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Rigorous justification of Taylor Dispersion via Center Manifold theory

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RIGOROUS JUSTIFICATION OF TAYLOR DISPERSION VIA CENTER MANIFOLD THEORY

by

OSMAN CHAUDHARY

BS, Millersville University, 2010

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Approved by

First Reader

C. Eugene Wayne, PhD
Professor of Mathematics

Second Reader

Margaret Beck, PhD
Associate Professor of Mathematics

Third Reader

Tasso Kaper, PhD
Professor of Mathematics

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Boston University, Graduate School of Arts and Sciences, 2017

ABSTRACT

Imagine fluid moving through a long pipe or channel, and we inject dye or solute into this pipe. What happens to the dye concentration after a long time? Initially, the dye just moves along downstream with the fluid. However, it is also slowly diffusing down the pipe and towards the edges as well. It turns out that after a long time, the combined effect of transport via the fluid and this slow diffusion results in what is effectively a much more rapid diffusion process, lengthwise down the stream. If $0 < \nu << 1$ is the slow diffusion coefficient, then the effective longitudinal diffusion coefficient is inversely proportional to ν, i.e. much larger. This phenomenon is called Taylor Dispersion, first studied by GI Taylor in the 1950s, and studied subsequently by many authors since, such as Aris, Chatwin, Smith, Roberts, and others. However, none of the approaches used in the past seem to have been mathematically rigorous. I'll propose a dynamical systems explanation of this phenomenon: specifically, I'll explain how one can use a Center Manifold reduction to obtain Taylor Dispersion as the dominant term in the long-time limit, and also explain how this Center Manifold can be used to provide any finite number of correction terms to Taylor Dispersion as well.

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List of Abbreviations

ν	....	Viscosity
A	....	Cross-stream average of fluid velocity
χ	....	Deviation from average of cross-stream fluid velocity
ν_T	....	Enhanced diffusion coefficient
ξ	....	Spatial coordinate in scaling variables
τ	....	Time coordinate in scaling variables
$\mathcal{L}$	....	Diffusion operator in scaling variables
$\mathcal{L}_T$	....	Taylor diffusion operator in scaling variables

Chapter 1

Introduction

My thesis research has focused on using Center Manifold reductions to explain Taylor Dispersion, in a mathematically rigorous way. Taylor Dispersion is a phenomenon occuring in pipe flows. Suppose one injects a small, concentrated blob of solute or dye into a very long, straight pipe with fluid flowing through it; see Figure 1·1. If we ride along with the blob as it flows down the pipe, eventually, we see the dye blob has evened itself out in the radial direction. After this happens, the dye starts to spread out very *rapidly* in the longitudinal direction, at a rate inversely proportional to the original, slow rate of diffusion. This rapid diffusion is what we mean when we say Taylor Dispersion. Taylor Dispersion has been analyzed many times over the years,

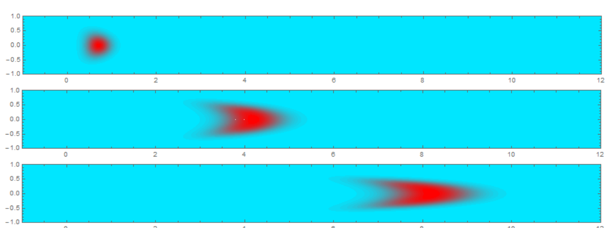

Figure 1·1: Illustration of the expected behavior of the dye blob at initial time, and several times just after the initial time.

including by British fluid mechanist GI Taylor himself (for whom the phenomenon is named) in the 1950s (Taylor, 1953) (Taylor, 1954), and later Aris (Aris, 1956), Chatwin (Chatwin and Allen, 1985), Mercer and Roberts (Mercer and Roberts, 1990), and others. All approaches conclude that if one looks in an appropriate moving frame, the distribution of solute eventually looks Gaussian, with variance increasing rapidly in time, and hence, spreading out quickly. However, though this problem has been extensively studied since the 1950s, rigorous analysis of Taylor Dispersion has been elusive, as each of these approaches seem to contain at least some non-rigorous steps. The approach presented here is mathematically rigorous.

1.1 Setup and formal argument

We consider the evolution of a solute distribution $u(x, y, z, t)$ advected by a fluid velocity field $\vec{V}(x, y, z) = (V(y, z), 0, 0)$ in an infinitely long, straight pipe with cross section Ω, as modeled by

$$u_t = \operatorname{div}(\nu \nabla u) - V(y, z)u_x. \tag{1.1}$$

Here, $x \in \mathbb{R}$, and $(y, z) \in \Omega$, with Neumann Boundary conditions on $\partial\Omega$, and ν is the kinematic viscosity of the fluid, i.e. $\nabla u \cdot \vec{n}|_{\partial\Omega} = 0$. We assume $0 < \nu << 1$. Note we can write the velocity $V(y, z) = A(1 + \chi(y, z))$, where A is the cross-stream average velocity, and χ is a zero average function. Continuing, we change to a frame moving longitudinally with speed A by setting $\tilde{x} = x + At$:

$$u_t = \operatorname{div}(\nu \nabla u) - A\chi(y, z)u_x. \tag{1.2}$$

Here we have suppressed the tildes for notational convenience. Next, owing to the fact that Taylor Dispersion occurs due to interaction between cross-stream variations in the velocity and radial diffusion (Taylor, 1953), we separate the radial diffusion term:

$$u_t = \nu u_{xx} + \nu(u_{yy} + u_{zz}) - A\chi(y, z)u_x. \tag{1.3}$$

Continuing, we can expand both the solution u and the fluid velocity χ in the eigenfunctions $\{\psi_n\}_{n=0}^{\infty}$ of $\partial_y^2 + \partial_z^2$:

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, t)\psi_n(y, z) \quad (1.4)$$

$$\chi(y, z) = \sum_{n=0}^{\infty} \chi_n \psi_n(y, z) \quad (1.5)$$

Note that these are the eigenfunctions of the operator $\partial_y^2 + \partial_z^2$ on Ω, with Neumann boundary conditions, which form an orthonormal basis for $L^2(\Omega)$. The corresponding eigenvalues are denoted $\{-\mu_n\}_{n=0}^{\infty}$, where $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots$ This eigenfunction expansion will turn (1.3) into an infinite system of PDEs:

$$(u_0)_t = \nu(u_0)_{xx} - A \sum_{m=1}^{\infty} \chi_m (u_m)_x \quad (1.6)$$

$$(u_n)_t = \nu(u_n)_{xx} - \nu\mu_n u_n - A\chi_n(u_0)_x - A \sum_{m=1}^{\infty} \chi_{n,m}(u_m)_x \quad (1.7)$$

Here $n = 1, 2, 3, \ldots$. Here with a slight abuse of notation, $\chi_{n,m} = (\psi_n, \chi\psi_m)_{L^2(\Omega)}$. To get this system, we have used the fact that the average of χ is zero, the leading eigenvalue $\mu_0 = 0$, and the corresponding eigenfunction ψ_0 is constant and can be taken to be identically 1.

Notice in the system (1.6) (1.7), the "leading order" linear terms correspond to longitudinal diffusion of the various cross-stream eigenmodes: $\nu\partial_x^2$ in the first equation, and $\nu\partial_x^2 - \nu\mu_n$ in the rest of the equations. However, the spectrum of $\nu\partial_x^2$ on $L^2(\mathbb{R})$ is the entire negative real axis, so there is no spectral gap. We cannot separate timescales into neutral directions and decaying directions. However, we can introduce "scaling variables" as in (Wayne, 1997) and (Gallay and Wayne, 2002), which turn $\nu\partial_x^2$ into an operator $\mathcal{L}$ which has a spectral gap on a certain function space. These variables will also tell us what the long-time leading order terms actually are. The scaling variables are introduced via

$$u_0(x,t) = \frac{1}{\sqrt{1+t}} w_0\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right) \tag{1.8}$$

$$u_n(x,t) = \frac{1}{(1+t)} w_n\left(\frac{x}{\sqrt{1+t}}, \log(1+t)\right), \quad n \neq 0 . \tag{1.9}$$

and $\xi = \frac{x}{\sqrt{1+t}}$, $\tau = \log(1+t)$. Scaling variables have been shown to be helpful in analyzing one-dimensional diffusion equations $u_t = au_{xx}$. This PDE has the Gaussian fundamental solution $\frac{1}{\sqrt{4\pi at}} e^{-x^2/4at}$. In fact, scaling variables will turn the PDE $u_t = au_{xx}$ into the PDE $w_\tau = \mathcal{L}w =: a{\partial_\xi}^2 w + \frac{1}{2}\partial_\xi(\xi w)$, with the Gaussian becoming an eigenfunction for $\mathcal{L}$ with eigenvalue zero. This operator $\mathcal{L}$ corresponds to longitudinal diffusion, with diffusivity a, in scaling variables.
The reason for the difference in the power of t above in the prefactors of u_0 and u_n is related to the intuitive discussion below. Additionally, we expect the dye distribution to even out in the radial direction. Therefore, it makes sense to separate u_0, which is the coefficient of the radial eigenfunction $\psi_0 \equiv 1$.

The resulting system of PDEs, in scaling variables, is

$$\begin{aligned} \partial_\tau w_0 &= \nu{\partial_\xi}^2 w_0 + \frac{1}{2}\partial_\xi(\xi w_0) - A\sum_{m=1}^{\infty}\chi_m \partial_\xi w_m \\ e^{-\tau}\partial_\tau w_n &= e^{-\tau}(\nu{\partial_\xi}^2 w_n + \frac{1}{2}\partial_\xi(\xi w_n) + \frac{1}{2}w_n) \\ &\quad - (\nu\mu_n w_n + A\chi_n\partial_\xi w_0) - Ae^{-\tau/2}\sum_{m=1}^{\infty}\chi_{n,m}\partial_\xi w_m. \end{aligned} \tag{1.10}$$

Notice that the operator $\mathcal{L} = \nu{\partial_\xi}^2 + \frac{1}{2}\partial_\xi(\xi\cdot)$ appears above. As noted above, the operator $\mathcal{L}$ corresponds to longitudinal diffusion, with diffusivity ν, in scaling variables.

