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WEAK CONVERGENCE OF SUMS OF MOVING AVERAGES IN THE α-STABLE DOMAIN OF ATTRACTION

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Skorohod has shown that the convergence of sums of i.i.d. random variables to an α-stable Lévy motion, with $0 < \alpha < 2$, holds in the weak-J_1 sense. J_1 is the commonly used Skorohod topology. We show that for sums of moving averages with at least two nonzero coefficients, weak-J_1 convergence cannot hold because adjacent jumps of the process can coalesce in the limit; however, if the moving average coefficients are positive, then the adjacent jumps are essentially monotone and one can have weak-M_1 convergence. M_1 is weaker than J_1, but it is strong enough for the sup and inf functionals to be continuous.

1. Introduction and statement of the results. The investigation of functional limit theorems for processes with paths in $D[0,1]$ (space of right-continuous functions on $[0,1]$ with left limits) was started by Skorohod (1956). In that paper, Skorohod introduced four topologies on $D[0,1]$, called J_1, J_2, M_1 and M_2. Our results can be best understood if one visualizes the differences between the J_1, M_1 and M_2 topologies. These topologies differ in the way convergent sequences of deterministic functions f_n approach their limit f in the neighborhood of a jump of f.

In the case of the J_1 topology, f_n must have a single jump around a jump of f close to the jump of f in location and magnitude [Figure 1(a)]. In the case of the M_2 topology, several jumps are allowed but the extended graph of f_n (graph + vertical segments) must be close to that of f [Figure 1(b)]. In the case of the M_1 topology, several jumps are allowed, but the graph of f_n must be, within ε, a "monotone staircase," which gets "compressed" into a single jump of f as $n \to \infty$ [Figure 1(c)].

J_1 convergence is thus appropriate when a jump of the limit arises from a single jump in f_n. Let $\mathscr{D}(a)$ denote the α-stable domain of attraction. As shown by Skorohod (1957), normalized and centered sums $(1/a_n)\sum_{i=1}^{[nt]}(X_i - b_{[nt]})$ of i.i.d. random variables X_i in $\mathscr{D}(\alpha)$, $0 < \alpha < 2$, converge weakly in the J_1 sense. The limit is the *Lévy α-stable motion*, whose increments are stationary, independent and have a stable distribution with index α. J_1 is the commonly used Skorohod topology.

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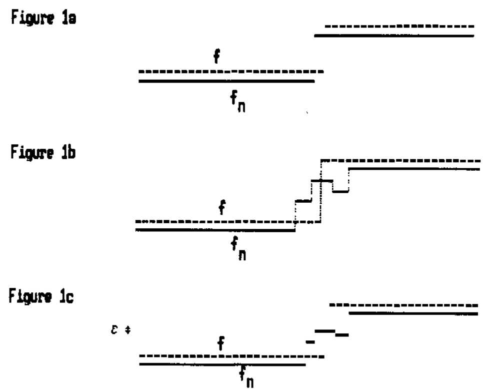

FIG. 1. *Modes of convergence for the Skorohod topologies J_1, M_2 and M_1.*

We will show, however, that in the case of normalized sums of moving averages of i.i.d. random variables in $\mathscr{D}(\alpha)$ with summable coefficients, weak-J_1 convergence does *not* hold, when at least two of these coefficients are nonzero (Theorem 1). The reason roughly is that each jump of the limit arises from a "staircase" with at least two steps.

If the coefficients of the moving average all have the same sign, then the steps of the staircase all go essentially in the same direction. We show that in this case weak-M_1 convergence holds (Theorem 2). Although M_1 is weaker than J_1, it is strong enough for the commonly used functions $\inf_{0 \le t \le 1}$ and $\sup_{0 \le t \le 1}$ to be continuous [Skorohod (1957), 2.2.10].

We now introduce some notation and give a precise statement of results.

Let X_i be an i.i.d. sequence belonging to $\mathscr{D}(\alpha)$, $0 < \alpha < 2$. Assume also that $EX_i = 0$ when $1 < \alpha < 2$ and that the X_i are symmetric when $\alpha = 1$.

Let $\mathbf{c} = \{c_i, i \in \mathbb{Z}\}$ be a sequence satisfying

$$\sum_{i=-\infty}^{\infty} |c_i|^\nu < \infty \tag{1.1}$$

for some $0 < \nu < \alpha$. Condition (1.1) ensures that the moving averages

$$Y_i = \sum_{j=-\infty}^{\infty} c_{i-j} X_j = \sum_{j=-\infty}^{\infty} c_j X_{i-j}, \qquad i \in \mathbb{Z}, \tag{1.2}$$

converge in L^ν [and, in fact, also a.s.; cf. Kawata (1972), Theorems 12.11.2 and 12.10.4].

Let a_n be normalization constants such that

$$\sum_{i=1}^{[nt]} X_i / a_n \xrightarrow{\text{f.d.d.}} Z_\alpha(t), \tag{1.3}$$

where $X_\alpha(t)$ is a Lévy α-stable motion and $\xrightarrow{\text{f.d.d.}}$ denotes convergence of the finite-dimensional distributions. Astrauskas (1983), Theorem 1i, and Davis and Resnick (1985), Theorem 4.1, show that the normalized sums of the moving average Y_i are also attracted to a Lévy α-stable process whenever $\sum_{i=-\infty}^{\infty}|c_i| < \infty$. More precisely, let

$$Z_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} Y_i, \qquad 0 \le t \le 1, \tag{1.4}$$

where a_n is as in (1.3).

LEMMA 1 [Astrauskas (1983) and Davis and Resnick (1985)]. *When* $\sum_{i=-\infty}^{\infty}|c_i| < \infty$, *then*

$$Z_n(t) \xrightarrow{\text{f.d.d.}} \left(\sum_{i=-\infty}^{\infty} c_i \right) X_\alpha(t), \tag{1.5}$$

where $X_\alpha(t)$ *is the same Lévy* α*-stable motion as in* (1.3).

Can the f.d.d. convergence in (1.5) be replaced by weak convergence in $D[0,1]$ with respect to one of the Skorohod topologies? We show that the answer is generally negative for J_1, the most commonly used Skorohod topology.

THEOREM 1. *Suppose that* Y_i *is a finite-order moving average with at least two nonzero coefficients. Then convergence in* (1.5) *does not hold in the weak-*J_1 *sense.*

Theorem 1 is proved in Section 2.

REMARK. When only one coefficient c_i is nonzero (i.e., when the summands are independent), weak J_1 convergence holds by Skorohod (1957).

Although in Theorem 1, weak-J_1 convergence does not hold, weak-M_1 convergence holds if one imposes some extra assumptions. The main one is that all the coefficients c_i's have the same sign.

THEOREM 2′. *Suppose that* Y_i *is a finite-order moving average with all nonnegative coefficients. Then convergence in* (1.5) *holds in the weak-*M_1 *sense.*

The proof of Theorem 2′ could be established through either a nonprobabilistic method or a probabilistic one. We choose the probabilistic method because it yields bounds (Proposition 3) that can be used to prove the more general result concerning the M_1 convergence of moving averages whose order is not necessarily finite. This more general result is stated in Theorem 2 below. It requires, when $\alpha > 1$, the following technical condition, which we refer to as

(T.C.). To formulate it, let $\alpha' \geq 1 \geq \nu$ and

$$s(\alpha', \mathbf{c}) = \left(\sum_{i=-\infty}^{\infty} |c_i|^{\nu} \right) \left(\sum_{i=-\infty}^{\infty} |c_i| \right)^{\alpha' - \nu}. \tag{1.6}$$

Observe that $s(\alpha', \mathbf{c}) = (\sum_{i=-\infty}^{\infty} |c_i|)^{\alpha'}$ when $\nu = 1$. Now introduce the truncated sequences $\mathbf{c}^{>n} = \{c_i^{>n}, -\infty < i < \infty\}$ and $\mathbf{c}^{\leq n} = \{c_i^{\leq n}, -\infty < i < \infty\}$, where

$$c_i^{>n} = \begin{cases} c_i & \text{if } |i| > n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c_i^{\leq n} = c_i - c_i^{>n}.$$

THE TECHNICAL CONDITION (T.C.). Let $\alpha > 1$. The sequence $\mathbf{c} = \{c_i, -\infty < i < \infty\}$ satisfies the condition (T.C.) if $\mathbf{c}$ satisfies (1.1) and if for some $0 < \eta \leq \alpha - 1$,

$$\lim_{n \to \infty} (\ln n)^{1+\alpha+\eta} s(\alpha - \eta, \mathbf{c}^{>n}) = 0. \tag{T.C.}$$

The condition (T.C.) is required only when $\alpha > 1$. It is always satisfied if the moving average is of finite order. It is also satisfied in many other cases of interest, for example, when $s(\alpha - \eta, \mathbf{c}^{>n})$ is dominated by a regularly varying sequence with strictly negative exponent. In fact, we obtain the following proposition.

