle^{2s}\langle k-k_1\rangle^{2l}\langle \tau_1+\alpha k_1^2+\epsilon^2 k_1^4\rangle^{2b}\langle |\tau-\tau_1|-\beta|k-k_1|\langle \epsilon(k-k_1)\rangle\rangle^{2b}} < \infty, \tag{5.36}$$

since

$$E \lesssim \left(\sup_{\tau,k} I\right)^{1/2} \|u\|_{X_S^{s,b}} \|n\|_{X_W^{l,b}} \|w\|_{L^2_{k,\tau}}. \tag{5.37}$$

Let $\frac{|k|}{2} \leq |k_1| \leq 2|k|$. Integrating in τ_1 via lemma 1.2.1 and noting that $\langle k\rangle^{2s}\langle k_1\rangle^{-2s} \simeq 1$, we have

$$I \lesssim \sum_{k_1\neq k;\pm} \frac{1}{\langle \tau+\alpha k^2+\epsilon^2 k^4\rangle\langle k-k_1\rangle^{2l}\langle \epsilon^2 k_1^4+\alpha k_1^2+\tau\pm\beta(k-k_1)\langle\epsilon(k-k_1)\rangle\rangle^{4b-1}}. \tag{5.38}$$

If $|k| \lesssim 1$, then $\langle k-k_1\rangle \simeq 1$, and therefore the sum above is finite by lemma 5.4.1. On the other hand, by lemma 5.4.2 if $|k| \geq C(\alpha,\beta,\epsilon)$,

$$I \lesssim \sum_{k_1\neq k;\pm} \frac{1}{\langle k-k_1\rangle^{2l+2}\langle \epsilon^2 k_1^4+\alpha k_1^2+\tau\pm\beta(k-k_1)\langle\epsilon(k-k_1)\rangle\rangle^{4b-1}} \leq c_1. \tag{5.39}$$

since $l \geq -1$. Now let $|k| \geq 2|k_1|$. In this region, we have $\frac{|k|}{2} \leq |k-k_1| \leq \frac{3|k|}{2}$ and by lemma 5.4.3, if $|k| \geq C(\alpha,\beta,\epsilon)$, then $\langle \tau+\alpha k^2+\epsilon^2 k^4\rangle \gtrsim \langle k\rangle^4$ and

$$I \lesssim \sum_{k_1\neq k;\pm} \frac{\langle k_1\rangle^{-2s}}{\langle k\rangle^{2l-2s+4}\langle \epsilon^2 k_1^4+\alpha k_1^2+\tau\pm\beta(k-k_1)\langle\epsilon(k-k_1)\rangle\rangle^{4b-1}} \leq c_1, \tag{5.40}$$

since $s-l \leq 2$ and $s \geq 0$. If $|k| \lesssim 1$, then by lemma 5.4.1, $I \lesssim \sigma_1 \leq c_1$. Lastly if $\frac{|k_1|}{2} \geq |k|$, then $\frac{|k_1|}{2} \leq |k-k_1| \leq \frac{3|k_1|}{2}$ and by treating $|k_1| \geq C(\alpha,\beta,\epsilon)$ and $|k_1| \leq C(\alpha,\beta,\epsilon)$ separately as above, we have the desired uniform bound.

II. $\max\Big(|\tau+\alpha k^2+\epsilon^2k^4|, |\tau_1+\alpha k_1^2+\epsilon^2 k_1^4|, ||\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle|\Big) = |\tau_1+\alpha k_1^2+\epsilon^2 k_1^4|$.

Arguing as above, it suffices to show

$$
\begin{aligned}
\sup_{\tau_1,k_1} II &:= \sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s}}{\langle \tau_1+\alpha k_1^2+\epsilon^2 k_1^4\rangle} \\
&\cdot \sum_{k\neq k_1}\int \frac{\langle k\rangle^{2s}d\tau}{\langle k-k_1\rangle^{2l}\langle \tau+\alpha k^2+\epsilon^2k^4\rangle\langle |\tau-\tau_1|-\beta|k-k_1|\langle \epsilon(k-k_1)\rangle\rangle^{2b_2}} \\
&\lesssim \sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s}}{\langle \tau_1+\alpha k_1^2+\epsilon^2 k_1^4\rangle} \sum_{k\neq k_1;\pm} \frac{\langle k-k_1\rangle^{2l}\langle k\rangle^{2s}}{\langle \epsilon^2k^4+\alpha k^2+\tau_1\pm\beta(k_1-k)\langle\epsilon(k-k_1)\rangle\rangle^{2b_2-}},
\end{aligned}
\tag{5.41}
$$

where we set $b_1=\frac{1}{2}$. By lemma 5.4.3, if $|k|\geq 2|k_1|$ and $|k|\geq C(\alpha,\beta,\epsilon)$, then $\langle\tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle\gtrsim\langle k\rangle^4$, and we have

$$
II \lesssim \sum_{k\neq k_1;\pm}\frac{\max(1,\langle k\rangle^{-2s})}{\langle k\rangle^{2l-2s+4}\langle \epsilon^2k^4+\alpha k^2+\tau_1\pm\beta(k_1-k)\langle\epsilon(k-k_1)\rangle\rangle^{2b_2-}}\leq c_1, \tag{5.42}
$$

and similarly, the desired uniform bound of II follows if $|k|\geq 2|k_1|$ and $|k|\lesssim 1$; by applying lemma 5.4.3 again, we can show that II is uniformly bounded for $\frac{|k_1|}{2}\geq|k|$ by treating $|k|\leq C(\alpha,\beta,\epsilon)$ and $|k|\geq C(\alpha,\beta,\epsilon)$ separately. For $\frac{|k|}{2}\leq|k_1|\leq 2|k|$, we can argue as (5.38).

III. $\max\Big(|\tau+\alpha k^2+\epsilon^2k^4|,|\tau_1+\alpha k_1^2+\epsilon^2k_1^4|,||\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle|\Big)=||\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle|.$

From (5.26), we have

$$
||\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle|\gtrsim|k_2|\left|(2k-k_2)(\alpha+\epsilon^2(k^2+(k-k_2)^2))\mp\beta\langle\epsilon k_2\rangle\right|. \tag{5.43}
$$

If suffices to show $\sup\limits_{\tau_2,k_2} III<\infty$ where $b_2=\frac{1}{2}$ and

$$
\begin{aligned}
III &:= \frac{1}{\langle k_2\rangle^{2l}\langle |\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle\rangle} \\
&\cdot\sum_k\int\frac{\langle k\rangle^{2s}d\tau}{\langle k-k_2\rangle^{2s}\langle\tau+\alpha k^2+\epsilon^2k^4\rangle\langle\tau-\tau_2+\alpha(k-k_2)^2+\epsilon^2(k-k_2)^4\rangle^{2b_1}} \\
&\lesssim\sum_k\frac{\langle k\rangle^{2s}}{\langle k_2\rangle^{2l}\langle k-k_2\rangle^{2s}\langle|\tau_2|-\beta|k_2|\langle\epsilon k_2\rangle\rangle\langle 4\epsilon^2k_2p(k)\rangle^{2b_1-}},
\end{aligned}
\tag{5.44}
$$

where

$$
p(k)=k^3-\frac{3k_2}{2}k^2+\Big(\frac{\alpha+2\epsilon^2k_2^2}{2\epsilon^2}\Big)k+\frac{\tau_2-\alpha k_2^2-\epsilon^2k_2^4}{4\epsilon^2k_2}. \tag{5.45}
$$

If $\frac{2}{3}|k_2| \le |k| \le 2|k_2|$, then $\frac{\langle k \rangle^{2s}}{\langle k_2 \rangle^{2l} \langle k-k_2 \rangle^{2s}} \lesssim \frac{1}{\langle k \rangle^{2l-2s} \langle k-k_2 \rangle^{2s}}$. If $|k| \lesssim 1$, $III \lesssim \sigma_2 \le c_2$ by lemma 5.4.1. If $|k| \gg 1$, we argue as in lemma 5.4.3 to obtain

$$\left|(2k - k_2)(\alpha + \epsilon^2(k^2 + (k - k_2)^2)) \mp \beta\langle \epsilon k_2 \rangle\right| \gtrsim |k|^3, \tag{5.46}$$

from which, we estimate

$$III \lesssim \sum_k \frac{\max(1, \langle k \rangle^{-2s})}{\langle k \rangle^{2l-2s+4} \langle k_2 p(k) \rangle^{2b_1-}} \lesssim \sigma_2 \le c_2, \tag{5.47}$$

by lemma 5.4.1 and $l \ge -2$.