Note that in the limit $\tau \to \infty$, the second set of equations reduces formally to $\nu\mu_n w_n + A\chi_n\partial_\xi w_0 = 0$. Plugging this into the equation for w_0, we obtain a simplified equation for w_0 only:

$$\partial_\tau w_0 = \nu {\partial_\xi}^2 w_0 + \frac{1}{2}\partial_\xi(\xi w_0) - A\sum_{m=1}^{\infty}\chi_m\left(-A\frac{\chi_m}{\nu\mu_m}{\partial_\xi}^2 w_0\right) \tag{1.11}$$

$$= \left(\nu + \frac{A^2}{\nu}\sum_{m=1}^{\infty}\frac{\chi_m^2}{\mu_m}\right){\partial_\xi}^2 w_0 + \frac{1}{2}\partial_\xi(\xi w_0) =: \mathcal{L}_T w_0. \tag{1.12}$$

Observe that the right-hand side of the last equation has the operator $\mathcal{L}_T$, which is corresponds to longitudinal diffusion with diffusivity $\nu + \frac{A^2}{\nu}\sum_{m=1}^{\infty}\frac{\chi_m^2}{\mu_m} =: \nu_T$. This new diffusivity is proportional to $1/\nu$, when ν is small. This is precisely what we were aiming for: w_0 is the coefficient of the leading radial eigenfunction (expressed in scaling variables), and we have formally shown that after a long time, it obeys a diffusion equation with diffusivity proportional to $1/\nu$. This formula for the enhanced diffusivity is the same as the one obtained non-rigorously in (Chatwin and Allen, 1985).

We need to explain how a Center Manifold arises here. Notice that in (1.10), the quantity $e^{-\tau}\partial_\tau w_n$ appears on the left-hand side of the second set of equations. In the formal calculation, we compute a limit as $\tau \to \infty$, and the result is an algebraic relation $\nu\mu_n w_n + A\chi_n\partial_\xi w_0 = 0$. We plug this result into the first equation, giving us a reduced system with the apparent long-time behavior. In singular perturbation theory, one typically has

$$x' = f(x, y) \tag{1.13}$$

$$\varepsilon y' = g(x, y). \tag{1.14}$$

Sending $\varepsilon \to 0$ gives an algebraic relation $g(x, y) = 0$, and plugging this into the x equation gives a reduced system. In our setting, letting $\tau \to \infty$ is analogous to letting $\varepsilon \to 0$. The analogy is promising, because in singularly perturbed problems, invariant manifolds are central to the analysis, and in our problem, our goal is to use an invariant manifold - a Center Manifold to explain the occurence of Taylor Dispersion.

However, there are several aspects of system (1.10) that are atypical of standard appliciations of Center Manifold theory. For instance, in the formal discussion, the apparent invariant set is given by a linear relation $\nu\mu_n w_n + A\chi_n \partial_\xi w_0 = 0$, as opposed to a quadratic or higher-order one, as is standard in Center Manifold theory. Additionally, if one rewrites (1.10) with only $\partial_\tau w_0$, $\partial_\tau w_n$ on the left-hand side, the prefactor of the "stable coordinate" $\nu\mu_n w_n + A\chi_n \partial_\xi w_0$ is e^τ, suggesting double-exponential in τ approach to the invariant set, which is not typical of attracting Center Manifolds (Chen et al., 1997). Therefore, in order to detect the approach to the invariant set, we have to rescale the time coordinate back, using $\tau = \log(1+t)$. However, the resulting system is still non-autonomous. Therefore we make it autonomous by introducing a new variable $\sigma = (1+t)^{-1/2}$. However, there is still a problem, as the "higher order" terms are proportional to $\sigma^2 \mathcal{L} w_n$ and hence contain unbounded operators.

We can get around these issues. In fact, we can prove the following theorem:

Theorem 1.1.1. *Given any $M > 0$, for initial conditions $u(x, y, z, 0)$ of (1.1) with sufficiently rapid spatial decay, there exists a system of ordinary differential equations possessing a finite dimensional center manifold, such that the long-time asymptotics of solutions of (1.1), up to terms of $\mathcal{O}(t^{-M})$, is given by the restriction of solutions of this system of ODEs to its center manifold. Moreover, the dynamics on this center manifold correspond to enhanced diffusion proportional to $\nu + A^2/\nu \sum_{m=1}^{\infty} \frac{\chi_m^2}{\mu_m}$.*

Remark 1.1.2. *The spatial decay required in the initial data is quantified in the weighted Hilbert spaces defined below in Section 2.*

The proof is motivated by my already published work in (Beck et al., 2015) and is in preparation in (Beck et al.,). In (Beck et al., 2015), my coauthors and I study a model analysis problem based on system (1.10) (which I review in Chapter 2). We outline the proof as follows. First, we need to state some information about the spectrum of $\mathcal{L}_T$. Recall that the operator $\mathcal{L}_T$ has an eigenfunction ϕ_0, which is a Gaussian in scaling variables. This eigenfunction corresponds to the zero eigenvalue of $\mathcal{L}_T$. Actually, $\mathcal{L}_T$, on the weighted Hilbert space $L^2(m) = \{f \in L^2(\mathbb{R}) | \int (1+\xi^2)^m |f(\xi)|^2 d\xi < \infty\}$ has eigenvalues at each negative half-integer $-k/2$, with corresponding eigenfunctions ϕ_k, along with a half-plane of essential spectrum satisfying $Re(\lambda) \leq (1-2m)/4$. The point is that as m increases, the space $L^2(m)$ is restricted to functions which decay more rapidly at $|\xi| = \infty$, and more and more of the eigenvalues are isolated.

We proceed by writing solutions to (1.10) using a finite eigenfunction expansion in the eigenfunctions of $\mathcal{L}_T$, and derive ODEs for the coefficients in this expansion. The idea is that the leading order behavior should correspond to the eigenfunctions with eigenvalues closest to zero, i.e. the Gaussian corresponding to Taylor Dispersion. Since the operator $\mathcal{L}_T$ is best suited to the scaling time variable τ, we derive these ODEs in scaling variables. We then diagonalize these ODEs, using the corresponding center and stable directions from the formal discussion. At this point, the ODEs are still in terms of the scaling time τ. Hence, they still suggest double-exponential in τ approach to the invariant set, and are still not autonomous. Therefore we rescale time back using $\tau = \log(1+t)$, and autonomize using $\sigma = (1+t)^{-1/2}$, as suggested in the previous paragraph. The resulting system of ODEs has a spectral gap of size $\mathcal{O}(\nu)$, now with bounded higher-order terms. Therefore the system of ODEs has a Center Manifold, with $t = \mathcal{O}(1/\nu)$ approach time. The dynamics on the Center Manifold are analyzed using the scaling time τ. This is reasonable, since motion on the Center Manifold occurs much more slowly than motion approaching the Center Manifold. The restriction of this system of ODEs to its Center Manifold give Taylor Dispersion as the leading-order behavior, along with a systematic hierarchy of decaying correction terms. Furthermore, the size of the spectral gap gives us approach time $t = \mathcal{O}(1/\nu)$ consistent with classical Taylor Dispersion estimates (Taylor, 1953).

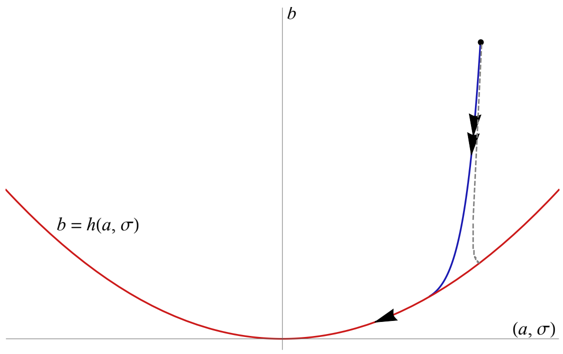

Figure 1·2: The Center Manifold, along with a typical trajectory, and the stable fiber (the dashed curve). The double arrows along the trajectory indicate fast transit towards the Center Manifold. Here a and b are "diagonalized" center and stable coordinates representing the coefficients in the finite eigenfunction expansion, and $\sigma = (1+t)^{-1/2}$ is the autonomizing coordinate.

This leading-order behavior will be only be valid if the discarded "error" terms in the finite eigenfunction expansion decay at the rate consistent with the Center Manifold analysis. Due to a possible lack of separation in timescales, we will choose not to analyze these error terms using Center Manifold theory. Instead, we perform the estimates in Fourier space of the *original* variables appearing in (1.6) (1.7). We split the analysis between low longitudinal-wavenumbers (for which Taylor Dispersion occurs) and high longitudinal-wavenumbers (which decay exponentially quickly in t).

The argument above, in which we change timescales several times throughout the analysis, is reminiscient of singular perturbation theory. Here, the Center Manifold plays the role of the slow manifold, and the stable fibers along which solutions approach the center manifold play the role of the fast fibers. See Figure 1·2.