PROPOSITION 1. *Suppose $\alpha > 1$ and $\sum_{i=-\infty}^{\infty} |c_i|^{\nu} < \infty$ for some $\nu < \alpha$.*

(i) *If $\nu < 1$ and $\{c_i, i \geq 0\}$ and $\{c_i, i < 0\}$ are monotone sequences, then* (T.C.) *holds.*

(ii) *If $\nu = 1$ but $\sum_{i=-\infty}^{\infty} |c_i|^{\nu'} = \infty$ for all $\nu' < 1$, then* (T.C.) *may not hold even when $\{c_i\}$ is a monotone sequence.*

This proposition is proved in Section 2. The next theorem extends Theorem 2′ to moving averages whose order is not necessarily finite.

THEOREM 2. *Suppose that $\sum_{i=-\infty}^{\infty} c_i < \infty$ with $c_i \geq 0$, and that either:*

(i) $$\alpha \leq 1$$

or

(ii) $$\alpha > 1 \textit{ and } (\text{T.C.}) \textit{ holds}.$$

Then, as $n \to \infty$,

$$Z_n(\cdot) \Rightarrow \left(\sum_{i=-\infty}^{+\infty} c_i \right) X_\alpha(\cdot)$$

in $D[0, 1]$ endowed with the M_1 topology.

If the X_i's are positive, the result is immediate because of Lemma 1 and the fact that the c_i's have the same sign. The general case $0 < \alpha < 1$ can be reduced to this one by expressing the sequence of partial sums as sums of their positive and negative parts and by using the fact that if $U_n \to_{M_1} U$, $V_n \to_{M_1} V$, and the processes U and V have disjoint discontinuities with probability 1, then $U_n + M_n \to_{M_1} U + V$ [see the remarks in Sections 1 and 6 of Whitt (1980)].

It is convenient, however, to give a unified proof of Theorem 2 for all $\alpha \in [0, 2]$. This can be done without much additional effort because the estimates needed for $\alpha = 1$ turn out to be typically similar to those for $\alpha < 1$. Theorem 2 is proved in Section 3 using auxiliary results established in Section 4.

As for weak-M_2 convergence, we make the following conjecture.

CONJECTURE. *If $c_i = 0$ for $i \le 0$, $c_1, c_2, \dots \in \mathbb{R}$ and if for every K,*

$$0 \le \sum_{i=1}^{K} c_i \bigg/ \sum_{i=1}^{\infty} c_i \le 1,$$

then (1.5) *holds in the sense of weak-M_2 convergence.*

We will now give a heuristic justification of our results. Let us assume that Y_i is the finite moving average

$$Y_i = \sum_{j=0}^{K} c_j X_{i-j}.$$

Heuristically, most of the sequence $X_{i,n} := X_i/a_n \approx 0$ (is negligible), except for a sequence of "big values" $X_{i_0,n}, X_{i_1,n}, \dots, X_{i_k,n}, \dots$, which are spread far apart, that is, for which $i_0 \ll i_1 \ll \cdots \ll i_k \ll \cdots$. It follows that most of the $Y_{i,n} := Y_i/a_n$, which are the increments of $Z_n(t) = \sum_{i=1}^{[nt]} Y_{i,n}$, are also asymptotically negligible; however, a big value $X_{i_0,n}$ produces $K+1$ successive big values in the sequence $Y_{i,n}$:

$$Y_{i_0,n} \approx c_0 X_{i_0,n}, \qquad Y_{i_0+1,n} \approx c_1 X_{i_0,n}, \dots, Y_{i_0+K,n} \approx c_K X_{i_0,n}.$$

Thus, asymptotically, $Z_n(t)$ is made out of "staircases," each covering an interval on the x axis of length $K/n \to 0$, and thus each staircase degenerates in the limit into a single jump. From the heuristics given at the beginning of this section, we see that:

If the staircase has at least two steps, J_1 convergence cannot hold (Theorem 1).

If the staircase is monotone (all steps go in the same direction), we have M_1 convergence (Theorem 2).

If the vertical size of each staircase in bounded between 0 and the size of the limiting jump, then we might have M_2 convergence (Conjecture).

The following counterexamples show that some conditions on the c_i are necessary in order to get at least one of the weak Skorohod convergences.

COUNTEREXAMPLES. If Y_i is a finite-order moving average with coefficients of both signs, then the "staircase" is not monotone, and M_1 convergence does not hold. The precise proof is similar to that of Theorem 1.

Consider now the even simpler example, where

$$c_0 = 1, \qquad c_1 = -1, \qquad c_k = 0 \quad \text{for } k \neq 0, 1,$$

so that $\sum_{i=-\infty}^{\infty} c_i = 0$ and

$$\frac{1}{a_n} \sum_{i=1}^{[nt]} Y_i = \frac{X_{[nt]} - X_0}{a_n} \xrightarrow{\text{f.d.d.}} 0.$$

But f.d.d. convergence cannot be replaced by weak convergence in any of the four topologies, because, as is known, $\sup_{0 \le t \le 1} X_{[nt]}/a_n$ converges in distribution to a nonzero limit, and $\sup_{0 \le t \le 1}$ is a continuous functional in all the four Skorohod topologies.

On the other hand, if we make the strong assumptions $c_i \ge 0$, $X_i \ge 0$ (assumptions that can hold when $\alpha < 1$), then, since $\sum_{i=1}^{[nt]} Y_i / a_n$ has monotone paths, weak-M_1 convergence holds automatically. In Theorem 2, $X_i \ge 0$ is not assumed.

The paper is organized as follows. In Section 2, we define the functions J and M that characterize the J_1 and M_1 topologies and we establish Theorem 1 and Proposition 1. In Section 3, we give the main steps leading to Theorem 2 and prove Theorem 2. The validity of these main steps is established in Section 4.

2. Proof of Theorem 1 and Proposition 1. The following functions enter in the definition of the Skorohod topologies:

$$J(x_1, x_2, x_3) = \min\{|x_2 - x_1|, |x_3 - x_2|\}, \tag{2.1}$$

$$M(x_1, x_2, x_3) = \text{the distance from } x_2 \text{ to } [x_1, x_3]$$
$$= \begin{cases} 0, & \text{if } x_2 \in [x_1, x_3], \\ J(x_1, x_2, x_3), & \text{otherwise.} \end{cases} \tag{2.2}$$

Let H stand for either J or M, and introduce the H oscillation of a function $Z(t)$:

$$w_\delta^H(Z) = \sup_{\substack{t_1 \le t \le t_2 \\ 0 \le t_2 - t_1 \le \delta}} H(Z(t_1), Z(t), Z(t_2)). \tag{2.3}$$

Refer to Skorohod (1956) for a definition of the Skorohod topologies and their properties. Here we will need only the following corollary of his Theorems 3.2.1 and 3.2.2. [For the J_1 version, see also Billingsley (1968), Theorems 15.3 and 15.4.]

PROPOSITION 2 [Skorohod (1956)]. *Let $Z_n(t)$ be processes in* $D[0,1]$ *whose finite-dimensional distributions converge to those of a process $Z(t)$ which is a.s. continuous at $t = 0$ and at $t = 1$. Let H stand for either J or M. Then weak-H_1 convergence holds if and only if for every $\varepsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\{(\omega_\delta^H(Z_n)) > \varepsilon\} = 0. \tag{2.4}$$

PROOF OF THEOREM 1. We will show that relation (2.4) does not hold for $H = J_1$ when

$$Z_n(t) = \sum_{j=1}^{[nt]} Y_{j,n}, \qquad Y_{j,n} = \sum_{i=0}^{K} c_i X_{j-i,n},$$

with $c_0 \neq 0$, $c_{i_1} \neq 0$. Let $i_1 \leq K$ denote the first nonzero coefficient after c_0; hence $c_1 = \cdots = c_{i_1 - 1} = 0$.