If $|k| \le \frac{2}{3}|k_2|$ or $|k| \ge 2|k_2|$, then $\frac{\langle k \rangle^{2s}}{\langle k_2 \rangle^{2l} \langle k-k_2 \rangle^{2s}} \lesssim \frac{1}{\langle k_2 \rangle^{2l}}$ since $|k - k_2| \ge \frac{|k|}{2}$. As in lemma 5.4.2, if $|k_2| \ge C(\alpha, \beta, \epsilon)$, we have

$$III \lesssim \sum_k \frac{1}{\langle k_2 \rangle^{2l+2} \langle k_2 p(k) \rangle^{2b_1-}} \lesssim \sigma_2 \le c_2. \tag{5.48}$$

Lastly if $|k_2| \lesssim 1$ (for $|k| \le \frac{2}{3}|k_2|$) or $|k| \lesssim 1$ (for $|k| \ge 2|k_2|$), then $\langle k_2 \rangle \simeq 1$ and we have $III \lesssim \sigma_2 \le c_2$, which concludes the proof of the first inequality of proposition 5.4.1. To show the second inequality by the duality argument, it suffices to estimate

$$\sum_{k_1+k_2-k=0} \int_{\tau_1+\tau_2-\tau=0} \frac{\langle k \rangle^{l+\rho} \langle k_1 \rangle^{-s} \langle k_2 \rangle^{-s} f(\tau_1, k_1) g(-\tau_2, -k_2) w(\tau, k)}{\langle |\tau| - \beta|k|\langle \epsilon k \rangle \rangle^{1/2} \langle \tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4 \rangle^{b_1} \langle \tau_2 - \alpha k_2^2 - \epsilon^2 k_2^4 \rangle^{b_2}}, \tag{5.49}$$

where $f(\tau, k) = |\hat{u}(\tau, k)| \langle k \rangle^s \langle \tau + \alpha k^2 + \epsilon^2 k^4 \rangle^{b_1}$, $g(\tau, k) = |\hat{v}(\tau, k)| \langle k \rangle^s \langle \tau + \alpha k^2 + \epsilon^2 k^4 \rangle^{b_2}$ and $b_1, b_2 \le \frac{1}{2}$. Since $\rho > 0$, we take $k \ne 0$ in the sum.

IV. $\max\Big(\left||\tau| - \beta|k|\langle \epsilon k \rangle\right|, |\tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4|, |\tau_2 - \alpha k_2^2 - \epsilon^2 k_2^4| \Big) = \left||\tau| - \beta|k|\langle \epsilon k \rangle\right|$.

In this region, the lower bound of the dispersive weight is similar to (5.43). For $b_1 = b_2 = b = \frac{1}{2}-$, it suffices to show

$$\begin{aligned}
\sup_{\tau,k} IV &:= \sup_{\tau,k} \frac{\langle k \rangle^{2l+2\rho}}{\langle |\tau| - \beta|k|\langle \epsilon k \rangle \rangle} \\
&\cdot \sum_{k_1} \int \frac{\langle k_1 \rangle^{-2s} \langle k - k_1 \rangle^{-2s} d\tau_1}{\langle \tau_1 + \alpha k_1^2 + \epsilon^2 k_1^4 \rangle^{2b} \langle \tau - \tau_1 - \alpha(k - k_1)^2 - \epsilon^2 (k - k_1)^4 \rangle^{2b}} \\
&\lesssim \sup_{\tau,k} \frac{\langle k \rangle^{2l+2\rho}}{\langle |\tau| - \beta|k|\langle \epsilon k \rangle \rangle} \sum_{k_1} \frac{1}{\langle k_1 \rangle^{2s} \langle k - k_1 \rangle^{2s} \langle \langle k \rangle p(k_1) \rangle^{4b-1}} < \infty
\end{aligned} \tag{5.50}$$

where p is defined in (5.45). If $\frac{2}{5}|k| \le |k_1| \le \frac{2}{3}|k|$, then $\frac{|k|}{3} \le |k-k_1| \le \frac{5}{3}|k|$ and $\frac{\langle k\rangle^{2l+2\rho}}{\langle k_1\rangle^{2s}\langle k-k_1\rangle^{2s}} \lesssim \frac{1}{\langle k\rangle^{4s-2l-2\rho}}$. For $|k| \lesssim 1$, (5.50) reduces to lemma 5.4.1. If $|k| \ge C(\alpha,\beta,\epsilon)$ as in lemma 5.4.2, we have

$$IV \lesssim \sum_{k_1} \frac{1}{\langle k\rangle^{4s-2l-2\rho+2}\langle\langle k\rangle p(k_1)\rangle^{4b-1}} \lesssim \sigma_2, \tag{5.51}$$

since $4s-2l-2\rho+2 \ge 0$. If $\frac{2}{3}|k| \le |k_1| \le \frac{3}{2}|k|$ and $|k| \lesssim 1$, then again (5.50) reduces to lemma 5.4.1. If $|k| \ge C(\alpha,\beta,\epsilon)$, then as in lemma 5.4.3,

$$IV \lesssim \sum_{k_1} \frac{1}{\langle k\rangle^{2s-2l-2\rho+4}\langle\langle k\rangle p(k_1)\rangle^{4b-1}} \le \sigma_2, \tag{5.52}$$

since $s-l \ge -2+\rho$. Similarly for $|k_1| \le \frac{2}{5}|k|$ or $|k_1| \ge \frac{3}{2}|k|$, we treat $|k| \lesssim 1$ and $|k| \ge C(\alpha,\beta,\epsilon)$ separately where for $|k| \ge C(\alpha,\beta,\epsilon)$, we have $\langle|\tau-\beta|k|\langle\epsilon k\rangle|\rangle \gtrsim \langle k\rangle^4$, and therefore we can argue as (5.52).

V. $\max\Big(\left||\tau|-\beta|k|\langle\epsilon k\rangle\right|, |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|, |\tau_2-\alpha k_2^2-\epsilon^2k_2^4|\Big) = |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|$.

It suffices to show

$$\begin{aligned}\sup_{\tau_1,k_1} V &:= \sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s}}{\langle \tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle} \\ &\cdot \sum_k \int \frac{\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s}d\tau}{\langle|\tau|-\beta|k|\langle\epsilon k\rangle\rangle\langle\tau-\tau_1-\alpha(k-k_1)^2-\epsilon^2(k-k_1)^4\rangle^{2b_2}} \\ &\lesssim \sup_{\tau_1,k_1} \frac{\langle k_1\rangle^{-2s}}{\langle \tau_1+\alpha k_1^2+\epsilon^2k_1^4\rangle} \sum_{k,\pm} \frac{\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s}}{\langle\epsilon^2(k-k_1)^4+\alpha(k-k_1)^2+\tau_1\mp\beta k\langle\epsilon k\rangle\rangle^{2b_2-}}\end{aligned} \tag{5.53}$$

where $|\tau_1+\alpha k_1^2+\epsilon^2k_1^4| \gtrsim |k|\cdot|\alpha(2k_1-k)(\alpha+\epsilon^2(k_1^2+(k-k_1)^2))\mp\beta\langle\epsilon k\rangle|$ and $b_1=\frac{1}{2}$. Note that for $\max\Big(\left||\tau|-\beta|k|\langle\epsilon k\rangle\right|, |\tau_1+\alpha k_1^2+\epsilon^2k_1^4|, |\tau_2-\alpha k_2^2-\epsilon^2k_2^4|\Big) = |\tau_2-\alpha k_2^2-\epsilon^2k_2^4|$, the corresponding $L^\infty L^1$ estimate reduces to the current case by an appropriate change of variable.