1.2 Splitting of solutions and statement of main theorem

We conclude this introductory section by rewriting system (1.10) in terms of the operator $\mathcal{L}_T$:

$$\begin{aligned}\partial_\tau w_0 &= \mathcal{L}_T w_0 - \frac{D_T}{\nu}\partial_\xi{}^2 w_0 - A\sum_{m=1}^{\infty}\chi_m \partial_\xi w_m \\ \partial_\tau w_n &= \mathcal{L}_T w_n + \frac{1}{2}w_n + \left(-\frac{D_T}{\nu}\right)\partial_\xi{}^2 w_n - e^\tau\left(\nu\mu_n w_n + A\chi_n\partial_\xi w_0\right) \\ &\quad - Ae^{\tau/2}\sum_{m=1}^{\infty}\chi_{n,m}\partial_\xi w_m\end{aligned} \tag{1.15}$$

where we have denoted $\nu_T = \nu + \frac{D_T}{\nu}$. Next, recall that in the formal asymptotic limit, $w_n = -\frac{A\chi_n}{\nu\mu_n}\partial_\xi w_0$; hence, w_n asymptotically behaves like a derivative, so we set $\partial_\xi u_n = w_n$. This gives us

$$\begin{aligned}\partial_\tau w_0 &= \mathcal{L}_T w_0 - \frac{D_T}{\nu}\partial_\xi{}^2 w_0 - A\sum_{m=1}^{\infty}\chi_m \partial_\xi{}^2 u_m \\ \partial_\tau u_n &= \mathcal{L}_T u_n + \left(-\frac{D_T}{\nu}\right)\partial_\xi{}^2 u_n - e^\tau\left(\nu\mu_n u_n + A\chi_n w_0\right) \\ &\quad - Ae^{\tau/2}\sum_{m=1}^{\infty}\chi_{n,m}\partial_\xi u_m\end{aligned} \tag{1.16}$$

Notice in this change of variable, we are implicitly assuming the w_n have zero average in ξ.

Remark 1.2.1. *We believe that, via minor modifications, our results can be extended to the case when* $\int_{-\infty}^{\infty} w_n^s(\xi,t)d\xi \neq 0$. *We plan to discuss such modifications in a future work*

We will return to the study of (1.16) in Chapter 3, but first, in order to provide additional insight into the approach, we discuss a simplified model in order to gain insight into the Taylor dispersion phenomenon in an analytically simpler setting.

Chapter 2

A model problem illustrating some ideas of the proof

In this chapter I introduce and treat a model problem consisting of just a pair of coupled PDEs which present an enhanced diffusion property similar to Taylor diffusion in an analytically simpler setting. This problem was first treated with my coauthors M. Beck and C.E. Wayne in (Beck and Wayne, 2013) and this chapter is largely taken from that source. The model system of coupled PDEs is

$$\begin{aligned} \partial_\tau w &= \mathcal{L}w - \partial_\xi v \\ \partial_\tau v &= (\mathcal{L}+1/2)v - e^{\tau}(\nu v + \partial_\xi w), \end{aligned} \tag{2.1}$$

which corresponds to just the modes w_0 and w_1 in (1.15) (or more generally, to w_0 and w_n, where n is the first integer for which $\hat{\chi}_n \neq 0$). The reader should note that this is not meant to be a physical model, but instead it is meant to be an analysis problem which reflects the core mathematical difficulties of analyzing the full Taylor Dispersion problem. Proceeding, note that the term proportional to $e^{\tau/2}$ has disappeared since, if only w_0 and w_1 are non-zero, that sum reduces to $\hat{\chi}_0 \partial_\xi w_1$, and $\hat{\chi}_0 = 0$. (We have also rescaled the variables so that the coefficient $A\hat{\chi}_1 = A\hat{\chi}_{-1} = 1$.) Also note that (2.1), written back in terms of the original variables, which we denote by $\tilde{w}(x,t)$ and

$\tilde{v}(x,t)$, is given by

$$\begin{aligned} \tilde{w}_t &= \nu \tilde{w}_{xx} - \tilde{v}_x \\ \tilde{v}_t &= \nu \tilde{v}_{xx} - \nu \tilde{v} - \tilde{w}_x . \end{aligned} \tag{2.2}$$

For this coupled system of two partial differential equations we will show that

- The long-time behavior of solutions can be computed to any degree of accuracy by the solution on a (finite-dimensional) invariant manifold.
- To leading order, the long-time behavior on this invariant manifold agrees with that given by a diffusion equation with the enhanced Taylor diffusion constant.
- The expressions for the invariant manifolds can be computed quite explicitly, but we are not able to show that these expressions converge as the dimension of the manifold goes to infinity.

In this section we focus on a center-manifold analysis of the model equation (2.1). Our analysis justifies the formal lowest order approximation $\nu v + \partial_x w = 0$ and shows that to this order the solutions behave as if w was the solution of a diffusion equation with "enhanced" diffusion coefficient $\nu_T = (\nu + \frac{1}{\nu})$. Furthermore, the center-manifold machinery allows one to systematically (and rigorously) compute corrections to these leading order asymptotics to any order in time.

Because we expect $v \approx -\frac{1}{\nu}\partial_\xi w$ - i.e. because we expect v to behave at least asymptotically as a derivative, we define a new dependent variable u as

$$v = \partial_\xi u \; . \tag{2.3}$$

Inserting into the $\partial_\tau v$ equation in (2.1), we get

$$\begin{aligned}\partial_\tau(\partial_\xi u) &= \partial_\tau v = (\mathcal{L}+1/2)\,v - e^\tau\,(\nu v + \partial_\xi w)\\ &= (\mathcal{L}+1/2)\,\partial_\xi u - e^\tau\,(\nu\partial_\xi u + \partial_\xi w)\\ &= \partial_\xi \mathcal{L} u - e^\tau\,(\nu\partial_\xi u + \partial_\xi w)\end{aligned}$$

where we have used the fact that $\partial_\xi \mathcal{L} u = \mathcal{L}\partial_\xi u + \frac{1}{2}\partial_\xi u$. After antidifferentiating the last line with respect to ξ, we get a system in terms of w and u:

$$\begin{aligned}\partial_\tau w &= \mathcal{L} w - \partial_\xi{}^2 u\\ \partial_\tau u &= \mathcal{L} u - e^\tau\,(\nu u + w)\,.\end{aligned} \tag{2.4}$$

Remark 2.0.1. *Note that, if $u \in L^2(m)$, the change of variables* (2.3) *implies that $\int_{-\infty}^{\infty} v(\xi,t)d\xi = 0$. We believe that, via minor modifications, our results can be extended to the case when $\int_{-\infty}^{\infty} v(\xi,t)d\xi \neq 0$. We plan to discuss such modifications in a future work.*

Studies of Taylor dispersion generally focus on localized tracer distributions. For that reason, and also because of the spectral properties of the operators $\mathcal{L}$ which we discuss further below, it is convenient to work in weighted Hilbert spaces.

Definition 2.0.2. *The Hilbert space $L^2(m)$ is defined as*

$$L^2(m) = \left\{ f \in L^2(\mathbb{R}) \mid \|f\|_m^2 = \int (1+\xi^2)^m |f(\xi)|^2 d\xi < \infty \right\}.$$

Note that we require the solutions of the equation to lie in these weighted Hilbert spaces when expressed in terms of the *scaling* variables. If we revert to the original variables then it is appropriate to study them in the time-dependent norms obtained from these as follows:

$$
\begin{aligned}
\|w(\xi,\tau)\|^2_{L^2(m)} &= \int (1+\xi^2)^m |w(\xi,\tau)|^2 d\xi \\
&= e^{\tau/2} \int (1+\xi^2)^m |\tilde{w}(e^{\tau/2}\xi, e^{\tau}-1)|^2 d\xi \\
&= \int (1+e^{-\tau}x^2)^m |\tilde{w}(x, e^{\tau}-1)|^2 dx \\
&= \sum_{\ell=0}^{m} \frac{C(m,\ell)}{(1+t)^{\ell}} \int x^{2\ell} |\tilde{w}(x,t)|^2 dx \ .
\end{aligned}
$$

Thus, when we study solutions of our model equations in the "original" variables, as opposed to the scaling variables, we will also consider the weighted L^2 norms of the functions, but the different powers of x will be weighted by a corresponding (inverse) power of t to account for the relationship between space and time encapsulated in the definition of the scaling variables.

Since we expect $\nu u + w \approx 0$, we further rewrite (2.4) by adding and subtracting $\frac{1}{\nu}\partial_\xi^2 w$ from the first equation and $\frac{1}{\nu}\partial_\xi^2 u$ from the second finally obtaining

$$
\begin{aligned}
\partial_\tau w &= \mathcal{L}_T w - \frac{1}{\nu}\left({\partial_\xi}^2 w + \nu {\partial_\xi}^2 u\right) \\
\partial_\tau u &= \mathcal{L}_T u - \frac{1}{\nu}{\partial_\xi}^2 u - e^{\tau}\left(\nu u + w\right),
\end{aligned}
\tag{2.5}
$$

where

$$
\mathcal{L}_T \phi = \left(\nu + \frac{1}{\nu}\right)\partial_{\xi^2}\phi + \frac{1}{2}\partial_\xi(\xi\phi) \ .
$$

Thus, $\mathcal{L}_T$ is just the diffusion operator, written in terms of scaling variables, but with the enhanced, Taylor diffusion rate, $\nu_T = \nu + 1/\nu$.