Consider the random variables $Y_{i',n}$ and $Y_{i'+i_1,n}$ where necessarily, $1 \leq i' < i' + i_1 \leq n$, so that $1 \leq i' \leq n - i_1$. Choose this $i' = i'(n)$ to be the index at which $\max_{1 \leq i \leq n - i_1} |X_{i,n}|$ is obtained.

Fix $\varepsilon > 0$ and introduce the events

$$A_{n,\varepsilon} = \{|X_{i',n}| > \varepsilon\} = \left\{ \max_{1 \leq i \leq n - i_1} |X_{i,n}| > \varepsilon \right\}$$

and

$$B_{n,\varepsilon} = \{|X_{i',n}| > \varepsilon \text{ and } \exists\, l \neq 0,\ -K \leq l \leq i_1, \text{ where } |X_{i'+l,n}| > \lambda\varepsilon\},$$

where λ is a constant to be specified later.

We will show that the four following statements hold:

$$\lim_{n \to \infty} P(A_{n,\varepsilon}) > 0. \tag{a}$$

$$\lim_{n \to \infty} P(B_{n,\varepsilon}) = 0. \tag{b}$$

(c) On $A_{n,\varepsilon} \setminus B_{n,\varepsilon}$, the random variables $Y_{j,n}$, $j = i'$ and $j = i' + i_1$, are "large" [specifically, $|Y_{j,n}| \geq \varepsilon(|c_{j-i'}| - \lambda\sum_{i=0}^{K}|c_i|)$], and the random variables $Y_{j,n}$, $i' < j < i' + i_1$, are "small" [specifically, $|Y_{j,n}| \leq \lambda\varepsilon\sum_{i=0}^{K}|c_i|$].

(d) On the events $A_{n,\varepsilon} \setminus B_{n,\varepsilon}$, the J oscillation $w_\delta^J(Z_n)$ with $\delta = (i_1 + 1)/n$ is bounded below.

Statements (a) and (b) ensure $\lim_{n \to \infty} P(A_{n,\varepsilon} \setminus B_{n,\varepsilon}) > 0$. Statement (c) is used to establish statement (d) which contradicts Skorohod's criterion for J_1-weak convergence.

We now verify the statements.

(a) This is a well-known property of the α-stable domain of attraction, a consequence of

$$\lim_{n \to \infty} nP\{|X_{1,n}| > x\} = kx^{-\alpha}, \qquad x > 0, \tag{2.5}$$

for $k > 0$. In fact, one has

$$\lim_{n\to\infty} P\Big\{\max_{1\le i\le n} |X_{i,n}| > x\Big\} = 1 - e^{-kx^{-\alpha}}, \qquad x \ge 0.$$

(b) Observe that

$$B_{n,\varepsilon} \subset \bigcup_{i=1}^{n-i_1} \bigcup_{\substack{l=-K\\ l\neq 0}}^{i_1} \{|X_{i,n}| > \varepsilon\} \cap \{|X_{i+l,n}| > \lambda\varepsilon\}.$$

Thus by (2.5) there is a constant M depending on ε such that

$$\begin{aligned} P(B_{n,\varepsilon}) &\le (n - i_1)(i_1 + K) P\{|X_{1,n}| > \varepsilon\} P\{|X_{1,n}| > \lambda\varepsilon\} \\ &\le M \frac{(n - i_1)}{n^2} \to 0 \end{aligned}$$

as $n \to \infty$.

(c) For any $k \in \{0, 1, \dots, i_1\}$,

$$Y_{i'+k,n} = c_k X_{i',n} + \sum_{\substack{j=0\\ j\neq k}}^{K} c_j X_{i'+k-j,n}.$$

On $A_{n,\varepsilon} \setminus B_{n,\varepsilon}$, one has $|X_{i'+l,n}| \le \lambda\varepsilon$, $\forall\, l \ne 0$, $-K \le l \le i_1$, so that

$$\left| \sum_{\substack{j=0\\ j\neq k}}^{K} c_j X_{i'+k-j,n} \right| \le \lambda\varepsilon \sum_{j=0}^{K} |c_j|.$$

Hence for $k = 0$ and $k = i_1$,

$$\begin{aligned} |Y_{i'+k,n}| &\ge |c_k X_{i',n}| - \left| \sum_{\substack{j=0\\ j\neq k}}^{K} c_j X_{i'+k-j,n} \right| \\ &\ge \varepsilon \left(|c_k| - \lambda \sum_{j=0}^{K} |c_j| \right), \end{aligned}$$

while for $0 < k < i_1$, $c_k = 0$ and thus $|Y_{i'+k,n}| \le \lambda\varepsilon\sum_{j=0}^{K}|c_j|$.

(d) Consider now the following two consecutive increments of the process $Z_n(t)$:

$$\left| Z_n\left(\frac{i'}{n}\right) - Z_n\left(\frac{i'-1}{n}\right) \right| = |Y_{i',n}| > \varepsilon\left(|c_0| - \lambda \sum_{j=0}^{K} |c_j| \right) \tag{2.6}$$

and

$$
\begin{aligned}
\left| Z_n\left(\frac{i'+i_1}{n}\right) - Z_n\left(\frac{i'}{n}\right)\right| &= \left|\sum_{k=1}^{i_1} Y_{i'+k,n}\right| \\
&\geq |Y_{i'+i_1,n}| - \sum_{k=1}^{i_1-1} |Y_{i'+k,n}| \\
&\geq \varepsilon\left(|c_{i_1}| - \lambda\sum_{j=0}^{K}|c_j| - (i_1-1)\lambda\sum_{j=0}^{K}|c_j|\right) \\
&= \varepsilon\left(|c_{i_1}| - \lambda i_1 \sum_{j=0}^{K}|c_j|\right).
\end{aligned} \tag{2.7}
$$

Now choose λ so that

$$
\lambda\sum_{k=0}^{K}|c_k| \leq \min\left(\frac{|c_0|}{2}, \frac{|c_{i_1}|}{2i_1}\right),
$$

that is, so that the absolute increments in (2.6) and (2.7) are respectively greater than $(\varepsilon/2)|c_0|$ and $(\varepsilon/2)|c_{i_1}|$. Then

$$
\begin{aligned}
\omega^J_{(i_1+1)/n}(Z_n) &\geq J\left(Z_n\left(\frac{i'-1}{n}\right), Z_n\left(\frac{i'}{n}\right), Z_n\left(\frac{i'+i_1}{n}\right)\right) \\
&\geq \frac{\varepsilon}{2}\min\left(|c_0|, |c_{i_1}|\right)
\end{aligned} \tag{2.8}
$$

on the event $A_{n,\varepsilon} \setminus B_{n,\varepsilon}$, hence verifying statement (d).

We now conclude the proof. By statements (a) and (b), we have $\lim_{n\to\infty} P(A_{n,\varepsilon} \setminus B_{n,\varepsilon}) > 0$, which implies

$$
\begin{aligned}
0 &< \liminf_{n\to\infty} P\left(\omega^J_{(i_1+1)/n}(Z_n) \geq \frac{\varepsilon}{2}\min\left(|c_0|, |c_{i_1}|\right)\right) \\
&\leq \lim_{\delta\to 0}\limsup_{n\to\infty} P\left(\omega^J_\delta(Z_n) \geq \frac{\varepsilon}{2}\min\left(|c_0|, |c_{i_1}|\right)\right),
\end{aligned} \tag{2.9}
$$

since $w^J_\delta(\cdot)$ is nondecreasing in δ. Hence weak-J_1 convergence does not hold (Proposition 2). □

REMARK. If all the c_i are nonnegative, then the big increments $Y_{i',n}$ $(\approx c_0 X_{i',n})$ and $Y_{i'+i_1,n}$ $(\approx c_{i_1} X_{i',n})$ have the same sign, and thus produce zero M oscillation. Thus, although they preclude J_1 convergence, they do not preclude M_1 convergence, which indeed holds, by Theorem 2.