If $|k| \le \frac{|k_1|}{2}$ or $\frac{3|k_1|}{2} \le |k| \le \frac{5|k_1|}{2}$, then $|k-k_1| \gtrsim |k_1|$, and therefore $\langle k_1\rangle^{-2s}\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s} \lesssim \langle k_1\rangle^{-4s}\langle k\rangle^{2l+2\rho}$. Hence $V \lesssim \sigma_1$ if $|k_1| \lesssim 1$, and by lemma 5.4.2,

$$V \lesssim \sum_{k,\pm} \frac{\langle k_1\rangle^{-4s}\max(1,\langle k_1\rangle^{2l+2\rho-2})}{\langle\epsilon^2(k-k_1)^4+\alpha(k-k_1)^2+\tau_1\mp\beta k\langle\epsilon k\rangle\rangle^{2b_2-}} \lesssim \sigma_1, \tag{5.54}$$

if $|k_1| \geq C(\alpha,\beta,\epsilon)$. If $\frac{|k_1|}{2} \leq |k| \leq \frac{3|k_1|}{2}$, then $\langle k_1\rangle^{-2s}\langle k\rangle^{2l+2\rho}\langle k-k_1\rangle^{-2s} \lesssim \frac{1}{\langle k\rangle^{2s-2l-2\rho}}$. If $|k| \lesssim 1$, then $V \lesssim \sigma_1$, and by lemma 5.4.3 if $|k| \geq C(\alpha,\beta,\epsilon)$, then $V \lesssim \sigma_1$ since $2s-2l-2\rho+4 \geq 0$. A similar statement follows for $\frac{5|k_1|}{2} \leq |k|$ if $s-l \geq -2+\rho$ since $|k-k_1| \gtrsim |k|$ and $\langle \tau_1+\alpha k_1^2+\epsilon^2 k_1^4\rangle \gtrsim \langle k\rangle^4$ for sufficiently large $|k|$ by lemma 5.4.3. □

We modify the examples of spacetime functions given in (Takaoka et al., 1999) to give a converse statement for proposition 5.4.1. From the next result, it is deduced that $s \geq -1+\frac{\rho}{2}$ is necessary for proposition 5.4.1 to hold. It is of interest to find out whether proposition 5.4.1 holds for $s \in [-1+\frac{\rho}{2}, 0)$ when $\rho \in [0,1]$. The proof of the following proposition is presented in the appendix.

Proposition 5.4.3. *Suppose* $\|un\|_{X_S^{s,b-1}} \lesssim \|u\|_{X_S^{s,b}}\|n\|_{X_W^{l,b}}$ *holds for all* $u,n \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ *for some* $s,l,b \in \mathbb{R}$. *Then* $l \geq \max(2(b-1),-2b) \geq -1$ *and* $s-l \leq \min(-4(b-1),4b) \leq 2$. *Furthermore suppose* $\|D^\rho(u\overline{v})\|_{X_W^{l,b-1}} \lesssim \|u\|_{X_S^{s,b}}\|v\|_{X_S^{s,b}}$ *holds for all* $u,v \in C_c^\infty(\mathbb{T}\times\mathbb{R})$ *for some* $s,l,b \in \mathbb{R}$, $\rho \in (0,1]$. *Then* $2s-l-\rho \geq \max(2(b-1),-2b) \geq -1$ *and* $s-l \geq \max(\rho+4(b-1),\rho-4b) \geq -2+\rho$.

5.5 Global well-posedness and semi-classical limit.

Here our goal is to extend the local solutions obtained in the previous section. For simplicity, fix $\alpha=\beta=1$. While Guo-Zhang-Guo (Guo et al., 2013a) used the energy method and a compactness argument to derive global well-posedness results on $\mathbb{R}^d$ for $d=1,2,3$ for initial data with integer Sobolev regularity, we extend their results to the compact domain $\mathbb{T}$ for all initial data in certain fractional Sobolev spaces with improved bounds. We show

Theorem 5.5.1. *If* $(u_0,n_0,n_1) \in \Omega_G$, *then the unique local solution obtained in theorem 5.4.1 can be extended to a global solution. More precisely, there exists* $(u,n,\partial_t n) \in C_{loc}([0,\infty),H^{s,l})$ *that satisfies (5.1) such that for all* $T>0$, $(u,n,\partial_t n)$ *is a unique solution in* $Y_{S,T}^s \times Y_{W,T}^l \times Y_{W,T}^{l-2}$.

To fully exploit the conservation law (5.8), we assume that n_0, n_1 are of mean zero. Recall that both n and $\partial_t n$, assumed to be sufficiently regular, are of mean zero whenever they are defined, and thus $H[u,n,\partial_t n](t) = H[u,n,\partial_t n](0) = H_0 < \infty$. In fact, the nonlinear part of energy is bounded above by the linear part by Gagliardo-Nirenberg inequality. Once we establish a global solution for mean zero data, then we invoke the change of variable (5.9) to conclude that such global extension holds without the mean zero assumption.

To discuss the semi-classical limit to ZS, we denote $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)$ by the QZS flow generated by $(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon)$ for $\epsilon \geq 0$. Given a solution $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)$, we denote H^ϵ by the corresponding energy and H_0^ϵ by H^ϵ at $t = 0$. We remark

Remark 5.5.1. *Let $s \geq 4$. If $\sup\limits_{\epsilon} \|(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon)\|_{H^{s,s-1}} \leq R < \infty$ and $(u_0^0, n_0^0, n_1^0) \in H_0^{s,s-1}$ where $(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon) \xrightarrow[\epsilon \to 0]{H_0^{s-2,s-3}} (u_0^\epsilon, n_0^\epsilon, n_1^\epsilon)$, then $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon) \xrightarrow[\epsilon \to 0]{} (u^0, n^0, \partial_t n^0)$ in $C([0,T], H_0^{s-2,s-3})$.*

It is crucial to obtain a uniform bound on $(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)$ that depends only on R, T (see the next lemma), after which one can adopt the proof of (Guo et al., 2013a, theorem 1.3) to prove remark 5.5.1.

Lemma 5.5.1. *Let $(s,l) \in \Omega_G$ and $\sup\limits_{\epsilon > 0} \|(u_0^\epsilon, n_0^\epsilon, n_1^\epsilon)\|_{H^{s,l}} \leq R$ for some $R > 0$. Then $\sup\limits_{\epsilon > 0} \sup\limits_{t \in [0,\infty)} \|(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)\|_{H_0^{1,0}} \leq C(R)$. If $s \geq 4$, then $\sup\limits_{\epsilon > 0} \|(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)\|_{C_T H_0^{s',s'-1}} \leq C(T,R)$ for all $1 \leq s' \leq s-2$ and $T > 0$.*

We observe that the argument in (Guo et al., 2013a) in studying the $\epsilon \to 0$ problem on $\mathbb{R}^d$ applies to $\mathbb{T}$ as well with certain subtleties, which we clarify in the concluding remarks. Now we state a useful Sobolev space inequality:

Lemma 5.5.2. *Let $d \in \mathbb{N}$, $s \in [-\frac{d}{2}, \frac{d}{2}]$ and consider $H^s(M)$ where $M = \mathbb{R}^d, \mathbb{T}^d$. Then*

$$\|fg\|_{H^s} \lesssim_{d,s} \|f\|_{H^{\frac{d}{2}+}} \|g\|_{H^s}.$$

Proof. If $s = 0$, the statement follows from the Hölder's inequality and the Sobolev embedding $H^{\frac{d}{2}+} \hookrightarrow L^\infty$. If $s < 0$, then

$$\|fg\|_{H^s} = \sup_{\|h\|_{H^{-s}}=1} |\langle fg, h\rangle| \leq \|g\|_{H^s} \sup_{\|h\|_{H^{-s}}=1} \|fh\|_{H^{-s}}, \tag{5.55}$$

and hence it suffices to show the statement for $s > 0$. By the Leibniz's rule,

$$\|fg\|_{H^s} \lesssim \|f\|_{W^{s,q}} \|g\|_{L^r} + \|f\|_{L^\infty} \|g\|_{H^s}, \tag{5.56}$$

where $q \in [2, \infty), r \in (2, \infty]$ are to be determined. The second term is bounded above by $\|f\|_{H^{\frac{d}{2}+}} \|g\|_{H^s}$ again by the Sobolev embedding. To obtain

$$H^{\frac{d}{2}+} \hookrightarrow W^{s,q},\ H^s \hookrightarrow L^r, \tag{5.57}$$

it suffices to have

$$\frac{1}{2} - \frac{1}{q} < \frac{(d/2+) - s}{d},\ \frac{1}{2} - \frac{1}{r} < \frac{s}{d} \tag{5.58}$$

and by noting $\frac{1}{2} = \frac{1}{q} + \frac{1}{r}$, we can pick $r \in (2, \infty]$ such that

$$\frac{1}{2} - \frac{s}{d} < \frac{1}{r} < \frac{(d/2+) - s}{d}, \tag{5.59}$$

which uniquely determines $q \in [2, \infty)$, and therefore validates (5.57). $\square$

proof of theorem 5.5.1. Assume $(s, l) = (2, 1)$. By the Gagliardo-Nirenberg inequality

$$\|f\|_{L^4}^4 \lesssim \|\partial_x f\|_{L^2} \|f\|_{L^2}^3, \tag{5.60}$$

(5.8), and the Young's inequality, we have

$$\left| \int n|u|^2 \right| \leq \frac{1}{4}\|n\|_{L^2}^2 + \frac{\epsilon^2}{2}\|\partial_x u\|_{L^2}^2 + C(\|u_0\|_{L^2}, \epsilon), \tag{5.61}$$

and since

$$\|u(t)\|_{H^2}^2 + \|n(t)\|_{H^1}^2 + \|\partial_t n(t)\|_{H^{-1}}^2 \lesssim \|u_0\|_{L^2}^2 + \|\partial_{xx} u(t)\|_{L^2}^2 + \|n(t)\|_{L^2}^2 \tag{5.62}$$
$$+ \|\partial_x n\|_{L^2}^2 + \|\partial_t n(t)\|_{\dot{H}^{-1}}^2 \lesssim \|u_0\|_{L^2}^2 + (1+\epsilon^{-2})|H_0| + (1+\epsilon^{-2})\left|\int n|u|^2\right|,$$

by using the bound (5.61) to absorb $\|n\|_{L^2}, \|\partial_x u\|_{L^2}$ to the LHS of (5.62), we have $\|(u,n,\partial_t n)\|_{H^{2,1}} \le C(\epsilon)$ for all $t \in \mathbb{R}$ where $C(\epsilon) \xrightarrow[\epsilon \to 0]{} \infty$. For $\epsilon > 0$, this yields a global solution for $(s,l) = (2,1)$.

Let $(s,l) \in \Omega_G \setminus \{(2,1)\}$ and denote $l = s - l_0$ for $0 \le l_0 \le 2$, where since $s + l \ge 4$, we have

$$s \ge 2 + \frac{l_0}{2} \ge 2,\ l \ge 2 - \frac{l_0}{2} \ge 1. \tag{5.63}$$

With $a = l - 2$, multiply $\langle \nabla \rangle^{2a} \partial_t n$ to the second equation of (5.1) and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t n\|_{H^a}^2 + \|\partial_x n\|_{H^a}^2 + \epsilon^2 \|\partial_{xx} n\|_{H^a}^2\Big) = \int (\langle \nabla \rangle^a \partial_t n)(\langle \nabla \rangle^a \partial_{xx} |u|^2)$$
$$\lesssim_a \|\partial_t n\|_{H^a}^2 + \|u\|_{H^{a+2}}^2. \tag{5.64}$$

We first assume $l_0 > 0$. For $T > 0$, we make the following inductive hypothesis:

$$\|u\|_{H^l} \le C(T, l_0, \epsilon) < \infty, \tag{5.65}$$

from which Gronwall's inequality on (5.64) yields

$$\|\partial_t n\|_{H^a}^2 + \|\partial_x n\|_{H^a}^2 + \epsilon^2 \|\partial_{xx} n\|_{H^a}^2 \le C(T), \tag{5.66}$$

which, together with the conservation of energy, controls $\|n\|_{H^l}$.

Now take ∂_t of the first equation of (5.1), multiply the resulting equation by $-i\langle \nabla \rangle^{2b} \overline{\partial_t u}$, integrate by parts, and take its real part to obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|_{H^b}^2 = Im \int \Big(\langle\nabla\rangle^b \overline{\partial_t u}\Big)\Big(\langle\nabla\rangle^b(\partial_t u \cdot n + u \cdot \partial_t n)\Big)$$
$$\leq \|\partial_t u\|_{H^b}(\|\partial_t u \cdot n\|_{H^b} + \|u \cdot \partial_t n\|_{H^b}). \tag{5.67}$$

We re-write the first equation of (5.1),

$$\Delta u = \langle\epsilon\nabla\rangle^{-2}(-i\partial_t u + un), \tag{5.68}$$

and let $b = s - 4$. Further, note that $\|\Delta u\|_{H^{s-2}}$ controls $\|u\|_{H^s}$ by mass conservation. We claim

$$\|\partial_t u \cdot n\|_{H^b} \lesssim \|\partial_t u\|_{H^b}\|n\|_{H^l}. \tag{5.69}$$

If $s > \frac{9}{2}$, then $b > \frac{1}{2}$ and we have

$$\|\partial_t u \cdot n\|_{H^b} \lesssim \|\partial_t u\|_{H^b}\|n\|_{H^b} \leq \|\partial_t u\|_{H^b}\|n\|_{H^{s-4}}. \tag{5.70}$$

If $s < \frac{7}{2}$, then $-b > \frac{1}{2}$ and we have

$$\|\partial_t u \cdot n\|_{H^b} = \sup_{\|\phi\|_{H^{-b}}=1} |\langle\partial_t u \cdot n, \phi\rangle| \leq \|\partial_t u\|_{H^b} \sup_{\|\phi\|_{H^{-b}}=1} \|n\phi\|_{H^{-b}} \lesssim \|\partial_t u\|_{H^b}\|n\|_{H^{-b}}$$
$$\leq \|\partial_t u\|_{H^b}\|n\|_{H^l}, \tag{5.71}$$

where the last inequality holds since $s \geq 2 + \frac{l_0}{2}$. If $\frac{7}{2} \leq s \leq \frac{9}{2}$, then $-\frac{1}{2} \leq b \leq \frac{1}{2}$ and by lemma 5.5.2

$$\|\partial_t u \cdot n\|_{H^b} \lesssim \|\partial_t u\|_{H^b}\|n\|_{H^l}. \tag{5.72}$$