The operators $\mathcal{L}_T$ have been analyzed in (Gallay and Wayne, 2002). In particular, their spectrum can be computed in the weighted Hilbert spaces $L^2(m)$ and one finds

$$
\sigma(\mathcal{L}_T) = \left\{\lambda \in \mathbb{C} \mid \Re(\lambda) \le \frac{1}{4} - \frac{m}{2}\right\} \cup \left\{-\frac{k}{2} \mid k \in \mathbb{N}\right\} \ .
$$

Furthermore, the eigenfunctions corresponding to the isolated eigenvalues $\lambda_k = -k/2$ are given by the Hermite functions

$$\phi_0(\xi) = \frac{1}{\sqrt{4\pi\nu_T}} e^{-\xi^2/(4\nu_T)} \ , \quad \text{and} \quad \phi_k(\xi) = \partial_\xi^k \phi_0(\xi)$$

and the corresponding spectral projections are given by the Hermite polynomials

$$H_k(\xi) = \frac{2^k(\nu_T)^k}{k!} e^{\xi^2/(4\nu_T)} \partial_\xi^k e^{-\xi^2/(4\nu_T)} \ .$$

Remark 2.0.3. *The expressions in (Gallay and Wayne, 2002) for ϕ_k and H_k are derived in the case when the diffusion coefficient is* 1*. The expressions given here follow easily by the change of variables $\xi \to \xi/\sqrt{\nu_T}$. More explicitly, for the classical Hermite functions $\tilde{\phi}_0(y) = \frac{1}{\sqrt{4\pi}} e^{-y^2/4}$, $\tilde{\phi}_k(y) = \partial_y^k \phi_0$, and $\tilde{H}_k(y) = \frac{2^k}{k!} e^{y^2/4} \partial_y^k e^{-y^2/4}$, one has the orthonormality relations $\int \tilde{H}_k(y)\tilde{\phi}_\ell(y)dy = \delta_{k,\ell}$. Changing variables to $y = \xi/\sqrt{\nu_T}$ leads to the formulas for the eigenfunctions and spectral projections for $\mathcal{L}_T$. Note further that with this definition, the Hilbert space adjoint of $\mathcal{L}_T$ satisfies ${\mathcal{L}_T}^\dagger H_k = -\frac{k}{2} H_k$.*

Given the spectrum of $\mathcal{L}_T$ discussed above, we expect that the leading order part of the solution as t tends to infinity will be associated with the eigenspace corresponding to eigenvalues closest to zero. With this in mind, fix an integer N and assume that $m > N + 1/2$. This insures that the spectrum of $\mathcal{L}_T$ has at least $N + 1$ isolated eigenvalues on the Hilbert space $L^2(m)$ and that the essential spectrum lies strictly to the left of the half-plane $\{\lambda \in \mathbb{C} \mid \Re(\lambda) < -N/2\}$. Now define P_N to be the spectral projection onto the first $N + 1$ eigenmodes

$$P_N w = \sum_{k=0}^{N} \alpha_k(\tau)\phi_k(\xi),$$

where

$$\alpha_k(\tau) = \langle H_k, w(\tau) \rangle_{L^2}.$$

We will write the solutions of (2.5) as

$$\begin{aligned} w &= P_N w + w_s \\ u &= P_N u + u_s. \end{aligned} \tag{2.6}$$

Based on the spectral picture and our discussion above, we expect that w^s and u^s will decay faster than $P_N w$ and $P_N u$, and hence, since we are interested in the leading order terms in the long time behavior, we focus our attention on $P_N w$ and $P_N u$.

2.1 Model Problem - Center Manifold calculations

We will show that for any N the equations for $P_N w$ and $P_N u$ have an attractive center manifold and that the motion on this manifold reproduces and refines the expected Taylor diffusion.

If we apply the projection operator P_N to both of the equations in (2.5), we obtain

$$\begin{aligned} \sum_{k=0}^{N} \dot{\alpha}_k \phi_k &= \sum_{k=1}^{N} -\frac{k}{2} \alpha_k \phi_k - \frac{1}{\nu} \sum_{k=0}^{N-2} (\alpha_k + \nu\beta_k) \phi_{k+2} \\ \sum_{k=0}^{N} \dot{\beta}_k \phi_k &= \sum_{k=0}^{N} -\frac{k}{2} \beta_k \phi_k - \frac{1}{\nu} \sum_{k=0}^{N-2} \beta_k \phi_{k+2} - e^{\tau} \left(\sum_{k=0}^{N} (\nu\beta_k + \alpha_k)\, \phi_k \right). \end{aligned}$$

Shifting indices and matching coefficients gives us the following system of ODEs for the coefficients α_k and β_k:

$$
\begin{aligned}
\dot{\alpha_0} &= 0 \\
\dot{\alpha_1} &= -\frac{1}{2}\alpha_1 \\
\dot{\alpha_k} &= -\frac{k}{2}\alpha_k - \left(\frac{1}{\nu}\alpha_{k-2} + \beta_{k-2}\right) \text{ for } 2 \le k \le N \\
\dot{\beta_0} &= -e^{\tau}\left(\nu\beta_0 + \alpha_0\right) \\
\dot{\beta_1} &= -\frac{1}{2}\beta_1 - e^{\tau}\left(\nu\beta_1 + \alpha_1\right) \\
\dot{\beta_k} &= -\frac{k}{2}\beta_k - \frac{1}{\nu}\beta_{k-2} - e^{\tau}\left(\nu\beta_k + \alpha_k\right) \text{ for } 2 \le k \le N
\end{aligned}
\tag{2.7}
$$

Note that these equations contain *no* contributions from the "stable" modes w^s and u^s. Note further that, because of the form of the equations, those with even indices k decouple from those with k odd. Thus, we can analyze these two cases separately. We'll provide the details for the case of k even below - the equations with k odd behave in a very similar fashion.

Remark 2.1.1. *Note that if we multiply all of the equations in* (2.7) *by* $e^{-\tau}$ *and set* $e^{-\tau} = \epsilon$ *(since we are interested in large times) we get equations that are formally of classical singularly perturbed form. (However, the small parameter* ϵ *is time dependent here.) Invariant manifold theory has been a powerful tool in the rigorous analysis of singularly perturbed problems and that analogy will guide our use of the center-manifold theory in what follows.*

In order to make the invariant manifold more apparent we rewrite the even index equations by rescaling the time variable as

$$\tau = \log(1 + t). \tag{2.8}$$

In analogy with the above remark about singularly perturbed systems, we are essentially switching to a "fast" version of our system by making this change of time variable. Continuing, we introduce a new dependent variable

$$\eta = e^{-\tau} = \frac{1}{1+t}\ . \tag{2.9}$$

Then, if we denote $\frac{d}{dt}$ by a prime $'$, we have

$$\begin{aligned}
\alpha_0' &= 0 \\
\alpha_k' &= -\eta\left(\frac{k}{2}\alpha_k + \frac{1}{\nu}\alpha_{k-2} + \beta_{k-2}\right) \\
\beta_0' &= -\left(\nu\beta_0 + \alpha_0\right) \\
\beta_k' &= -\left(\nu\beta_k + \alpha_k\right) - \eta\left(\frac{k}{2}\beta_k + \frac{1}{\nu}\beta_{k-2}\right) \\
\eta' &= -\eta^2,
\end{aligned} \tag{2.10}$$

where the values $2 \le k \le N$ are even. Notice the linearization of this system at the fixed point $\alpha_k = \beta_k = \eta = 0$ has eigenvalues $\lambda^c = 0$, with an $[N/2]+2$ dimensional eigenspace and $\lambda^s = -\nu$, with an $[N/2]+1$ dimensional eigenspace (here $[M]$ refers to the greatest integer less than or equal to M). We proceed by diagonalizing the linear part of the system via

$$\begin{aligned}
a_k &= \alpha_k \\
b_k &= \frac{1}{\nu}\alpha_k + \beta_k
\end{aligned} \tag{2.11}$$

which transforms (2.10) into

$$\begin{aligned}
a_0' &= 0 \\
a_k' &= -\eta\left(\frac{k}{2}a_k + b_{k-2}\right) \\
b_0' &= -\nu b_0 \\
b_k' &= -\nu b_k - \eta\left(\frac{k}{2}b_k - \frac{1}{\nu^2}a_{k-2} + \frac{2}{\nu}b_{k-2}\right) \\
\eta' &= -\eta^2,
\end{aligned} \tag{2.12}$$

where again $2 \leq k \leq N$ are even.

The remainder of this section is devoted to the analysis of these equations and we prove two main results:

- We first show that, for any N, (2.12) has a center-manifold of the type described in the introduction, and we derive explicit expressions for the functions whose graphs give the manifold. (See Propositions 2.1.2 and 2.1.3.)

- We derive the asymptotic (in τ) behavior of solutions of these equations. (See Propositions 2.1.4 and 2.1.6, and Corollary 2.1.5.)

We begin by noting that the linearization of (2.12) at the fixed point $a_k = b_k = \eta = 0$ has eigenvalues $\lambda^c = 0$, with an $[N/2]+2$ dimensional eigenspace and $\lambda^s = -\nu$, with an $[N/2]+1$ dimensional eigenspace. Thus, from the classical center-manifold theorem (say, for example, the center-manifold theorem proven in (Chen et al., 1997)), we know that (at least in a neighborhood of this point), there will be an invariant $[N/2]+2$ dimensional center manifold. We also know that, in a neighborhood of the origin, the center-manifold can be written as the graph of a function with components

$$b_k = h_k(a_N, \ldots, a_0, \eta). \tag{2.13}$$

In addition, because of the "lower triangular" form of the equations (i.e. the fact that the equations for a_k' and b_k' depend only on a_ℓ and b_ℓ with $\ell \leq k$), we find that we can express the manifold as

$$b_k = h_k(a_k, a_{k-2}, \ldots, a_0, \eta) \ .$$

We now show that we can find explicit expressions for the functions h_k successively, starting with h_0 and then progressing through h_2, h_4, etc. What's more, these expressions hold for all $a_k, a_{k-2}, \ldots, a_0, \eta$, i.e. without the restriction to a small neighborhood that is inherent in general center-manifold theorems like that of (Chen et al., 1997).

We start with the equations for a_0 and b_0 which are just

$$
\begin{aligned}
a_0' &= 0 \\
b_0' &= -\nu b_0 \ .
\end{aligned}
$$

From this we see immediately that we can choose the invariant manifold to be the graph of $h_0 \equiv 0$. However, note that this example also reminds us that the center manifold is not unique, since we could also choose the center manifold to be given by the graph of $\tilde{h}_0(a_0, \eta) = K_0 e^{-\nu/\eta} a_0$. This is consistent with the theorems on the existence of center manifolds, since both of these manifolds have the same Taylor expansion to any finite order about $a_0 = \eta = 0$. For simplicity, in what follows we will always use the first function - i.e. we will take $h_0 \equiv 0$.