PROOF OF PROPOSITION 1. (i) Assume w.l.o.g. that $\sum_i |c_i|^\nu < 1$ and thus $\sum_i |c_i| < 1$ and $|c_i| < 1$. Choose η satisfying $0 < \eta \leq \min(\alpha - 1, 1 - \nu)$, so that

$\nu + \eta \le 1 \le \alpha - \eta$. Since $\{c_i,\ i \ge 0\}$ and $\{c_i,\ i < 0\}$ are monotone sequences,

$$\begin{aligned} s(\alpha - \eta, \mathbf{c}^{\ge n}) &= \Big(\sum_{|i| \ge n} |c_i|^{\nu}\Big)\Big(\sum_{|i| \ge n} |c_i|\Big)^{\alpha - \eta - \nu} \\ &\le \Big[|c_{\pm n}|^{\eta} \sum_{|i| \ge n} |c_i|^{1-\eta}\Big]^{\alpha - \eta - \nu} \\ &\le \Big[|c_{\pm n}|^{\eta} \sum_{|i| \ge n} |c_i|^{\nu}\Big]^{\eta} \\ &\le (n|c_{\pm n}|)^{\eta^2} n^{-\eta^2} \\ &= O(n^{-\eta^2}), \end{aligned}$$

where $c_{\pm n} = \max(c_{-n}, c_n)$. Thus $s(\alpha - \eta, \mathbf{c}^{\ge n})$ is bounded by n to a negative power and (T.C.) is satisfied.

(ii) Consider the following counterexample:

$$c_i = \frac{1}{|i|(\ln |i|)^{1+\beta}}, \qquad 0 < \beta \le 1 + \frac{1}{\alpha}.$$

Here

$$\sum_{|i| \ge n} c_i = O\Big(\frac{1}{(\ln n)^{\beta}}\Big).$$

Thus $\sum_i |c_i| < \infty$, but $\sum_i |c_i|^{1-\eta} = \infty$, $\forall\ \eta > 0$. Then, if $\alpha - \eta \ge 1$, we have

$$s(\alpha - \eta, \mathbf{c} > {}^{\ge n}) = \Big(\sum_{|i| \ge n} c_i\Big)^{\alpha - \eta} = O\Big(\frac{1}{(\ln n)^{\beta(\alpha - \eta)}}\Big).$$

To satisfy (T.C.), we have to find $\eta > 0$, such that $\beta(\alpha - \eta) > 1 + \alpha + \eta$; that is, $\eta\beta + \eta < \beta\alpha - (1 + \alpha)$. This is possible if and only if $\beta\alpha - (1 + \alpha) > 0$, that is, if and only if $\beta > 1 + 1/\alpha$. Since $\beta \le 1 + 1/\alpha$, (T.C.) cannot be satisfied. □

3. Proof of Theorem 2. Theorem 2 will be established by showing that in the case $H = M$, relation (2.4) holds for

$$Z_n(t) = \sum_{i=1}^{[nt]} \frac{Y_i}{a_n} = \sum_{i=1}^{[nt]} \sum_{j=-\infty}^{+\infty} c_{i-j} \frac{X_j}{a_n}. \tag{3.1}$$

This is accomplished by approximating Y_i/a_n by a moving average of order K_n for a suitably chosen sequence of constants $K_n \to \infty$.

Let M and ω_δ^M be defined as in (2.2) and (2.3) and let

$$M_n(t_1, t, t_2) = M(Z_n(t_1), Z_n(t), Z_n(t_2)), \tag{3.2}$$

so that

$$\omega_\delta^M(Z_n) = \sup_{\substack{t_1 \le t \le t_2 \\ 0 \le t_2 - t_1 \le \delta}} M_n(t_1, t, t_2). \tag{3.3}$$

Let $\eta > 0$ be a constant satisfying

$$\begin{cases} \alpha - \eta > \nu, \alpha + \eta < 1, & \text{if } \alpha < 1, \\ \alpha - \eta > \nu, & \text{if } \alpha = 1, \\ \alpha - \eta > 1, & \text{if } \alpha > 1. \end{cases} \tag{3.4}$$

PROPOSITION 3. *Let $0 < \eta < 1/2$ satisfy* (3.4). *If $c_i \ge 0$, and $c_i = 0$ when $|i| > K$, for some finite K, then for n satisfying*

$$n^{(1/2-\eta)/(1+\alpha+\eta)} > K, \tag{3.5}$$

there exists a constant L independent of K and n such that for $0 \le t_1 \le t \le t_2 \le 1$ and all $\varepsilon > 0$,

(i)

$$P\{M_n(t_1, t, t_2) > \varepsilon\} \le L\varepsilon^{-2(\alpha+\eta)}(t_2 - t_1)^{1+2\eta}. \tag{3.6}$$

(ii) *Furthermore, there exists a constant k independent of K and n such that*

$$P\{\omega_\delta^M(Z_n) > \varepsilon\} \le Lk\varepsilon^{-2(\alpha+\eta)}\delta^{2\eta}. \tag{3.7}$$

Part (i) of Proposition 3 is established in Section 4. Part (ii) follows from Theorem 1 of Avram and Taqqu (1989).

By using Lemma 1 of Section 1, Skorohod's Proposition 2 of Section 2 and Proposition 3, we will be able to conclude that Theorem 2 holds when the Y_i's are finite moving averages. To deal with the general case, we decompose $Y_{i,n} = Y_i/a_n$ as follows.

Let K_n be a sequence increasing to ∞,

$$c_i^{\le K_n} = \begin{cases} c_i, & \text{if } |i| \le K_n, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$c_i^{> K_n} = c_i - c_i^{\le K_n} = \begin{cases} 0, & \text{if } |i| \le K_n, \\ c_i, & \text{if } |i| > K_n. \end{cases}$$

Let $Y_{i,n}^{\le K_n}$ and $Y_{i,n}^{> K_n}$ be the moving averages with coefficients $c_i^{\le K_n}$ and $c_i^{> K_n}$ respectively, and let their sum from $i = 1$ to $[nt]$ be denoted $Z_n^{\le K_n}(t)$ and $Z_n^{> K_n}(t)$ respectively.

The $Z_n^{\le K_n}$ are sums of finite moving averages, to which Proposition 3 applies, while the $Z_n^{> K_n}$ are sums of moving averages with "small" coefficients. They will be handled by the use of the following proposition.

PROPOSITION 4. *Let Z_n be defined as in* (3.1), *and let $\eta > 0$ satisfy* (3.4). *Then there exist constants L' and k', independent of n, and of the sequence* $\mathbf{c}$, *such that*:

(i) $$P\{|Z_n(t_2) - Z_n(t_1)| > \varepsilon\} \le L'\varepsilon^{-(\alpha+\eta)}(t_2 - t_1)s(\alpha - \eta, \mathbf{c}),$$

where

$$(3.8)\qquad s(\alpha', \mathbf{c}) := \begin{cases} \sum_{i=-\infty}^{\infty} |c_i|^{\alpha'}, & \textit{if } \alpha' \le 1, \\ \left(\sum_{i=-\infty}^{\infty} |c_i|^{\nu}\right)\left(\sum_{i=-\infty}^{\infty} |c_i|\right)^{\alpha'-\nu}, & \textit{if } \alpha' > 1 \ge \nu; \end{cases}$$

(ii) $$P\Big\{\sup_{0\le t\le 1} |Z_n(t)| > \varepsilon\Big\} \le \begin{cases} L'k'\varepsilon^{-(\alpha+\eta)} s(\alpha - \eta, \mathbf{c}^{(n)}), & \textit{if } \alpha \le 1, \\ L'k'\varepsilon^{-(\alpha+\eta)}(\ln n)^{1+\alpha+\eta} s(\alpha - \eta, \mathbf{c}^{(n)}), & \textit{if } \alpha > 1. \end{cases}$$

Proposition 4 is proved in Section 4. The definition of $s(\alpha', \mathbf{c})$ in (3.8) extends the one given in (1.6) to $\alpha' \le 1$.