Similarly we have $\|u \cdot \partial_t n\|_{H^b} \lesssim 1$, and hence

$$\frac{d}{dt}\|\partial_t u\|_{H^b}^2 \lesssim (\|\partial_t u\|_{H^b}^2 + \|\partial_t u\|_{H^b}), \tag{5.73}$$

from which Gronwall's inequality yields $\|\partial_t u\|_{H^{s-4}} \leq C(T)$. Using similar arguments, we have

$$\|un\|_{H^b} \leq C(T), \tag{5.74}$$

and from

$$\|\Delta u\|_{H^{s-2}} \leq \|\langle \epsilon\nabla\rangle^{-2}\partial_t u\|_{H^{s-2}} + \|\langle \epsilon\nabla\rangle^{-2}(un)\|_{H^{s-2}} \tag{5.75}$$

follows $\|u\|_{H^s} \leq C(T)$. To show (5.65), consider the base case $s_0 = 2 + \frac{l_0}{2}$, where by conservation of energy, $\|u\|_{H^{2-\frac{l_0}{2}}} \leq C$. Then for all $s \in [s_0, s_1]$ where $s_1 = s_0 + l_0$, we have $\|u\|_{H^{s-l_0}} \leq \|u\|_{H^{s_0}}$. We iterate this process by an increment of l_0 to cover the entire range of $s \geq 2 + \frac{l_0}{2}$. It remains to prove the $l_0 = 0$ case. Let $s \geq 2 + \frac{\epsilon_0}{2}$ where $0 \leq \epsilon_0 \leq 1$. As before, we consider the energy estimate

$$\begin{aligned} \frac{d}{dt}\Big(\|\partial_t n\|^2_{H^{s-2}} + \|\partial_x n\|^2_{H^{s-2}} + \epsilon^2\|\partial_{xx} n\|^2_{H^{s-2}}\Big) &\lesssim \|\partial_t n\|^2_{H^{s-2}} + \|u\|^2_{H^s} \\ \frac{d}{dt}\|\partial_t u\|^2_{H^{s-4}} &\lesssim \|\partial_t u\|_{H^{s-4}}(\|\partial_t u \cdot n\|_{H^{s-4}} + \|u \cdot \partial_t n\|_{H^{s-4}}), \end{aligned} \tag{5.76}$$

where by a similar argument as before we derive

$$\|\partial_t u \cdot n\|_{H^{s-4}} \lesssim \|\partial_t u\|_{H^{s-4}}\|n\|_{H^{s-\epsilon_0}}; \ \|u\partial_t n\|_{H^{s-4}} \lesssim \|u\|_{H^s}\|\partial_t n\|_{H^{s-4}}. \tag{5.77}$$

Recall

$$\|\Delta u\|_{H^{s-2}} \lesssim \|\partial_t u\|_{H^{s-4}} + \|un\|_{H^{s-4}} \lesssim \|\partial_t u\|_{H^{s-4}} + \|u\|_{H^{s-\epsilon_0}}\|n\|_{H^{s-\epsilon_0}}, \tag{5.78}$$

and hence

$$\begin{aligned} \frac{d}{dt}\Big(\|\partial_t n\|^2_{H^{s-2}} + \|\partial_x n\|^2_{H^{s-2}} + \epsilon^2\|\partial_{xx} n\|^2_{H^{s-2}}\Big) &\lesssim \|\partial_t n\|^2_{H^{s-2}} + \|\partial_t u\|^2_{H^{s-4}} + \|u_0\|^2_{L^2} \\ &\quad + (\|u\|_{H^{s-\epsilon_0}}\|n\|_{H^{s-\epsilon_0}})^2 \\ \frac{d}{dt}\|\partial_t u\|^2_{H^{s-4}} &\lesssim \|\partial_t u\|^2_{H^{s-4}}\|n\|_{H^{s-\epsilon_0}} \\ &\quad + \|\partial_t u\|_{H^{s-4}}(\|u_0\|_{L^2} + \|\partial_t u\|_{H^{s-4}} + \|u\|_{H^{s-\epsilon_0}}\|n\|_{H^{s-\epsilon_0}})\|\partial_t n\|_{H^{s-4}}. \end{aligned} \tag{5.79}$$

If $\epsilon_0 = 0$, then integrate the first differential inequality of (5.76) to obtain an exponential growth bound on $\|n\|_{H^2} + \|\partial_t n\|_{L^2}$, and then apply the Gronwall's inequality again to the second differential inequality of (5.76). If $s > 2$, use the exponential growth bound for $s = 2$ for the base case $s_0 = 2 + \frac{\epsilon_0}{2}$. Such exponential bound is obtained for all $s \geq 2 + \frac{\epsilon_0}{2}$ by iterating the Gronwall's inequality. Since $\epsilon_0 > 0$ is arbitrary, we have an exponential bound on the Sobolev norms of solutions for all $s \geq 2$. □

proof of lemma 5.5.1. By an inspection, $|H_0^\epsilon| \le C(R)$ uniformly in ϵ. Since mass is conserved and

$$\begin{aligned}\|\partial_x u^\epsilon\|_{L^2}^2 + \frac{1}{2}\|n^\epsilon\|_{L^2}^2 + \frac{1}{2}\|\partial_t n^\epsilon\|_{H^{-1}} &\le |H_0^\epsilon| + \left|\int n^\epsilon |u^\epsilon|^2\right| \\ &\le |H_0^\epsilon| + \frac{1}{4}\|n^\epsilon\|_{L^2}^2 + \frac{1}{2}\|\partial_x u^\epsilon\|_{L^2}^2 + C', \qquad (5.80)\end{aligned}$$

where the last inequality is by the Gagliardo-Nirenberg inequality, and C' is independent of ϵ, we have

$$\sup_{\epsilon} \sup_{t\in[0,\infty)} \|(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)\|_{H_0^{1,0}} \le C(R). \qquad (5.81)$$

Now assume $s \ge 4, T > 0$ and the following inductive hypothesis:

$$\|u^\epsilon\|_{H^{s'-2}}, \|n^\epsilon\|_{H^{s'-2}}, \|n^\epsilon\|_{H^1} \le C(T, R), \qquad (5.82)$$

uniformly in $\epsilon > 0$ and $t \in [0, T]$. Energy conservation and (5.82) yield

$$\|u^\epsilon n^\epsilon\|_{H^{s'-2}} \lesssim C(T, R)$$

since $\|u^\epsilon n^\epsilon\|_{H^{s'-2}} \lesssim \|u^\epsilon\|_{H^{s'-2}} \|n^\epsilon\|_{H^{s'-2}} \le C(T, R)$ for $s' > \frac{5}{2}$ and

$$\|u^\epsilon n^\epsilon\|_{H^{s'-2}} \le \|u^\epsilon n^\epsilon\|_{H^1} \lesssim \|u^\epsilon\|_{H^1} \|n^\epsilon\|_{H^1} \le C(T, R)$$

for $s' \in [1, \frac{5}{2}]$. Moreover since $\|u^\epsilon\|_{\dot{H}^{s'}} \le \|\langle\epsilon\nabla\rangle^2 \Delta u^\epsilon\|_{H^{s'-2}}$ for all $\epsilon \ge 0$, we have

$$\|u^\epsilon\|_{H^{s'}}^2 \lesssim \|u_0^\epsilon\|_{L^2}^2 + \|\partial_t u^\epsilon\|_{H^{s'-2}}^2 + \|u^\epsilon n^\epsilon\|_{H^{s'-2}}^2 \le \|u_0^\epsilon\|_{L^2}^2 + \|\partial_t u^\epsilon\|_{H^{s'-2}}^2 + C(T, R), \qquad (5.83)$$

and hence the differential inequality obtained from the first equation of (5.1) is

$$\begin{aligned}\frac{d}{dt}\Big(\|\partial_t n^\epsilon\|_{H^{s'-2}}^2 + \|\partial_x n^\epsilon\|_{H^{s'-2}}^2 + \epsilon^2 \|\partial_{xx} n^\epsilon\|_{H^{s'-2}}^2\Big) &\lesssim \|\partial_t n^\epsilon\|_{H^{s'-2}}^2 \\ &+ \|\partial_t u^\epsilon\|_{H^{s'-2}}^2 + C(T, R), \qquad (5.84)\end{aligned}$$

where its LHS is well-defined since $s' \le s - 2 \le l$ from $(s, l) \in \Omega_G$.