Now consider the center manifold for a_2 and b_2. Since the equations for a_0, a_2, b_0, b_2, η decouple from all other a_k and b_k, we expect the center manifold to be given by the graph of a function $b_2 = h_2(a_2, a_0, \eta)$. In fact, as we show below, it has no dependence on a_2 - i.e. we can take $b_2 = h_2(a_0, \eta)$. In this case the equation for the invariance of the graph of this function takes the form

$$
(D_{a_0} h_2) a_0' + (D_\eta h_2) \eta' = -\eta h_2 - \nu h_2 - \frac{2\eta}{\nu} h_0 + \frac{1}{\nu^2} \eta a_0 \ .
$$

Inserting the equations for a_0' and η' and using the fact that $h_0 \equiv 0$, we find

$$
-\eta^2 (D_\eta h_2) = -\eta h_2 - \nu h_2 + \frac{1}{\nu^2} \eta a_0 \ .
$$

We now show that h_2 is linear in a_0, so we write

$$
h_2(a_0, \eta) = \phi_{2,0}(\eta) a_0 \ ,
$$

and find

$$-\eta^2\phi'_{2,0} = -(\eta+\nu)\phi_{2,0} + \frac{1}{\nu^2}\eta \ .$$

This equation is hard to solve in general due to the singular point at $\eta = 0$, but remarkably,

$$\phi_{2,0}(\eta) = \frac{\eta}{\nu^3}$$

is an exact solution (which goes to zero as $\eta \to 0$), so

$$h_2(a_0,\eta) = \frac{\eta a_0}{\nu^3}$$

is a function whose graph (together with that of $h_0 \equiv 0$) gives us the center manifold for the equations for a_0, a_2, b_0, b_2, η. Due to the singular point at $\eta = 0$, this may not be the only solution (just as in the case for h_0), but we are free to choose this special solution for h_2.

Next we consider the case of $h_4(a_4, a_2, a_0, \eta)$. Building on the examples above we show that

- h_4 is independent of a_4;
- h_4 is linear in a_2 and a_0.

If this is the case we can write

$$h_4(a_2, a_0, \eta) = \phi_{4,2}(\eta)a_2 + \phi_{4,0}(\eta)a_0 \ .$$

Inserting this form of the solution into the equation for the center-manifold, we find

$$\begin{aligned}\phi_{4,2}(\eta)a_2' + \phi_{4,0}(\eta)a_0' + (\phi'_{4,2}(\eta)a_2 + \phi'_{4,0}(\eta)a_0)\eta' \\ = -(\nu+2\eta)(\phi_{4,2}(\eta)a_2 + \phi_{4,0}(\eta)a_0) + \frac{\eta a_2}{\nu^2} - \frac{2\eta^2 a_0}{\nu^4}\end{aligned}$$

where in the last term we have plugged in the expression for h_2. Inserting the equations for a_2' and η' and grouping the terms proportional to a_2 and a_0 we find two ODE's for the $\phi's$, namely

$$\begin{aligned}
-\eta^2\phi_{4,2}'(\eta) &= -(\nu+\eta)\phi_{4,2}(\eta) + \frac{\eta}{\nu^2} \\
-\eta^2\phi_{4,0}'(\eta) &= -(\nu+2\eta)\phi_{4,0}(\eta) - \frac{2\eta^2}{\nu^4} \ .
\end{aligned}$$

The first of these equations is the same as the equation for $\phi_{2,0}$ above so we have

$$\phi_{4,2}(\eta) = \frac{\eta}{\nu^3} \ .$$

The second equation is very similar and we find that it again has a simple, exact solution, namely

$$\phi_{4,0}(\eta) = -\frac{2\eta^2}{\nu^5}.$$

Thus, we also have an exact expression for the center-manifold in this case:

$$h_4(a_2, a_0, \eta) = \frac{\eta a_2}{\nu^3} - \frac{2\eta^2 a_0}{\nu^5}.$$

One can continue this procedure. For instance, for the function h_6, one obtains the formula

$$h_6(a_4, a_2, a_0, \eta) = \frac{\eta a_4}{\nu^3} - \frac{2\eta^2 a_2}{\nu^5} + \frac{5\eta^3 a_0}{\nu^7}.$$

This leads to the following

Proposition 2.1.2. *For any $k = 0, 2, 4, \ldots$, there exist constants $\{\hat{H}(k, k-2\ell)\}$ such that the graph of the function*

$$h_k(a_{k-2}, a_{k-4}, \ldots, a_0, \eta) = \sum_{\ell=1}^{k/2} \hat{H}(k, k-2\ell)\eta^\ell a_{k-2\ell} \tag{2.14}$$

gives the invariant manifold for b_k. Furthermore, for any fixed k, the coefficients $\{\hat{H}(k, k-2\ell)\}$ can be explicitly determined, and the coefficients $\hat{H}(k,p) \sim \mathcal{O}(\nu^{-(k-p)-1})$.

Proof. The proof proceeds inductively. Note that we have already verified the inductive hypothesis for $k = 0, 2, 4$. (We take the empty sum that occurs on the RHS of (2.14) when $k = 0$ to correspond to $h_0 \equiv 0$.) Assume that it holds for all even integers less than or equal to $k - 2$. We now show that it holds for h_k.

Inserting our inductive hypothesis into the invariance equation we find

$$\begin{aligned}
&\sum_{\ell=1}^{k/2} \hat{H}(k, k-2\ell)\eta^{\ell} a'_{k-2\ell} + \sum_{\ell=1}^{k/2} \ell \hat{H}(k, k-2\ell)\eta^{\ell-1} a_{k-2\ell}\eta' \\
&\quad = -\frac{k}{2}\eta h_k - \nu h_k - \frac{2}{\nu}\eta h_{k-2} + \frac{1}{\nu^2}\eta a_{k-2} \\
&\quad = -\sum_{\ell=1}^{k/2} \frac{k}{2}\hat{H}(k, k-2\ell)\eta^{\ell+1} a_{k-2\ell} - \sum_{\ell=1}^{k/2} \nu \hat{H}(k, k-2\ell)\eta^{\ell} a_{k-2\ell} \\
&\qquad -\frac{2}{\nu}\sum_{\ell=1}^{k/2-1} \hat{H}(k-2, k-2-2\ell)\eta^{\ell+1} a_{k-2-2\ell} + \frac{1}{\nu^2}\eta a_{k-2}.
\end{aligned} \tag{2.15}$$

Inserting the equations for $a'_{k-2\ell}$ and η' into the first line of (2.15), one finds

$$\begin{aligned}
&\sum_{\ell=1}^{k/2} \hat{H}(k, k-2\ell)\eta^{\ell} \left(-\left(\frac{k-2\ell}{2}\right)\eta a_{k-2\ell} - \eta h_{k-2\ell-2} \right) - \sum_{\ell=1}^{k/2} \ell \hat{H}(k, k-2\ell)\eta^{\ell+1} a_{k-2\ell} \\
&\quad = -\sum_{\ell=1}^{k/2} \frac{k}{2}\hat{H}(k, k-2\ell)\eta^{\ell+1} a_{k-2\ell} - \sum_{\ell=1}^{k/2} \hat{H}(k, k-2\ell)\eta^{\ell+1} h_{k-2\ell-2}.
\end{aligned} \tag{2.16}$$

Note that the first sum in the last line of (2.16) cancels the first sum on the RHS of (2.15). Thus, we can rewrite (2.15)-(2.16) as

$$\begin{aligned}
\sum_{\ell=1}^{k/2} \nu \hat{H}(k, k-2\ell)\eta^{\ell} a_{k-2\ell} &= \frac{1}{\nu^2}\eta a_{k-2} - \frac{2}{\nu}\sum_{\ell=1}^{k/2-1} \hat{H}(k-2, k-2\ell-2)\eta^{\ell+1} a_{k-2-2\ell} \\
&\quad + \sum_{\ell=1}^{k/2} \hat{H}(k, k-2\ell)\eta^{\ell+1} h_{k-2\ell-2}.
\end{aligned} \tag{2.17}$$

We now rewrite the last sum in this expression by using the inductive form of $h_{k-2\ell-2}$,

$$h_{k-2\ell-2} = \sum_{m=1}^{k/2-(\ell+1)} \hat{H}(k-2(\ell+1), k-2(\ell+m+1))\eta^{m} a_{k-2(\ell+m+1)} \ .$$

Thus,

$$\sum_{\ell=1}^{k/2} \hat{H}(k,k-2\ell)\eta^{\ell+1}h_{k-2\ell-2}$$
$$= \sum_{\ell=1}^{k/2} \sum_{m=1}^{k/2-(\ell+1)} \hat{H}(k,k-2\ell)\hat{H}(k-2(\ell+1),k-2(\ell+m+1))\eta^{\ell+m+1}a_{k-2(\ell+m+1)}$$
$$= \sum_{p=3}^{k/2} \sum_{\ell=1}^{p-2} \hat{H}(k,k-2\ell)\hat{H}(k-2(\ell+1),k-2p)\eta^{p}a_{k-2p},$$

where in the last term we set $p = \ell+m+1$ and interchanged the order of summation. If in the last sum in the first line of (2.17) we also change the summation variable to $p = \ell+1$ we find that (2.17) can finally be rewritten as

$$\sum_{\ell=1}^{k/2} \nu\hat{H}(k,k-2\ell)\eta^{\ell}a_{k-2\ell} = \frac{1}{\nu^2}\eta a_{k-2} - \frac{2}{\nu}\sum_{p=2}^{k/2} \hat{H}(k-2,k-2p))\eta^{p}a_{k-2p}$$
$$+ \sum_{p=3}^{k/2} \sum_{\ell=1}^{p-2} \hat{H}(k,k-2\ell)\hat{H}(k-2(\ell+1),k-2p)\eta^{p}a_{k-2p}. \tag{2.18}$$