PROOF OF THEOREM 2. We look for a sequence $\{K_n\}_{n=1}^{\infty}$, where K_n is small enough [satisfying (3.5)] so that Proposition 3 can be applied to the process $Z_n^{\le K_n}$, but large enough so that the estimate for $P\{\sup_{0\le t\le 1} |Z_n(t)^{>K_n}| > \varepsilon\}$ given in Proposition 4(ii), namely

$$e_n := \begin{cases} s(\alpha - \eta, \mathbf{c}^{>K_n}), & \text{if } \alpha \le 1, \\ (\ln n)^{1+\alpha+\eta} s(\alpha - \eta, \mathbf{c}^{>K_n}), & \text{if } \alpha > 1, \end{cases}$$

tends to 0, as $N \to \infty$. An adequate sequence is $K_n = n^{1/6}$. This K_n satisfies (3.5) since for η small enough, $n^{1/6} < n^{(1/2-\eta)/(1+\alpha+\eta)}$. When $\alpha \le 1$, $e_n \to 0$, since $K_n \to \infty$. On the other hand, if $\alpha > 1$, by assumption (T.C.) of Theorem 2,

$$\lim_{n\to\infty} s(\alpha - \eta, \mathbf{c}^{\ge K_n})(\ln K_n)^{1+\alpha+\eta}$$
$$= 0 = \left(\tfrac{1}{6}\right)^{1+\alpha+\eta} \lim_{n\to\infty} s(\alpha - \eta, \mathbf{c}^{\ge K_n})(\ln n)^{1+\alpha+\eta}$$

and thus the estimate $e_n \to 0$. Now it remains only to note that if $\omega_\delta^M(Z_n^{\le K_n}) \le \varepsilon/2$, and $\sup_{0<t\le 1} |Z_n^{>K_n}(t)| \le \varepsilon/4$, then $\omega_\delta^M(Z_n) \le \varepsilon$. This is so because M involves the process at only three time points. Thus

$$P\{\omega_\delta^M(Z_n) > \varepsilon\} \le P\Big\{\omega_\delta^M(Z_n^{\le K_n}) > \frac{\varepsilon}{2}\Big\} + P\Big\{\sup_{0\le t\le 1} |Z_n^{>K_n}(t)| > \frac{\varepsilon}{4}\Big\}$$
$$\le Lk\left(\frac{\varepsilon}{2}\right)^{-2(\alpha+\eta)} \delta^{2\eta} + L'k'\left(\frac{\varepsilon}{4}\right)^{-(\alpha+\eta)} e_n$$

(by Propositions 3 and 4). Hence

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\{\omega_\delta^M(Z_n) > \varepsilon\} \le \lim_{\delta \to 0} Lk\left(\frac{\varepsilon}{2}\right)^{-2(\alpha+\eta)} \delta^{2\eta} = 0.$$

Theorem 2 follows by applying Lemma 1 and Proposition 2. □

4. Proof of the auxiliary results. We establish here Propositions 3(i) and 4 of Section 3. We need two lemmas. Recall that X_i is an i.i.d. sequence in $\mathscr{D}(\alpha)$, $0 < \alpha < 2$, with $EX_i = 0$ if $\alpha > 1$ and with X_i symmetric if $\alpha = 1$. Also $X_{i,n} = X_i/a_n$, where a_n are the normalization constants in the central limit theorem (1.3).

LEMMA 2. *Let $\eta > 0$ be a constant satisfying* (3.4) *and let $|b_{i,n}| < 1$, $i, n = 1, 2, \dots$. Then there exists a constant M depending only on the distribution of X_i and η, such that for all $\varepsilon > 0$,*

$$P\left\{\sup_{1 \le k \le m} \left|\sum_{i=1}^{k} b_{i,n} X_{i,n}\right| \ge \varepsilon\right\} \le M \frac{\varepsilon^{-(\alpha+\eta)}}{n} \sum_{i=1}^{m} |b_{i,n}|^{\alpha-\eta} \tag{4.1}$$

for all $m \le \infty$.

PROOF. We treat separately the cases $\alpha < 1$, $1 < \alpha < 2$ and $\alpha = 1$.

(a) If $\alpha < 1$, we let

$$X_{i,n}^{\le} = X_{i,n} 1\{|X_{i,n}| \le 1\},$$
$$X_{i,n}^{>} = X_{i,n} 1\{|X_{i,n}| > 1\}.$$

Let $\eta > 0$ be such that $\alpha - \eta > 0$, $\alpha + \eta < 1$, and consider

$$\begin{aligned} P\left\{\sup_{1 \le k \le m} \left|\sum_{i=1}^{k} b_{i,n} X_{i,n}^{>}\right| \ge \frac{\varepsilon}{2}\right\} &\le P\left\{\sum_{i=1}^{m} |b_{i,n}|\, |X_{i,n}^{>}| \ge \frac{\varepsilon}{2}\right\} \\ &\le \left(\frac{\varepsilon}{2}\right)^{-(\alpha-\eta)} E\left(\sum_{i=1}^{m} |b_{i,n}|\, |X_{i,n}^{>}|\right)^{\alpha-\eta} \\ &\le \left(\frac{\varepsilon}{2}\right)^{-(\alpha-\eta)} \sum_{i=1}^{m} |b_{i,n}|^{\alpha-\eta} E|X_{i,n}^{>}|^{\alpha-\eta}. \end{aligned} \tag{4.2}$$

Similarly,

$$P\left\{\sup_{1 \le k \le m} \left|\sum_{i=1}^{k} b_{i,n} X_{i,n}^{\le}\right| \ge \frac{\varepsilon}{2}\right\} \le \left(\frac{\varepsilon}{2}\right)^{-(\alpha+\eta)} E|X_{i,n}^{\le}|^{\alpha+\eta} \sum_{i=1}^{m} |b_{i,n}|^{\alpha+\eta}. \tag{4.3}$$

Let

$$M' = \sup_n \{nE|X_{1,n}^{>}|^{\alpha-\eta}\} \vee \sup_n \{nE|X_{1,n}^{\le}|^{\alpha+\eta}\}.$$

Since $M' < \infty$ [see also Astrauskas (1983), Lemma 1], we see that (4.2) and (4.3) imply (4.1), with $M = 2^{1+\alpha+\eta}M'$, whether m is finite or not.

(b) When $\alpha > 1$, let η be such that $\alpha - \eta > 1$, and let

$$\begin{aligned} \overline{X}_{i,n}^{\leq} &= X_{i,n}^{\leq} - EX_{i,n}^{\leq}, \\ \overline{X}_{i,n}^{>} &= X_{i,n}^{>} + EX_{i,n}^{\leq}. \end{aligned} \tag{4.4}$$

Thus $E(\overline{X}_{i,n}^{\leq}) = 0$, and also $E\overline{X}_{i,n}^{\geq} = EX_{i,n}^{>} + EX_{i,n}^{\leq} = EX_{i,n} = 0$.