Similarly we derive another differential inequality as in (5.67) with $b = s' - 2$. A similar calculation as before shows

$$\|\partial_t u^\epsilon \cdot n^\epsilon\|_{H^{s'-2}} \lesssim \|\partial_t u^\epsilon\|_{H^{s'-2}};\ \|u^\epsilon \partial_t n^\epsilon\|_{H^{s'-2}} \lesssim \|\partial_t n^\epsilon\|_{H^{s'-2}}, \tag{5.85}$$

by the inductive hypothesis where the implicit constants are independent of ϵ, and hence by the Young's inequality

$$\frac{d}{dt}\|\partial_t u^\epsilon\|^2_{H^{s'-2}} \lesssim \|\partial_t u^\epsilon\|^2_{H^{s'-2}} + \|\partial_t n^\epsilon\|^2_{H^{s'-2}}. \tag{5.86}$$

Integrating (5.84) and (5.86), we obtain

$$\|\partial_t n^\epsilon\|^2_{H^{s'-2}} + \|\partial_x n^\epsilon\|^2_{H^{s'-2}} + \epsilon^2\|\partial_{xx} n^\epsilon\|^2_{H^{s'-2}} + \|\partial_t u^\epsilon\|^2_{H^{s'-2}} \leq C(T,R). \tag{5.87}$$

Now we check (5.82). By (Guo et al., 2013a, proposition 2.4), we have $\sup_\epsilon \|n^\epsilon\|_{C_T H^1_x} \leq C(T,R)$. On the other hand, if $1 \leq s' \leq 2$, then $\|u^\epsilon\|_{H^{s'-2}}, \|n^\epsilon\|_{H^{s'-2}} \leq C$, independent of ϵ, by (5.81). Hence for such s'

$$\sup_{\epsilon>0}\|(u^\epsilon, n^\epsilon, \partial_t n^\epsilon)\|_{C_T H_0^{s',s'-1}} \leq C(T,R), \tag{5.88}$$

and using $s'_0 = 2$ as a base case, we can extend the uniform bound to all $2 \leq s' \leq 3$ from which we iterate to cover the entire $1 \leq s' \leq s-2$. □

Remark 5.5.2. *When $T = \infty$, we do not expect continuity as $\epsilon \to 0$. In fact, the data-to-solution map (where we extend the domain from D to $D \times [0,\infty)$ where D is the data space and $\epsilon \in [0,\infty)$) fails to be continuous at any $\epsilon \geq 0$ as the following example shows. Let $D = H_0^{s,l}$ or $H^{s,l}$ and $(u_0, n_0, n_1) = (\langle N\rangle^{-s} e^{iNx}, 0, 0)$ for $N \in \mathbb{R} \setminus \{0\}$. Then one can show $(u, n, \partial_t n)(x,t) = (\langle N\rangle^{-s} e^{-it(N^2+\epsilon^2 N^4)+iNx}, 0, 0)$ is the (classical) solution. One can explicitly compute*

$$\begin{aligned}
&\sup_{t\in[0,\infty)} \|\langle N\rangle^{-s} e^{-it(N^2+\epsilon^2N^4)+iNx} - \langle N\rangle^{-s} e^{-it(N^2+\epsilon_0^2N^4)+iNx}\|_{H^s_x} \\
&= \sup_{t\in[0,\infty)} |1 - e^{it(\epsilon^2-\epsilon_0^2)N^4}| = 2.
\end{aligned} \tag{5.89}$$

Note that this example is not valid on $\mathbb{R}^d$.

Remark 5.5.3. *On $\mathbb{R}^d$, the derivative flow $\partial_t n$ is split into a low and high-frequency; see (Guo et al., 2013a). On $\mathbb{T}$, we simply integrate out the mean by (5.9).*

Remark 5.5.4. *One way to prove remark 5.5.1 is to regularize the intial data, which is done via a particular convolution kernel on $\mathbb{R}^d$ in (Guo et al., 2013a), and use the triangle inequality on*

$$\begin{aligned} &(u^\epsilon - u^0, n^\epsilon - n^0, \partial_t n^\epsilon - \partial_t n^0) \\ =&(u^\epsilon - u^{0,h}, n^\epsilon - n^{0,h}, \partial_t n^\epsilon - \partial_t n^{0,h}) + (u^{0,h} - u^0, n^{0,h} - n^0, \partial_t n^{0,h} - \partial_t n^0), \end{aligned} \tag{5.90}$$

where on a periodic domain, one can define a family of mollifiers as a Fourier multiplier as follows: let $\eta \in C_c^\infty(\mathbb{R})$ that is identically one in the neighborhood of the origin. For $h > 0$, define $\widehat{J_h f}(k) = \eta(hk)\widehat{f}(k)$ for all $f \in L^1(\mathbb{T})$. Then $\|J_h f - f\|_{H^s} \xrightarrow[h \to 0]{} 0$ and for $\sigma > 0$

$$\|J_h f - f\|_{H^{s-\sigma}} \leq C(\sigma) h^\sigma \|f\|_{H^s}; \ \|J_h f\|_{H^{s+\sigma}} \leq \frac{C(\sigma)}{h^\sigma} \|f\|_{H^s}. \tag{5.91}$$

5.6 Appendix.

proof of proposition 5.4.3. The main idea is to add a fourth-order perturbation to the spacetime functions constructed in (Takaoka et al., 1999). Once those examples are given, one can directly substitute the examples to the inequalities in proposition 5.4.3 to derive a set of necessary conditions on the scaling parameter $N \gg 1$. Let $\delta(k)$ be the Kronecker delta function defined on $\mathbb{Z}$ and let $\phi(\tau)$ be a smooth bump function on $\mathbb{R}$ with a compact support. It suffices to consider u_i, $1 \leq i \leq 8$, n_i, $1 \leq i \leq 4$, and

$v_i,\ 5 \leq i \leq 8$ where

$$
\begin{aligned}
\widehat{u}_1(k,\tau) &= \delta(k+N)\phi(\tau+\alpha N^2+\epsilon^2 N^4),\\
\widehat{n}_1(k,\tau) &= \delta(k-2N)\phi(|\tau|-2\beta N\langle 2\epsilon N\rangle),\\
\widehat{u}_2(k,\tau) &= \delta(k+N)\phi(\tau+\alpha N^2+\epsilon^2 N^4+2\beta N\langle 2\epsilon N\rangle),\\
\widehat{n}_2(k,\tau) &= \delta(k-2N)\phi(|\tau|-2\beta N\langle 2\epsilon N\rangle),\\
\widehat{u}_3(k,\tau) &= \delta(k)\phi(\tau),\\
\widehat{n}_3(k,\tau) &= \delta(k-N)\phi(|\tau|-\beta N\langle \epsilon N\rangle),\\
\widehat{u}_4(k,\tau) &= \delta(k)\phi(\tau+\alpha N^2+\epsilon^2 N^4+\beta N\langle \epsilon N\rangle),\\
\widehat{n}_4(k,\tau) &= \delta(k-N)\phi(|\tau|-\beta N\langle \epsilon N\rangle),\\
\widehat{u}_5(k,\tau) &= \delta(k-N)\phi(\tau+\alpha N^2+\epsilon^2 N^4),\\
\widehat{v}_5(k,\tau) &= \delta(k+N)\phi(\tau+\alpha N^2+\epsilon^2 N^4),\\
\widehat{u}_6(k,\tau) &= \delta(k-N)\phi(\tau+\alpha N^2+\epsilon^2 N^4-2\beta N\langle 2\epsilon N\rangle),\\
\widehat{v}_6(k,\tau) &= \delta(k+N)\phi(\tau+\alpha N^2+\epsilon^2 N^4),\\
\widehat{u}_7(k,\tau) &= \delta(k)\phi(\tau),\\
\widehat{v}_7(k,\tau) &= \delta(k-N)\phi(\tau+\alpha N^2+\epsilon^2 N^4),\\
\widehat{u}_8(k,\tau) &= \delta(k)\phi(\tau+\alpha N^2+\epsilon^2 N^4+\beta N\langle \epsilon N\rangle),\\
\widehat{v}_8(k,\tau) &= \delta(k-N)\phi(\tau+\alpha N^2+\epsilon^2 N^4).
\end{aligned}
$$

$\square$

References

Babin, A. V., Ilyin, A. A., and Titi, E. S. (2011). On the regularization mechanism for the periodic Korteweg–de Vries equation. *Communications on Pure and Applied Mathematics*, 64(5):591–648.