We solve (2.18) for $\hat{H}(k,k-2\ell)$, beginning with $\hat{H}(k,k-2)$. Since the only term on the RHS of (2.18) proportional to a_{k-2} is the first term, and we obtain $\hat{H}(k,k-2) = \frac{1}{\nu^3}$, consistent with the inductive hypothesis. Next consider $\hat{H}(k,k-4)$. In this case, we consider all terms in (2.18) proportional to a_{k-4}. The only one comes from the second term on the RHS of the equation and we have $\hat{H}(k,k-4) = -\frac{2}{\nu^2}\hat{H}(k-2,k-4)$. The inductive hypothesis implies that $\hat{H}(k-2,k-4) \sim \mathcal{O}(\nu^{-3})$, so we find $\hat{H}(k,k-4) \sim \mathcal{O}(\nu^{-5})$ as required by the inductive hypothesis. We now continue to solve for the coefficients $\hat{H}(k,k-2\ell)$, $\ell = 3,4,\ldots$, noting that in each case, the terms on the RHS of the equation proportional to $a_{k-2\ell}$ have coefficients that have already been determined at prior stages of the inductive process and that they are all $\mathcal{O}(\nu^{-2\ell-1}) = \mathcal{O}(\nu^{-(k-p)-1})$. $\square$

We now describe the entirely analogous results for the modes α_k and β_k with k odd. If we introduce new variables t and η as in (2.8), (2.9), and diagonalize the linear part of the resulting equations using the change of variables (2.11), we find:

$$\begin{aligned} a_1' &= -\frac{1}{2}\eta a_1 \\ a_k' &= -\eta\left(\frac{k}{2}a_k + b_{k-2}\right) \\ b_1' &= -\left(\nu + \frac{1}{2}\eta\right) b_1 \\ b_k' &= -\nu b_k - \eta\left(\frac{k}{2}b_k - \frac{1}{\nu^2}a_{k-2} + \frac{2}{\nu}b_{k-2}\right) \\ \eta' &= -\eta^2, \end{aligned} \tag{2.19}$$

where the values $3 \leq k \leq N$ are odd this time.

Proceeding as before, consider first the equations for a_1, b_1, and η which decouple from all the rest of the equations. Then by inspection we see that, just as for b_0, the graph of the function $h_1(a_1, \eta) \equiv 0$ is an invariant center manifold for these equations. We now include the equations for a_3 and b_3 and, building on the experience from the even case, look for an invariant manifold of the form

$$b_3 = h_3(a_1, \eta) = \phi_{3,1}(\eta)a_1 \ .$$

Inserting this into the equations, we see that in order for this graph to be invariant, $\phi_{3,1}$ must satisfy

$$\phi_{3,1}a_1' + a_1\phi_{3,1}'\eta' = -(\nu + \frac{3}{2}\eta)\phi_{3,1}a_1 + \frac{\eta}{\nu^2}a_1 - \frac{2\eta}{\nu}h_1 \ .$$

From the fact that $h_1 \equiv 0$ and the equation for a_1', we see that this reduces to the ODE for $\phi_{3,1}$

$$-\eta^2\phi_{3,1}' = -(\nu + \eta)\phi_{3,1} + \frac{\eta}{\nu^2} \ .$$

This is the same equation satisfied by $\phi_{2,0}$ and thus we find

$$h_3(a_1, \eta) = \frac{\eta a_1}{\nu^3} \ .$$

Proceeding now as in the even case, we establish the following proposition by induction.

Proposition 2.1.3. *For any $k = 1, 3, 5, \ldots,$ there exist constants $\{\hat{H}^{odd}(k, k-2\ell)\}$*

such that the graph of the function

$$h_k(a_{k-2}, a_{k-4}, \ldots, a_1, \eta) = \sum_{\ell=1}^{\frac{k-1}{2}} \hat{H}^{odd}(k, k-2\ell)\eta^{\ell} a_{k-2\ell}$$

gives the equation for the invariant manifold for b_k*. Furthermore, for any fixed* k*, the coefficients* $\{\hat{H}^{odd}(k, k-2\ell)\}$ *can be explicitly determined, and the coefficients* $\hat{H}^{odd}(k,p) \sim \mathcal{O}(\nu^{-(k-p)-1})$.

We conclude this section by using our expressions for the center-manifold to derive the asymptotic behavior of the coefficient functions a_k and b_k (or equivalently α_k and β_k.)

Begin by noting that from the general theory of center-manifolds, any solution with initial conditions in a neighborhood of the invariant manifold will approach the manifold at a rate $\sim \mathcal{O}(e^{-\nu t}) = \mathcal{O}(e^{-\nu(e^{\tau}-1)})$. Thus, we can determine the long time asymptotics of all solutions in this neighborhood by focusing on the behavior of solutions on the invariant manifold. Note that this means, for solutions with sufficiently small initial conditions, that after a time τ such that $\nu e^{\tau} \gg 1$, we will be very close to the center-manifold and the behavior of solutions on this manifold will determine the asymptotic behavior of solutions after this time. Reverting from our rescaled time τ to the original time t in the problem this means that solutions on the center-manifold will determine the behavior of solutions for times $t > \mathcal{O}(\frac{1}{\nu})$, which is the expected timescale for Taylor Dispersion to occur. At the moment, it appears our results only hold for solutions with small initial conditions. However, it turns out our formulas for the center manifolds (which are defined globally) are also *globally attracting* on the timescale $t > \mathcal{O}(\frac{1}{\nu})$. We provide details in the Appendix.

We proceed with our calculation of the asymptotics of the quantities a_k and b_k. As in the case of the calculation of the manifold we focus separately on the coefficients with even and odd indices. Starting with the coefficients with k even, note that we obviously have $\alpha_0 =$ constant, so we begin with $k = 2$.

Given

$$a_2' = -\eta(a_2 + b_0),$$

we can simplify this by noting that $b_0 = h_0 \equiv 0$ on the center-manifold. Finally, it's simpler to solve this differential equation by reverting from the t variables to $\tau = \log(1 + t)$; keeping Remark 2.1.1 about singularly perturbed systems in mind, notice we are essentially switching to the "slow" version of the system (which gives the dynamics on the center manifold). The equation then reduces to

$$\dot{a}_2 = -a_2 \ ,$$

from which we can immediately conclude that

$$a_2(\tau) \sim \mathcal{O}(e^{-\tau}) \ .$$

Next consider a_4, for which we have (again, rewriting things in terms of the temporal variable τ)

$$\dot{a}_4 = -2a_4 - b_2 = -2a_4 - \frac{e^{-\tau} a_0}{\nu^3} \ ,$$

where the last equality used the fact that $b_2 = h_2(a_0, \eta) = \frac{\eta a_0}{\nu^2}$ on the center-manifold. Solving this equation using the method of variation of constants, we find that

$$a_4(\tau) \sim \mathcal{O}(\frac{e^{-\tau}}{\nu^3}) \ .$$

As a last explicit example, consider the case of a_6 where we have

$$\dot{a}_6 = -3a_6 - b_4 = -3a_6 - \frac{e^{-\tau} a_2}{\nu^3} + \frac{2e^{-2\tau} a_0}{\nu^5} \ .$$

Finally, since a_0 is constant and $a_2(\tau) \sim \mathcal{O}(e^{-\tau})$, we see that the asymptotic behavior of a_6 is

$$a_6(\tau) \sim \mathcal{O}(\frac{e^{-2\tau}}{\nu^5}) \ .$$

We can generalize these results in the following

Proposition 2.1.4. *Suppose $k = 4, 6, \dots$ is an even, positive integer. On the center manifold of the system of equations* (2.12)*, the variables a_k have the following asymptotic behavior:*

$$|a_k(\tau)| \leq \begin{cases} \frac{C(N,k) e^{-\frac{k}{4}\tau}}{\nu^{k-1}} & : k = 0 \ mod \ 4 \\ \frac{C(N,k) e^{-\frac{k+2}{4}\tau}}{\nu^{k-1}} & : k = 2 \ mod \ 4 \end{cases}$$

Note that once we have these formulas, the expressions for the center-manifold immediately imply the following.

Corollary 2.1.5. *Suppose $k = 4, 6, \dots$ is an even, positive integer. On the cen-*

ter manifold of the system of equations (2.12)*, the variables* b_k *have the following asymptotic behavior:*

$$|b_k(\tau)| \leq \begin{cases} \frac{C(N,k)e^{-\frac{k+4}{4}\tau}}{\nu^{k+1}} & : k = 0 \ mod \ 4 \\ \frac{C(N,k)e^{-\frac{k+2}{4}\tau}}{\nu^{k+1}} & : k = 2 \ mod \ 4 \end{cases}$$

Proof. The proof of Proposition 2.1.4 is a straightforward induction argument. Suppose that we have demonstrated that the estimates hold for $k = 4, 6, \ldots, k_0$. We then show that it holds for $k_0 + 2$. The equation of motion for a_{k_0+2} is

$$\dot{a}_{k_0+2} = -\frac{k_0+2}{2}a_{k_0+2} - h_{k_0}(a_{k_0-2}, a_{k_0-4}, \ldots, a_0, e^{-\tau}).$$

Inserting the formula for h_{k_0} from Proposition 2.1.2 and solving using Duhamel's formula, we obtain the bound

$$|a_{k_0+2}| \leq \frac{C(N)}{\nu}\sum_{\ell=1}^{k_0/2} a_{k_0-2\ell}\frac{\eta^\ell}{\nu^{2\ell}}. \tag{2.20}$$

Consider the case $k_0 = 0$ mod 4. Then

$$k_0 - 2\ell = \begin{cases} 2 \text{ mod } 4 & \text{if } \ell \text{ is odd} \\ 0 \text{ mod } 4 & \text{if } \ell \text{ is even} \end{cases}$$

and correspondingly from the induction hypothesis,

$$|a_{k_0-2\ell}| \leq \begin{cases} \frac{C(N)e^{-\frac{-k_0-2\ell+2}{4}\tau}}{\nu^{k-1}} & \text{if } \ell \text{ is odd} \\ \frac{C(N)e^{-\frac{-k_0-2\ell}{4}\tau}}{\nu^{k-1}} & \text{if } \ell \text{ is even.} \end{cases}$$