We will show that (4.2) and (4.3) continue to hold, with $\overline{X}_{i,n}^{>}$ and $\overline{X}_{i,n}^{<}$ replacing $X_{i,n}^{>}$ and $X_{i,n}^{\leq}$. Note first that $\sum_{i=1}^{k} b_{i,n}\overline{X}_{i,n}^{\leq}$ and $\sum_{i=1}^{k} b_{i,n}\overline{X}_{i,n}^{>}$ are martingales (as k varies). Using the maximal inequality

$$P\left\{\sup_{1\leq k\leq m} |S_k| \geq \lambda\right\} \leq \lambda^{-p}\frac{p}{p-1}E|S_m|^p,$$

which holds for $p > 1$ and S_n a martingale, and the von Bahr–Esseen inequality [Chatterji (1969), Lemma 1], $E|\sum_{j=1}^{m}\eta_j|^p \leq 2\sum_{j=1}^{m}E|\eta_j|^p$ which holds for $1 \leq p \leq 2$ and $\{\eta_j\}$ a martingale-difference sequence, we have

$$\begin{aligned} &P\left\{\sup_{1\leq k\leq m}\left|\sum_{i=1}^{k} b_{i,n}\overline{X}_{i,n}^{>}\right| \geq \frac{\varepsilon}{2}\right\} \\ &\quad\leq \left(\frac{\varepsilon}{2}\right)^{-(\alpha-\eta)}\kappa(\alpha-\eta)E\left|\sum_{i=1}^{m} b_{i,n}\overline{X}_{i,n}^{>}\right|^{\alpha-\eta} \\ &\quad\leq \left(\frac{\varepsilon}{2}\right)^{-(\alpha-\eta)}2\kappa(\alpha-\eta)\sum_{i=1}^{m}|b_{i,n}|^{\alpha-\eta}E|\overline{X}_{i,n}^{>}|^{\alpha-\eta}, \end{aligned} \tag{4.5}$$

where $\kappa(p) = p/(p-1)$. Similarly,

$$\begin{aligned} &P\left\{\sup_{1\leq k\leq m}\left|\sum_{i=1}^{k} b_{i,n}\overline{X}_{i,n}^{\leq}\right| \geq \frac{\varepsilon}{2}\right\} \\ &\quad\leq \left(\frac{\varepsilon}{2}\right)^{-(\alpha+\eta)}2\kappa(\alpha+\eta)\sum_{i=1}^{m}|b_{i,n}|^{\alpha+\eta}E|\overline{X}_{i,n}^{\leq}|^{\alpha+\eta}. \end{aligned} \tag{4.6}$$

Since, by Jensen's inequality,

$$\begin{aligned} E|X_{i,n}^{\leq} - EX_{i,n}^{\leq}|^{\alpha+\eta} &\leq 2^{\alpha+\eta-1}\left\{E|X_{i,n}^{\leq}|^{\alpha+\eta} + |EX_{i,n}^{\leq}|^{\alpha+\eta}\right\} \\ &\leq 2^{\alpha+\eta}E|X_{i,n}^{\leq}|^{\alpha+\eta}, \end{aligned} \tag{4.7}$$

and similarly,

$$E|X_{i,n}^{>} + EX_{i,n}^{\leq}|^{\alpha-\eta} = E|X_{i,n}^{>} - EX_{i,n}^{>}|^{\alpha-\eta} \leq 2^{\alpha-\eta}E|X_{i,n}^{>}|^{\alpha-\eta},$$

we see that

$$\sup_n\left\{nE|\overline{X}_{1,n}^{\leq}|^{\alpha+\eta}\right\} \vee \sup_n\left\{nE|\overline{X}_{1,n}^{>}|^{\alpha-\eta}\right\} < 2^{\alpha+\eta}M' < \infty.$$

Hence (4.5) and (4.6) imply (4.1), but this time with

$$M = 2^{2+2\alpha+2\eta}\kappa(\alpha - \eta)M' < \infty.$$

The case $m = \infty$ follows trivially by letting $m \to \infty$ in (4.5) and (4.6).

(c) When $\alpha = 1$, we use a "mixed" proof: We define $\overline{X}_{i,n}^{\le}$ and $\overline{X}_{i,n}^{>}$ as in (4.4) but these are equal to $X_{i,n}^{\le}$ and $X_{i,n}^{>}$ respectively, because X_i is symmetric. Then we majorize, on one hand, $\sup_{1\le k\le m}|\sum_{i=1}^{k} b_{i,n}\overline{X}_{i,n}^{>}|$ by $\sum_{i=1}^{m}|b_{i,n}|\,|\overline{X}_{i,n}^{>}|$, and proceed as in the case $\alpha < 1$ in (4.2). On the other hand, we apply to $P\{\sup_{1\le k\le m}|\sum_{i=1}^{k} b_{i,n}\overline{X}_{i,n}^{\le}| > \varepsilon/2\}$ the maximal inequality, as in the case $\alpha > 1$, and obtain (4.6) and (4.7). The conclusion (4.1) follows with $M = (2^{\alpha-\eta} + 2^{1+2\alpha+2\eta}\kappa(\alpha + \eta))M' < \infty$. □

Note that by applying Lemma 2 to $Z_n(1)$, we get the bound

$$\begin{aligned} P\{|Z_n(1)| > \varepsilon\} &= P\left\{\left|\sum_{i=1}^{n}\sum_{j=-\infty}^{\infty} c_{i-j}X_{j,n}\right| > \varepsilon\right\} \\ &= P\left\{\left|\sum_{j=-\infty}^{\infty} X_{j,n}\left(\sum_{i=1-j}^{n-j} c_i\right)\right| > \varepsilon\right\} \\ &\le \frac{M}{n}\varepsilon^{-(\alpha+\eta)}D_n^{(\alpha-\eta)}(\mathbf{c}), \end{aligned}$$

where

$$D_n^{(\alpha-\eta)}(\mathbf{c}) := \sum_{j=-\infty}^{\infty}\left|\sum_{i=1-j}^{n-j} c_i\right|^{\alpha-\eta}. \tag{4.8}$$

The following lemma shows that this quantity grows at most linearly in n, when $\nu \le 1$.

Lemma 3. *If $\nu \le 1$, then for every $\alpha \ge \nu$ we have*

$$D_n^{(\alpha)}(\mathbf{c}) \le ns(\alpha, \mathbf{c}), \tag{4.9}$$

where $s(\alpha, \mathbf{c})$ is defined in (3.8).

Proof. (a) If $\alpha \le 1$,

$$D_n^{(\alpha)}(\mathbf{c}) = \sum_{j=-\infty}^{\infty}\left|\sum_{i=1-j}^{n-j} c_i\right|^{\alpha} \le \sum_{j=-\infty}^{\infty}\sum_{i=1-j}^{n-j}|c_i|^{\alpha} = n\sum_{i=-\infty}^{\infty}|c_i|^{\alpha}.$$

If $\alpha > 1$,

$$\begin{aligned} D_n^{(\alpha)}(\mathbf{c}) &= \sum_{j=-\infty}^{\infty}\left|\sum_{i=1-j}^{n-j} c_i\right|^{\alpha} \le \left(\sum_{i=-\infty}^{\infty}|c_i|\right)^{\alpha-\nu}\left(\sum_{j=-\infty}^{\infty}\left(\sum_{i=1-j}^{n-j}|c_i|\right)^{\nu}\right) \\ &\le \left(\sum_{i=-\infty}^{\infty}|c_i|\right)^{\alpha-\nu} n^{\nu}\sum_{i=-\infty}^{\infty}|c_i|^{\nu} \le ns(\alpha, \mathbf{c}). \end{aligned}$$

□

REMARK. If $\sum_i |c_i| \le 1$, then $s(\alpha, \mathbf{c}) \le 1$ and thus (4.9) becomes

$$D_n^{(\alpha)}(\mathbf{c}) \le n. \tag{4.10}$$

PROOF OF PROPOSITION 3(i). We assume w.l.o.g. that $\sum_i c_i \le 1$. Expression (3.6) is obvious if $[nt_1] = [nt]$ or $[nt_2] = [nt]$. Hence we assume that $[nt] - [nt_1] \ge 1$ and $[nt_2] - [nt] \ge 1$, so that $t_2 - t_1 \ge 1/n$.