Barceló, J. A., Bennett, J., Carbery, A., and Rogers, K. M. (2011). On the dimension of divergence sets of dispersive equations. *Mathematische Annalen*, 349(3):599–622.

Beals, M. (1983). Self-spreading and strength of singularities for solutions to semilinear wave equations. *Annals of mathematics*, pages 187–214.

Behzadan, A. and Holst, M. (2015). Multiplication in Sobolev spaces, revisited. *arXiv preprint arXiv:1512.07379*.

Bejenaru, I. and Herr, S. (2011). Convolutions of singular measures and applications to the Zakharov system. *Journal of Functional Analysis*, 261(2):478–506.

Bejenaru, I., Herr, S., Holmer, J., and Tataru, D. (2009). On the 2D Zakharov system with L^2 Schrödinger data. *Nonlinearity*, 22(5):1063.

Bejenaru, I. and Tao, T. (2006). Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation. *Journal of functional analysis*, 233(1):228–259.

Ben-Artzi, M., Koch, H., and Saut, J.-C. (2000). Dispersion estimates for fourth order Schrödinger equations. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 330(2):87–92.

Bombieri, E., Pila, J., et al. (1989). The number of integral points on arcs and ovals. *Duke Math. J*, 59(2):337–357.

Bongioanni, B. and Torrea, J. L. (2006). Sobolev spaces associated to the harmonic oscillator. In *Proceedings of the Indian Academy of Sciences-Mathematical Sciences*, volume 116, pages 337–360. Springer.

Bourgain, J. (1993a). Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. *Geometric and Functional Analysis*, 3(3):209–262.

Bourgain, J. (1993b). On the Cauchy and invariant measure problem for the periodic Zakharov system. Technical report, P00021850.

Bourgain, J. (2013). On the Schrödinger maximal function in higher dimension. *Proceedings of the Steklov Institute of Mathematics*, 280(1):46–60.

Bourgain, J. (2016). A note on the Schrödinger maximal function. *Journal d'Analyse Mathématique*, 130(1):393–396.

Bourgain, J. and Colliander, J. (1996). On wellposedness of the Zakharov system. *International Mathematics Research Notices*, 1996(11):515–546.

Brezis, H. and Mironescu, P. (2018). Gagliardo–Nirenberg inequalities and non-inequalities: The full story. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 35, pages 1355–1376. Elsevier.

Burq, N., Gérard, P., and Tzvetkov, N. (2002). An instability property of the nonlinear Schrödinger equation on S^d. *Mathematical Research Letters*, 9(3):323–335.

Burq, N., Gérard, P., and Tzvetkov, N. (2007). Global solutions for the nonlinear Schrödinger equation on three-dimensional compact manifolds. *Mathematical Aspects of Nonlinear Dispersive Equations, in: Ann. of Math. Stud*, 163:111–129.

Carleson, L. (1966). On convergence and growth of partial sums of Fourier series. *Acta Mathematica*, 116(1):135–157.

Carleson, L. (1980). Some analytic problems related to statistical mechanics. In *Euclidean harmonic analysis*, pages 5–45. Springer.

Cazenave, T. (2003). *Semilinear Schrödinger equations*, volume 10. American Mathematical Soc.

Chen, T.-J., Fang, Y.-F., and Wang, K.-H. (2017). Low regularity global well-posedness for the quantum Zakharov system in 1D. *Taiwanese Journal of Mathematics*, 21(2):341–361.

Cho, C.-h. and Ko, H. (2018). A note on maximal estimates of generalized Schrödinger equation. *arXiv preprint arXiv:1809.03246*.

Choi, B. J. (2019). Global well-posedness of the adiabatic limit of quantum Zakharov system in 1D. *arXiv preprint arXiv:1906.10807*.

Christ, M., Colliander, J., and Tao, T. (2003a). Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. *American journal of mathematics*, 125(6):1235–1293.

Christ, M., Colliander, J., and Tao, T. (2003b). Ill-posedness for nonlinear Schrödinger and wave equations. *arXiv preprint math/0311048*.

Colliander, J., Holmer, J., and Tzirakis, N. (2008). Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems. *Transactions of the American Mathematical Society*, 360(9):4619–4638.

Cowling, M. G. (1983). Pointwise behaviour of solutions to Schrödinger equations. In *Harmonic analysis*, pages 83–90. Springer.

Dahlberg, B. E. and Kenig, C. E. (1982). A note on the almost everywhere behavior of solutions to the Schrödinger equation. In *Harmonic analysis*, pages 205–209. Springer.

DeVille, R. L., Harkin, A., Holzer, M., Josić, K., and Kaper, T. J. (2008). Analysis of a renormalization group method and normal form theory for perturbed ordinary differential equations. *Physica D: Nonlinear Phenomena*, 237(8):1029–1052.

Du, X., Guth, L., and Li, X. (2017). A sharp Schrödinger maximal estimate in $\mathbb{R}^2$. *Annals of Mathematics*, pages 607–640.

Du, X., Guth, L., Li, X., and Zhang, R. (2018). Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates. In *Forum of Mathematics, Sigma*, volume 6. Cambridge University Press.

Du, X. and Zhang, R. (2018). Sharp L^2 estimate of Schrödinger maximal function in higher dimensions. *arXiv preprint arXiv:1805.02775*.

Erdoğan, B. and Tzirakis, N. (2013). Smoothing and global attractors for the Zakharov system on the torus. *Analysis & PDE*, 6(3):723–750.

Erdogan, M. and Tzirakis, N. (2013). Talbot effect for the cubic non-linear Schrödinger equation on the torus. *Mathematical Research Letters*, 20(6):1081–1090.

Erdoğan, M. B. and Tzirakis, N. (2016). *Dispersive partial differential equations: wellposedness and applications*, volume 86. Cambridge University Press.

Fang, Y.-F., Kuo, H.-W., Shih, H.-W., Wang, K.-H., et al. (2019). Semi-classical limit for the quantum Zakharov system. *Taiwanese Journal of Mathematics*.

Fang, Y.-F., Lin, C.-K., and Segata, J.-I. (2016). The fourth-order nonlinear Schrödinger limit for quantum Zakharov system. *Zeitschrift für angewandte Mathematik und Physik*, 67(6):145.

Fang, Y.-F., Segata, J.-i., and Wu, T.-F. (2018). On the standing waves of quantum Zakharov system. *Journal of Mathematical Analysis and Applications*, 458(2):1427–1448.

Fang, Y.-F., Shih, H.-W., and Wang, K.-H. (2017). Local well-posedness for the quantum Zakharov system in one spatial dimension. *Journal of Hyperbolic Differential Equations*, 14(01):157–192.

Faris, W. G. and Lavine, R. B. (1974). Commutators and self-adjointness of Hamiltonian operators. *Communications in Mathematical Physics*, 35(1):39–48.

Fujiwara, D. (1983). Remarks on convergence of Feynman path integrals. In *Theory and Application of Random Fields*, pages 40–48. Springer.

Garcia, L., Haas, F., De Oliveira, L., and Goedert, J. (2005). Modified Zakharov equations for plasmas with a quantum correction. *Physics of Plasmas*, 12(1):012302.

Gidas, B., Ni, W.-M., and Nirenberg, L. (1979). Symmetry and related properties via the maximum principle. *Communications in Mathematical Physics*, 68(3):209–243.

Ginibre, J., Tsutsumi, Y., and Velo, G. (1997). On the Cauchy problem for the Zakharov system. *Journal of Functional Analysis*, 151(2):384–436.