Inserting into (2.20), using the fact that $\eta = e^{-\tau}$, and splitting the sum into even and odd ℓ, we obtain

$$|a_{k_0+2}| \leq \frac{C(N)}{\nu}\left\{\sum_{\ell=1,\ell odd}^{k_0/2-1}\frac{e^{-\frac{(k_0-2\ell+2)\tau}{4}}e^{-\ell\tau}}{\nu^{k_0-2\ell-1}\nu^{2\ell}} + \sum_{\ell=2,\ell even}^{k_0/2-2}\frac{e^{-\frac{(k_0-2\ell)\tau}{4}}e^{-\ell\tau}}{\nu^{k_0-2\ell-1}\nu^{2\ell}} + \frac{a_0e^{-\frac{k_0\tau}{2}}}{\nu^{k_0}}\right\} \tag{2.21}$$

Notice we have to separate out the $\ell = k_0/2$ term because this corresponds to a_0, which is actually constant. We are interested in locating the slowest decaying terms. These terms will have, in the exponent, the least negative coefficients on τ. For $\ell \geq 1$

odd, the coefficients in the exponent are

$$-\frac{k_0 - 2\ell + 2}{4} - \ell = -\frac{k_0}{4} - \frac{1}{2} - \frac{\ell}{2} \tag{2.22}$$

which are least negative when $\ell = 1$. The corresponding coefficient in the exponent is $-\frac{k_0+4}{4}$, and so the slowest decaying term from the ℓ odd sum is $\mathcal{O}(e^{-\frac{k_0+4}{4}\tau})$. We determine the slowest decaying term in the ℓ even sum. For $\ell \geq 2$ even, the coefficients in the exponent are

$$-\frac{k_0 - 2\ell}{4} - \ell = -\frac{k_0}{4} - \frac{\ell}{2} \tag{2.23}$$

which are least negative when $\ell = 2$. The corresponding coefficient in the exponent is again $-\frac{k_0+4}{4}$, and so the slowest decaying term from the ℓ odd sum is again $\mathcal{O}(e^{-\frac{k_0+4}{4}\tau})$. Lastly, we determine the ν dependence of the constant. The largest power of ν in the denominator comes from $\ell = k_0/2$ and is $\frac{1}{\nu^{k_0+1}}$. Therefore we have

$$|a_{k_0+2}| \leq \frac{C(N)}{\nu^{k_0+1}} e^{-\frac{k_0+4}{4}\tau}.$$

Recalling that we are in the case $k_0 = 0 \bmod 4$ (so that $k_0 + 2 = 2 \bmod 4$), we have verified the claim in this case. The case $k_0 = 2 \bmod 4$ follows similarly. Once Proposition 2.1.4 is established, a nearly identical calculation establishes Corollary 2.1.5.

□

The coefficients a_k and b_k, with k odd, can be estimated in an entirely analogous fashion to obtain the following proposition.

Proposition 2.1.6. *Suppose $k = 1, 3, \ldots$ is an odd, positive integer. On the center manifold of the system of equations* (2.19)*, the variables a_k have the following asymptotic behavior:*

$$|a_k(\tau)| \leq \begin{cases} \frac{C(N,k)e^{-\frac{k+1}{4}\tau}}{\nu^{k-1}} & : k = 1 \ mod\ 4 \\ \frac{C(N,k)e^{-\frac{k+3}{4}\tau}}{\nu^{k-1}} & : k = 3 \ mod\ 4. \end{cases} \tag{2.24}$$

If $k = 3, 5, \ldots$ (recall that $b_1 \equiv 0$ on the center manifold), the corresponding *coefficients b_k satisfy the estimates*

$$|b_k(\tau)| \leq \begin{cases} \frac{C(N,k)e^{-\frac{5+k}{4}\tau}}{\nu^{k+1}} & : k = 1 \ mod\ 4 \\ \frac{C(N,k)e^{-\frac{3+k}{4}\tau}}{\nu^{k+1}} & : k = 3 \ mod\ 4. \end{cases} \tag{2.25}$$

Next, we treat the error terms using the Fourier Transform.

2.2 Model Problem - a priori estimates via the Fourier Transform

In order to show that the center manifold, discussed in the previous section, really does describe the leading order large-time behavior of solutions of (2.2), we need to make our discussion before Theorem 1.4 in the introduction more precise (which basically says Taylor Dispersion only happens for low wavenumbers). We'll have to undo the scaling variables, and switch to the Fourier side; this way we can precisely cut-off wavenumbers larger than, say $|k_0| \approx \frac{\nu}{2}$ and quantify how fast these "high" wavenumber terms decay. To do this in a way that is consistent with the analysis in section 2.1, we need to introduce a new norm $|||\cdot|||$, which, when applied to functions on the Fourier side, is equivalent to the $L^2(m)$ norm applied to their real-space scaling variables counterparts.

The main result (see Theorem 2.3.1 in Section 4) depends on estimates of the solution in $L^2(m)$. With this in mind, we note that

$$\|w(\tau)\|_{L^2(m)} \leq C(m)(t+1)^{1/4}\sqrt{\sum_{j=0}^{m}\left\|\frac{1}{(1+t)^{j/2}}\partial_k^j\widehat{w}(\cdot,t)\right\|_{L^2}^2} =: |||\tilde{w}(t)|||,$$

and below we will bound each partial derivative of $\hat{w}(k,t)$. Note that $\|\cdot\|_{L^2(m)}$ and $|||\cdot|||$ are indeed equivalent norms, which follows from the fact that

$$\|\partial_k^j\hat{w}(\cdot,t)\|_{L^2}^2 \leq C\int(1+x^j)^2|\tilde{w}(x,t)|^2dx \leq C(t+1)^{j-1/2}\|w(\tau)\|_{L^2(j)}^2,$$

which in turn implies that $|||\tilde{w}(t)||| \leq C(m)\|w(\tau)\|_{L^2(m)}$.

Consider equation (2.2). Let $\hat{w} = \mathcal{F}\tilde{w}$ and $\hat{v} = \mathcal{F}\tilde{v}$, where $\mathcal{F}$ sends a function to its Fourier transform. We obtain

$$\frac{d}{dt}\begin{pmatrix}\hat{w}\\ \hat{v}\end{pmatrix} = A(k)\begin{pmatrix}\hat{w}\\ \hat{v}\end{pmatrix}, \qquad A(k) = \begin{pmatrix}-\nu k^2 & -\mathrm{i}k\\ -\mathrm{i}k & -\nu(k^2+1)\end{pmatrix}.$$

The solution to this equation is

$$\begin{pmatrix}\hat{w}(k,t)\\ \hat{v}(k,t)\end{pmatrix} = e^{A(k)t}\begin{pmatrix}\hat{w}_0(k)\\ \hat{v}_0(k)\end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix}\tilde{w}(x,t)\\ \tilde{v}(x,t)\end{pmatrix} = \mathcal{F}^{-1}[e^{A(k)t}] * \begin{pmatrix}\tilde{w}_0(x)\\ \tilde{v}_0(x)\end{pmatrix}.$$

To understand these solutions, we must understand $e^{A(k)t}$, which we'll do by diagonalizing $A(k)$. The eigenvalues of A are given by

$$\lambda_\pm(k,\nu) = -\nu k^2 - \frac{\nu}{2} \pm \frac{1}{2}\sqrt{\nu^2 - 4k^2},$$

and the corresponding eigenvectors are

$$v_\pm(\lambda,k) = \begin{pmatrix}\mathrm{i}k\\ -\nu k^2 - \lambda_\pm(k,\nu)\end{pmatrix} = \begin{pmatrix}\mathrm{i}k\\ \frac{\nu}{2} \mp \frac{1}{2}\sqrt{\nu^2-4k^2}\end{pmatrix}.$$

We put these into the columns of a matrix $S = [v_+, v_-]$ and obtain

$$\begin{aligned} S &= \begin{pmatrix}\mathrm{i}k & \mathrm{i}k\\ \frac{1}{2}[\nu - \sqrt{\nu^2-4k^2}] & \frac{1}{2}[\nu + \sqrt{\nu^2-4k^2}]\end{pmatrix}\\ S^{-1} &= \frac{1}{\mathrm{i}k\sqrt{\nu^2-4k^2}}\begin{pmatrix}\frac{1}{2}[\nu + \sqrt{\nu^2-4k^2}] & -\mathrm{i}k\\ \frac{1}{2}[-\nu + \sqrt{\nu^2-4k^2}] & \mathrm{i}k\end{pmatrix}.\end{aligned}$$

We then have $A = S\Lambda S^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_+, \lambda_-)$.

Remark 2.2.1. *Note that S becomes singular when $k = \pm\nu/2$, because for that value of k there is a double eigenvalue, and a slightly different decomposition of A, reflecting the resultant Jordan block structure, is necessary. This will be dealt with in the proof of Proposition 2.2.3. We do not highlight this issue in the below formulas for the solution, as we wish to focus on the intuition for how to decompose solutions, which does not depend on this singularity.*

Hence,

$$e^{A(k,\nu)t} = S(k,\nu)\begin{pmatrix}e^{\lambda_+(k,\nu)t} & 0\\ 0 & e^{\lambda_-(k,\nu)t}\end{pmatrix} S^{-1}(k,\nu),$$

or explicitly

$$\begin{aligned}
\hat{w}(k,t) &= \frac{ik(e^{\lambda_- t}-e^{\lambda_+ t})}{\sqrt{\nu^2-4k^2}}\hat{v}_0 + \frac{1}{2}\left(\frac{(-\nu+\sqrt{\nu^2-4k^2})e^{\lambda_- t}+(\nu+\sqrt{\nu^2-4k^2})e^{\lambda_+ t}}{\sqrt{\nu^2-4k^2}}\right)\hat{w}_0 \\
\hat{v}(k,t) &= \frac{1}{2}\left(\frac{(-\nu+\sqrt{\nu^2-4k^2})e^{\lambda_+ t}+(\nu+\sqrt{\nu^2-4k^2})e^{\lambda_- t}}{\sqrt{\nu^2-4k^2}}\right)\hat{v}_0 \\
&\qquad -\frac{ik(e^{\lambda_+ t}-e^{\lambda_- t})}{\sqrt{\nu^2-4k^2}}\hat{w}_0, \qquad (2.26)
\end{aligned}$$

which we'll abbreviate as

$$\hat{w}(k,t) = \left(f_1(k)\hat{w}_0(k) + f_2(k)\hat{v}_0(k)\right)e^{\lambda_+(k)t} + g(k)e^{\lambda_-(k)t} \qquad (2.27)$$

and similarly for $\hat{v}$. The motivation for separating the solution in this way is the fact that $\mathrm{Re}(\lambda_-(k)) \leq -\nu/2$, and so any component of the solution that includes a factor of $e^{\lambda_-(k)t}$ will decay exponentially in time, even for k near zero. Hence, it is primarily the first term, above, involving $e^{\lambda_+(k)t}$ that we must focus our attention on. We'll proceed with the analysis only for $\hat{w}$; all of the results for $\hat{v}$ are analogous.