Consider the increments $Z_n(t) - Z_n(t_1)$ and $Z_n(t_2) - Z_n(t)$. Since $c_k = 0$ for $|k| > K$, the first increment

$$Z_n(t) - Z_n(t_1) = \sum_{i=-\infty}^{\infty} X_{i,n} \sum_{k=[nt_1]-i+1}^{[nt]-i} c_k$$

involves only $X_{i,n}$'s whose index i satisfies $[nt_1] - K + 1 \le i \le [nt] + K$ (the others have zero coefficient). Similarly, the second increment

$$Z_n(t_2) - Z_n(t) = \sum_{i=-\infty}^{\infty} X_{i,n} \sum_{k=[nt]-i+1}^{[nt_2]-i} c_k$$

involves only $X_{i,n}$'s with index i satisfying $[nt] - K + 1 \le i \le [nt_2] + K$. Since

$$\begin{aligned}[nt_1] - K + 1 &\le [nt] - K < [nt] - K + 1 \\ &< [nt] + K < [nt] + K + 1 \le [nt_2] + K,\end{aligned}$$

we can write

$$Z_n(t) - Z_n(t_1) = S_1(t_1) + \sum_{i=[nt]-K+1}^{[nt]+K} X_{i,n} \sum_{k=[nt_1]-i+1}^{[nt]-i} c_k,$$

where

$$S_1(t_1) = \sum_{i=[nt_1]-K+1}^{[nt]-K} X_{i,n} \sum_{k=[nt_1]-i+1}^{K} c_k$$

and

$$Z_n(t_2) - Z_n(t) = \sum_{i=[nt]-K+1}^{[nt]+K} X_{i,n} \sum_{k=[nt]-i+1}^{[nt_2]-i} c_k + S_2(t_2),$$

where

$$S_2(t_2) = \sum_{i=[nt]+K+1}^{[nt_2]+K} X_{i,n} \sum_{k=-K}^{[nt_2]-i} c_k.$$

The terms $S_1(t_1)$ and $S_2(t_2)$ involve $X_{i,n}$'s that appear only in one of the two increments.

The $X_{i,n}$'s that appear in both increments are components of the vector

$$\mathbf{X} = \left(X_{[nt]-K+1,n}, \dots, X_{[nt]+K,n}\right)$$

and they appear as a scalar product of $\mathbf{X}$ with the vector

$$\mathbf{b}_1(t_1) = \{b_1^{(i)}(t_1)\}_{i=[nt]-K+1}^{[nt]+K}, \qquad b_1^{(i)}(t_1) = \sum_{k=[nt_1]-i+1}^{[nt]-i} c_k,$$

for the first increment, and with the vector

$$\mathbf{b}_2(t_2) = \{b_2^{(i)}(t_2)\}_{i=[nt]-K+1}^{[nt]+K}, \qquad b_2^{(i)}(t_2) = \sum_{k=[nt]-i+1}^{[nt_2]-i} c_k,$$

for the second increment. Therefore,

$$Z_n(t) - Z_n(t_1) = S_1(t_1) + \mathbf{b}_1(t_1) \cdot \mathbf{X}, \tag{4.11}$$

$$Z_n(t_2) - Z_n(t) = S_2(t_2) + \mathbf{b}_2(t_2) \cdot \mathbf{X}. \tag{4.12}$$

These decompositions are such that $S_1(t_1)$, $S_2(t_2)$ and $\mathbf{X}$ are independent.

Since $M_n(t_1, t, t_2)$ is 0 when the increments $Z_n(t) - Z_n(t_1)$ and $Z_n(t_2) - Z_n(t)$ have the same sign,

$$\begin{aligned} P\{M_n(t_1, t, t_2) \geq \varepsilon\} \\ &= P\{S_1(t_1) + \mathbf{b}_1(t_1) \cdot \mathbf{X} \geq \varepsilon,\ S_2(t_2) + \mathbf{b}_2(t_2) \cdot \mathbf{X} \leq -\varepsilon\} \\ &\quad + P\{S_1(t_1) + \mathbf{b}_1(t_1) \cdot \mathbf{X} \leq -\varepsilon,\ S_2(t_2) + \mathbf{b}_2(t_2) \cdot X \geq \varepsilon\}. \end{aligned}$$

We shall estimate each term separately, and since the proofs are similar, we consider only the first term. Introduce the events

$$\begin{aligned} \hat{S}_1 &= \left\{S_1(t_1) \geq \frac{\varepsilon}{2}\right\}, \\ \hat{S}_2 &= \left\{S_2(t_2) \leq -\frac{\varepsilon}{2}\right\}, \\ \hat{X}_1 &= \left\{\mathbf{b}_1(t_1) \cdot \mathbf{X} \geq \frac{\varepsilon}{2}\right\}, \\ \hat{X}_2 &= \left\{\mathbf{b}_2(t_2) \cdot \mathbf{X} \leq -\frac{\varepsilon}{2}\right\}, \\ E &= \{S_1(t_1) + \mathbf{b}_1(t_1) \cdot \mathbf{X} \geq \varepsilon,\ S_2(t_2) + \mathbf{b}(t_2) \cdot \mathbf{X} \leq -\varepsilon\}. \end{aligned}$$

Then

$$E \subset \left(\hat{S}_1 \cup \hat{X}_1\right) \cap \left(\hat{S}_2 \cup \hat{X}_2\right)$$

and hence, by independence and distributivity,

$$\begin{aligned} P(E) &\leq P(\hat{S}_1)P(\hat{S}_2) + P(\hat{S}_1)P(\hat{X}_2) \\ &\quad + P(\hat{S}_2)P(\hat{X}_1) + P\left(\hat{X}_1 \cap \hat{X}_2\right). \end{aligned} \tag{4.13}$$

The idea of the proof is now as follows. We must show that $P(E) = O(t_2 - t_1)^{1+2\eta}$. Each of $P(\hat{S}_1)$ and $P(\hat{S}_2)$ ought to be $O(t_2 - t_1)$ by Proposition 4(i) say, but we must also estimate $P(\hat{X}_1)$, $P(\hat{X}_2)$ and $P(\hat{X}_1 \cap \hat{X}_2)$. Note that

$t_2 - t_1$ and K are related to n because $1/n < t_2 - t_1 \le 1$ and K satisfies (3.5). It turns out that it is the term $P(\hat{X}_1 \cap \hat{X}_2)$ which gives the main contribution. To estimate it, we use the fact that since all the components of $\mathbf{b}_1(t_1)$ and $\mathbf{b}_2(t_2)$ are nonnegative, we can have at the same time $\mathbf{b}_1(t_1) \cdot \mathbf{X} > \varepsilon/2$ and $\mathbf{b}_2(t_2) \cdot \mathbf{X} < -\varepsilon/2$ only if at least two components of $\mathbf{X}$ are large.

We start with $P(\hat{S}_1)$. By Lemmas 2 and 3,

$$
\begin{aligned}
P(\hat{S}_1) &\le P\{S_1(t_1) \ge \varepsilon/2\} \\
&\le M \frac{\varepsilon^{-(\alpha+\eta)}}{n} \sum_{i=[nt_1]-K+1}^{[nt]-K} \left| \sum_{k=[nt_1]-i+1}^{[nt]-i} c_k \right|^{\alpha-\eta} \\
&\le M \frac{\varepsilon^{-(\alpha+\eta)}}{n} D^{(\alpha-\eta)}_{[nt]-[nt_1]}(\mathbf{c}) \\
&\le 2M\varepsilon^{-(\alpha+\eta)}(t - t_1) s(\alpha - \eta, \mathbf{c}) \\
&\le 2M\varepsilon^{-(\alpha+\eta)}(t_2 - t_1)
\end{aligned} \tag{4.14}
$$

since $\sum_{i=-\infty}^{\infty} c_i < 1$. Observe that $P(\hat{S}_2)$ satisfies the same inequality.

To analyze the events $\hat{X}_1$ and $\hat{X}_2$, introduce the events

$$A_i = \left\{ |X_{i,n}| \le \frac{\varepsilon}{8K} \right\}, \qquad i = [nt] - K + 1, \dots, [nt] + K.$$

($X_{i,n}$ is "small' on A_i.) Note first that $\bigcap_{i=[nt]-K+1}^{[nt]+K} A_i \subset \hat{X}_1^c$ because on this intersection

$$|\mathbf{b}_1(t_1) \cdot \mathbf{X}| \le \sum_{i=[nt]-K+1}^{[nt]+K} b_1^{(i)}(t_1) \frac{\varepsilon}{8K} \le \sum_{i=[nt]-K+1}^{[nt]+K} \frac{\varepsilon}{8K} \le \frac{\varepsilon}{4}. \tag{4.15}$$

Therefore,

$$\hat{X}_1 \subset \bigcup_{\langle i \rangle} A_i^c, \tag{4.16}$$

where, for convenience, we write $\langle i \rangle$ to mean the range $i = [nt] - K + 1, \dots, [nt] + K$. Thus $\hat{X}_i$ occurs only if $X_{i,n}$ is "large" for i in the range. Similarly,

$$\hat{X}_2 \subset \bigcup_{\langle i \rangle} A_i^c. \tag{4.17}$$

Applying Lemma 2 with $b_{i,n} = 1$, $m = 1$ yields the estimate

$$P(A_i^c) = P\left\{ |X_{i,n}| > \frac{\varepsilon}{8K} \right\} \le \frac{M}{n} \left(\frac{\varepsilon}{8K} \right)^{-(\alpha+\eta)}, \tag{4.18}$$

so that

$$P(\hat{X}_1) \le 2KP(A_i^c) \le M' \varepsilon^{-(\alpha+\eta)} \frac{K^{1+\alpha+\eta}}{n} \tag{4.19}$$

for some constant M'. Observe that $P(\hat{X}_2)$ satisfies the same inequality.