Ginibre, J. and Velo, G. (1979). On a class of nonlinear Schrödinger equations. i. the Cauchy problem, general case. *Journal of Functional Analysis*, 32(1):1–32.

Gulisashvili, A. and Kon, M. A. (1996). Exact smoothing properties of Schrödinger semigroups. *American Journal of Mathematics*, 118(6):1215–1248.

Guo, Y., Zhang, J., and Guo, B. (2013a). Global well-posedness and the classical limit of the solution for the quantum Zakharov system. *Zeitschrift für angewandte Mathematik und Physik*, 64(1):53–68.

Guo, Z., Kwon, S., and Oh, T. (2013b). Poincaré-Dulac normal form reduction for unconditional well-posedness of the periodic cubic NLS. *Communications in Mathematical Physics*, 322(1):19–48.

Haas, F. (2011). *Quantum plasmas: An hydrodynamic approach*, volume 65. Springer Science & Business Media.

Haas, F. and Shukla, P. K. (2009). Quantum and classical dynamics of Langmuir wave packets. *Physical Review E*, 79(6):066402.

Hall, B. C. (2013). *Quantum theory for mathematicians*, volume 267. Springer.

Jiang, J.-C., Lin, C.-K., and Shao, S. (2014). On one dimensional quantum Zakharov system. *arXiv preprint arXiv:1412.2882*.

Kato, T. (1987). On nonlinear Schrödinger equations. In *Annales de l'IHP Physique théorique*, volume 46, pages 113–129.

Keel, M. and Tao, T. (1998). Endpoint Strichartz estimates. *American Journal of Mathematics*, 120(5):955–980.

Kenig, C., Ponce, G., and Vega, L. (1996a). Quadratic forms for the 1-d semilinear Schrödinger equation. *Transactions of the American Mathematical Society*, 348(8):3323–3353.

Kenig, C. E., Ponce, G., and Vega, L. (1995). On the Zakharov and Zakharov-Schulman systems. *Journal of Functional Analysis*, 127(1):204–234.

Kenig, C. E., Ponce, G., and Vega, L. (1996b). A bilinear estimate with applications to the KdV equation. *Journal of the American Mathematical Society*, 9(2):573–603.

Kenig, C. E., Ponce, G., Vega, L., et al. (2001). On the ill-posedness of some canonical dispersive equations. *Duke Mathematical Journal*, 106(3):617–633.

Kishimoto, N. (2013). Local well-posedness for the Zakharov system on the multidimensional torus. *Journal d'Analyse Mathématique*, 119(1):213–253.

Kishimoto, N. (2014). Remark on the periodic mass critical nonlinear Schrödinger equation. *Proceedings of the American Mathematical Society*, 142(8):2649–2660.

Kishimoto, N. (2018). A remark on norm inflation for nonlinear Schrödinger equations. *arXiv preprint arXiv:1806.10066*.

Klainerman, S. (1997). On the regularity of classical field theories in Minkowski space-time r 3+ 1. In *Nonlinear Partial Differential Equations in Geometry and Physics*, pages 29–69. Springer.

Klainerman, S. and Machedon, M. (1993). Space-time estimates for null forms and the local existence theorem. *Communications on Pure and Applied Mathematics*, 46(9):1221–1268.

Koch, H., Tataru, D., et al. (2005). L^p eigenfunction bounds for the Hermite operator. *Duke Mathematical Journal*, 128(2):369–392.

Lee, S. (2006). On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$. *International Mathematics Research Notices*, 2006(9):32597–32597.

Lions, P.-L. (1984). The concentration-compactness principle in the calculus of variations. the locally compact case, part 2. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume 1, pages 223–283. Elsevier.

Lucà, R. and Rogers, K. M. (2017). A note on pointwise convergence for the Schrödinger equation. In *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 1–10. Cambridge University Press.

Nikishin, E. M. (1972). A resonance theorem and series in eigenfunctions of the Laplacian. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 36(4):795–813.

Oh, T. and Tzvetkov, N. (2017). Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation. *Probability theory and related fields*, 169(3-4):1121–1168.

Ozawa, T. and Tsutsumi, Y. (1992). Existence and smoothing effect of solutions for the Zakharov equations. *Publications of the Research Institute for Mathematical Sciences*, 28(3):329–361.

Ozawa, T., Tsutsumi, Y., et al. (1992). The nonlinear Schrödinger limit and the initial layer of the Zakharov equations. *Differential and Integral Equations*, 5(4):721–745.

Rudin, W. (2006). *Real and complex analysis*. Tata McGraw-Hill Education.

Schochet, S. H. and Weinstein, M. I. (1986). The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence. *Communications in Mathematical Physics*, 106(4):569–580.

Segal, I. (1963). Non-linear semi-groups. *Annals of Mathematics*, pages 339–364.

Simon, B. (1980). *Methods of modern mathematical physics: Functional analysis*. Academic Press.

Simon, B. (1982). Schrödinger semigroups. *Bulletin of the American Mathematical Society*, 7(3):447–526.

Simpson, G., Sulem, C., and Sulem, P.-L. (2009). Arrest of Langmuir wave collapse by quantum effects. *Physical Review E*, 80(5):056405.

Sjögren, P. and Torrea, J. L. (2009). On the boundary convergence of solutions to the Hermite-Schrödinger equation. *arXiv preprint arXiv:0906.3642*.

Sjölin, P. (2012). Some remarks on localization of Schrödinger means. *Bulletin des Sciences Mathématiques*, 136(6):638–647.

Sjölin, P. (2013). Nonlocalization of operators of Schrödinger type. In *Annales Academiae Scientiarum Fennicae Mathematica*, volume 38, pages 141–147.

Sjölin, P. et al. (1987). Regularity of solutions to the Schrödinger equation. *Duke mathematical journal*, 55(3):699–715.

Strauss, W. A. (1970). On weak solutions of semi-linear hyperbolic equtions. *An Acad. Brasil. Cienc.*, 42:645–651.

Strichartz, R. S. et al. (1977). Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Mathematical Journal*, 44(3):705–714.

Takaoka, H. et al. (1999). Well-posedness for the Zakharov system with the periodic boundary condition. *Differential and Integral Equations*, 12(6):789–810.

Tao, T. (2001). Multilinear weighted convolution of L^2 functions, and applications to nonlinear dispersive equations. *American Journal of Mathematics*, 123(5):839–908.

Tao, T. (2006). *Nonlinear dispersive equations: local and global analysis*. Number 106. American Mathematical Soc.

Taylor, M. (2003). The Schrödinger equation on spheres. *Pacific journal of mathematics*, 209(1):145–155.

Taylor, M. E. (2007). *Tools for PDE: pseudodifferential operators, paradifferential operators, and layer potentials*. Number 81. American Mathematical Soc.

Temam, R. (2001). *Navier-Stokes equations: theory and numerical analysis*, volume 343. American Mathematical Soc.

Tsutsumi, Y. (1987). Schrödinger equation. *Funkcialaj Ekvacioj*, 30:115–125.

Vega, L. (1988). Schrödinger equations: pointwise convergence to the initial data. *Proceedings of the American Mathematical Society*, 102(4):874–878.

Von Neumann, J. (2018). *Mathematical foundations of quantum mechanics: New edition*. Princeton university press.

Willem, M. (1997). *Minimax theorems*, volume 24. Springer Science & Business Media.

Yajima, K. (1987). Existence of solutions for Schrödinger evolution equations. *Communications in Mathematical Physics*, 110(3):415–426.

Zakharov, V. E. et al. (1972). Collapse of Langmuir waves. *Sov. Phys. JETP*, 35(5):908–914.

Zhao, L., Zhao, F., and Shi, J. (2015). Higher dimensional solitary waves generated by second-harmonic generation in quadratic media. *Calculus of Variations and Partial Differential Equations*, 54(3):2657–2691.

CURRICULUM VITAE

[redacted]