We now estimate $P(\hat{X}_1 \cap \hat{X}_2)$. Note first that

$$\bigcap_{\substack{\langle i \rangle \\ i \neq i_0}} A_i \cap A_{i_0}^c \cap \hat{X}_1 \cap \hat{X}_2 = \varnothing. \tag{4.20}$$

Indeed, suppose $X_{i_0, n} > 0$. Then $\bigcap_{\langle i \rangle, i \neq i_0} A_i \cap A_{i_0}^c \cap \hat{X}_2 = \varnothing$ since

$$\begin{aligned} -\frac{\varepsilon}{2} &\geq \mathbf{b}_2(t_2) \cdot \mathbf{X} \\ &= b_2^{(i_0)}(t_2) X_{i_0, n} + \sum_{\substack{\langle i \rangle \\ i \neq i_0}} b_2^{(i)}(t_2) X_{i,n} \\ &> b_2^{(i_0)}(t_2) \frac{\varepsilon}{8K} - \frac{\varepsilon}{4} \end{aligned}$$

by (4.15), contradicting $b_2^{(i_0)}(t_2) \geq 0$. A similar argument holds if $X_{i_0, n} \leq 0$.

Suppose that the event $\hat{X}_1 \cap \hat{X}_2$ occurs. Then at least one A_i^c occurs because by (4.16) and (4.17), $\hat{X}_1 \cap \hat{X}_2 \subset \bigcup_{\langle i \rangle} A_i^c$. Relation (4.20), however, states that it is impossible that exactly one A_i^c occurs. Therefore A_i^c must occur for at least two different i's (i.e., at least two $X_{i,n}$'s must be "large"). Hence by (4.18),

$$\begin{aligned} P\left(\hat{X}_1 \cap \hat{X}_2\right) &\leq \sum_{\substack{\langle i \rangle \\ i_0 \neq i_1}} P\left(A_{i_0}^c \cap A_{i_1}^c\right) \\ &\leq 2K^2 [P(A_i^c)]^2 \\ &\leq 2K^2 \left(\frac{M}{n} \left(\frac{\varepsilon}{8K} \right)^{-(\alpha + \eta)} \right)^2 \\ &\leq M'' \varepsilon^{-2(\alpha + \eta)} \left(\frac{K^{1 + \alpha + \eta}}{n} \right)^2 \end{aligned}$$

for some constant $M'' > 0$.

Putting together (4.13), (4.14), (4.20) and (4.21) and introducing a new constant L, we get

$$P(E) \leq L\varepsilon^{-2(\alpha + \eta)} \left[(t_2 - t_1)^2 + 2(t_2 - t_1) \frac{K^{1 + \alpha + \eta}}{n} + \left(\frac{K^{(1 + \alpha + \eta)}}{n} \right)^2 \right].$$

Since $1/n \leq t_2 - t_1$ and $n^{(1/2 - \eta)/(1 + \alpha + \eta)} > K$, we have

$$\frac{K^{1 + \alpha + \eta}}{n} < n^{-(1/2 + \eta)} < (t_2 - t_1)^{1/2 + \eta},$$

and since $t_2 - t_1 \leq 1$, we get

$$\begin{aligned} P(E) &\leq L\varepsilon^{-2(\alpha + \eta)} \left[(t_2 - t_1)^2 + 2(t_2 - t_1)^{3/2 + \eta} + (t_2 - t_1)^{1 + 2\eta} \right] \\ &\leq 4L\varepsilon^{-2(\alpha + \eta)} (t_2 - t_1)^{1 + 2\eta}. \end{aligned}$$

□

REMARK. If instead of the M function in (2.2), we used the J function in (2.1), then we would have had to define $\hat{X}_1 = \{|\mathbf{b}_1(t_1)\cdot\mathbf{X}| \ge \varepsilon/2\}$, $\hat{X}_2 = \{|\mathbf{b}_2(t)\cdot\mathbf{X}| \ge \varepsilon/2\}$. Coefficients could be chosen so that $\hat{X}_1 \cong \hat{X}_2$, making $P(\hat{X}_1 \cap \hat{X}_2) \cong P(\hat{X}_1) = (t_2 - t_1)^{1/2+\eta}$, which is not enough to make $P(E) = O(t_2 - t_1)^{1+\beta}$, $\beta > 0$.

PROOF OF PROPOSITION 4.

$$P\{|Z_n(t_2) - Z_n(t_1)| > \varepsilon\}$$

$$= P\left\{\left|\sum_j X_{j,n}\left(\sum_{i=[nt_1]+1-j}^{[nt_2]-j} c_i^{(n)}\right)\right| > \varepsilon\right\}$$

(i) $$\le \frac{M}{n}\varepsilon^{-(\alpha+\eta)}D^{(\alpha-\eta)}_{[nt_2]-[nt_1]}(\mathbf{c}^{(n)}) \quad [\text{by Lemma 2 and } (4.8)]$$

$$\le M\varepsilon^{-(\alpha+\eta)}s(\alpha-\eta,\mathbf{c}^{(n)})\frac{[nt_2]-[nt_1]}{n} \quad (\text{by Lemma 3})$$

$$\le 2M\varepsilon^{-(\alpha+\eta)}s(\alpha-\eta,\mathbf{c}^{(n)})(t_2-t_1).$$

(ii) In the case $\alpha \le 1$, we can take absolute values:

$$P\left\{\sup_{0\le t\le 1}|Z_n(t)| > \varepsilon\right\} \le P\left\{\sum_j |X_{j,n}|\left(\sum_{i=1-j}^{n-j}|c_i^{(n)}|\right) \le \varepsilon\right\}$$

$$\le \frac{M}{n}\varepsilon^{-(\alpha+\eta)}D_n^{(\alpha-\eta)}(|\mathbf{c}^{(n)}|) \quad (\text{by Lemma 2})$$

$$\le M\varepsilon^{-(\alpha+\eta)}s(\alpha-\eta,\mathbf{c}^{(n)}) \quad (\text{by Lemma 3}).$$

When $\alpha > 1$, consider $J_n(t_1, t, t_2) = J(Z_n(t_1), Z_n(t), Z_n(t_2))$ with J defined in (2.1). Let

$$\bar{J}_n(t_1,t_2) := \sup_{t\in[t_1,t_2]} J_n(t_1,t,t_2).$$

Then

$$P\left\{\sup_{0\le t\le 1}|Z_n(t)| > \varepsilon\right\} \le P\left\{|Z_n(1)| > \frac{\varepsilon}{2}\right\} + P\left\{\bar{J}_n(0,1) > \frac{\varepsilon}{2}\right\}$$

$$\le M\left(\frac{\varepsilon}{2}\right)^{-(\alpha+\eta)}s(\alpha-\eta,\mathbf{c}^{(n)}) + P\left\{\bar{J}_n(0,1) > \frac{\varepsilon}{2}\right\}$$

by Lemmas 2 and 3. By part (i) and Avram and Taqqu (1989), Theorem 2(a) for example, there exists a constant k depending only on $\alpha + \eta$ such that

$$P\left\{\bar{J}_n(t_1,t_2) > \frac{\varepsilon}{2}\right\} \le L'k\left(\frac{\varepsilon}{2}\right)^{-(\alpha+\eta)}(t_2-t_1)s(\alpha-\eta,\mathbf{c}^{(n)})(\ln n)^{1+\alpha+\eta},$$

and hence the result follows with $L'k' = 2^{\alpha+\eta}(M + L'k)$. □

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