Remark 2.2.2. *In order to justify the difference of $(t+1)^{-1/2}$ in the scaling variables for $\tilde{w}$ and $\tilde{v}$, corresponding to (1.8), we need to show that $\tilde{v}$ decays faster than $\tilde{w}$ by this amount. This can be seen from the above expression for solutions. In particular, for k near zero, say $|k| < \nu/2$, we have*

$$e^{A(k,\nu)t} \sim \frac{e^{-\nu k^2 t}}{ik\nu}\begin{pmatrix} 1 & \frac{k}{\nu} \\ \frac{k}{\nu} & -\frac{k^2}{\nu^2}\end{pmatrix}.$$

An extra factor of k corresponds to an x-derivative, and so the v component does decay faster by a factor of $t^{-1/2}$.

We will split the analysis into "high" and "low" frequencies using a cutoff function and Taylor expansion about $k = 0$. Define

$$\Omega^{>} = \{|k| > \frac{\sqrt{15}\nu}{8}\}, \quad \Omega^{<} = \{|k| \leq \frac{\sqrt{15}\nu}{8}\},$$

and let $\psi(k)$ be a smooth cutoff function equalling 1 on $\Omega^{<}$ and zero for $|k| \geq \frac{\sqrt{15}\nu}{8} + \nu^2$.

We then write $\hat{w}$ as

$$\begin{aligned}\hat{w}(k,t) &= \psi(k)\hat{w}(k,t) + (1-\psi(k))\,\hat{w}(k,t)\\ &=: \psi(k)\left(f_1(k)\hat{w}_0(k) + f_2(k)\hat{v}_0(k)\right)e^{\lambda_+(k)t} + \psi(k)g(k)e^{\lambda_-(k)t} + \hat{w}_{high}(k,t).\end{aligned}$$

Again, the motivation is to focus on the part of the solution that does not decay exponentially in time. This does not necessarily occur if k is small, which is exactly where $\psi(k) \neq 0$.

Notice that, on $\Omega^<$, we can write $\lambda_+(k) = -\left(\nu + \frac{1}{\nu}\right)k^2 + \Lambda(k)$, where

$$\Lambda(k) = \frac{\nu}{2}\sum_{n=2}^{\infty}\binom{1/2}{n}(-1)^n\left(\frac{4k^2}{\nu^2}\right)^n. \tag{2.28}$$

In $\Omega^<$, $4k^2/\nu^2 < 15/16 < 1$, and so the above series is convergent. It will also be important that it starts with four powers of k. More precisely,

$$\Lambda(k) = \frac{8k^4}{\nu^3}\sum_{n=0}^{\infty}\binom{1/2}{n+2}(-1)^n\left(\frac{4k^2}{\nu^2}\right)^n.$$

This representation for $\Lambda(k)$ holds by similar reasoning whenever $\psi(k) \neq 0$. We now write

$$\begin{aligned}\hat{w}(k,t) &= \psi(k)e^{-\nu_T k^2 t}e^{\Lambda(k)t}\left(f_1(k)\hat{w}_0(k) + f_2(k)\hat{v}_0(k)\right) + \psi(k)g(k)e^{\lambda_-(k)t} + \hat{w}_{high}(k,t)\\ &=: \psi(k)e^{-\nu_T k^2 t}\bar{w}(k,t) + \psi(k)g(k)e^{\lambda_-(k)t} + \hat{w}_{high}(k,t),\end{aligned}$$

where $\nu_T = \nu + \frac{1}{\nu}$ and

$$\bar{w}(k,t) = e^{\Lambda(k)t}\left(f_1(k)\hat{w}_0(k) + f_2(k)\hat{v}_0(k)\right). \tag{2.29}$$

The purpose of this last part of our decomposition of solutions is to emphasize that, to leading order, the decay of the low modes will be determined by the term $e^{-\nu_T k^2 t}$. Therefore, the Taylor dispersion phenomenon is also apparent in Fourier space.

Finally, we Taylor expand the quantity $\bar{w}$ into a polynomial of degree N, plus a remainder term:

$$\bar{w}(k,t) = \sum_{j=0}^{N}\frac{\partial_k^j\bar{w}(0,t)}{j!}k^j + \left[\bar{w}(k,t) - \sum_{j=0}^{N}\frac{\partial_k^j\bar{w}(0,t)}{j!}k^j\right] =: \bar{w}_{low}^N + \bar{w}_{low}^{res}.$$

Thus, we have (suppressing some of the k and t dependence for notational convenience)

$$\hat{w}(k,t) = \psi e^{-\nu_T k^2 t}\left(\bar{w}^N_{low} + \bar{w}^{res}_{low}\right) + \psi g e^{\lambda_-(k)t} + \hat{w}_{high}. \tag{2.30}$$

The main results of this section are

Proposition 2.2.3. *There exists a constant C, independent of ν and the initial data, such that*

$$\|\partial_k^j \hat{w}_{high}\|_{L^2} + \|\partial_k^j(\psi g e^{\lambda_- t})\|_{L^2} \le C\nu^{-2-j} e^{-\frac{\nu}{8}t}(\|\hat{w}_0\|_{C^j} + \|\hat{v}_0\|_{C^j}).$$

Proposition 2.2.4. *There exists a constant C such that*

$$\left\|\frac{1}{(1+t)^{\frac{j}{2}}}\partial_k^j\left(\psi e^{-\nu_T k^2 t}\bar{w}^{res}_{low}\right)\right\|_{L^2} \le \frac{C}{\nu^{\frac{N}{4}+\frac{j}{2}} t^{\frac{N}{4}+\frac{1}{2}}}(\|\hat{w}_0\|_{C^{N+j}} + \|\hat{v}_0\|_{C^{N+j}}).$$

The constant C depends on N, but it is independent of ν.

Here $\|f\|_{C^j} = \sum_{s=0}^{j}\sup_{k\in\mathbb{R}}|\partial_k^s f(k)|$. These results imply that $\hat{w}_{high}$ and $g\psi e^{\lambda_- t}$ decay exponentially in t, and are thus higher-order, while $\hat{w}^{res}_{low}$ decays algebraically, at a rate that can be made large by choosing N (which will correspond to the dimension of the center manifold from Section 2.1) large. In the next section, §2.3, it will be shown that the behavior of the remaining term, $\bar{w}^N_{low}$, is governed by the dynamics on the center manifold, in which one can directly observe the Taylor dispersion phenomenon.

Proof of Proposition 2.2.3

Notice that, for $k \in \Omega^>$ (the support of $\hat{w}_{high}$), the eigenvalues $\lambda_\pm(k)$ both lie in a sector with vertex at $(\mathrm{Re}\lambda, \mathrm{Im}\lambda) = (-\nu k^2 - \nu/4, 0)$. Therefore, to obtain the desired bound, we need to determine the effect of the derivatives ∂_k^j. Such a derivative could potentially be problematic, due to the factors of $\sqrt{\nu^2 - 4k^2}$, which can be zero in $\Omega^>$. (This is exactly due to the Jordan block structure at $k = \pm\nu/2$.) To work around this, we use the fact that we can equivalently write

$$\begin{pmatrix}\hat{w}(k,t)\\ \hat{v}(k,t)\end{pmatrix} = e^{A(k)t}\begin{pmatrix}\hat{w}_0(k)\\ \hat{v}_0(k)\end{pmatrix}$$

and bound derivatives of this expression for $k \in \Omega^>$. Such derivatives either fall on the initial conditions, which leads to the dependence of the constant on the C^j norms of $\hat{v}_0$ and $\hat{w}_0$, or the derivatives can fall on the exponential. In the latter case, using the fact that

$$A'(k) = \begin{pmatrix} -2\nu k & -i \\ -i & -2\nu k \end{pmatrix},$$

which behaves no worse that $\mathcal{O}(k)$, we obtain terms of the form (writing $\hat{U} = (\hat{w}_0, \hat{v}_0)$ for convenience)

$$\|(kt)^p e^{A(k)t}\partial_k^q \hat{U}_0\|_{L^2}^2 \leq C\|\hat{U}_0\|_{C^q}^2 \int |kt|^{2p}\|e^{A(k)t}\|^2 dk.$$

Next, note that $\|e^{A(k)t}\| \leq C\nu^{-2}e^{-\nu(k^2+1/4)t}$. This follows essentially from the above-mentioned bound on the real part of $\lambda_\pm$ in $\Omega^>$. One needs to be a bit careful when $k = \pm\nu/2$, as there $\lambda_+ = \lambda_-$. This changes the bound from $\sim e^{\lambda_+(k)t}$ to $\sim \nu t e^{\lambda_+(k)t}$, but this power of t can be absorbed into the exponential since $\mathrm{Re}(\lambda_+) < -\nu k^2 - \nu/4 - \delta\nu$ for some $\delta > 0$ that is independent of ν. The factor of ν^{-2} that appears is related to the fact that $\|S^{-1}\| = \mathcal{O}(\nu^{-2})$ for $k \in \Omega^>$, $k \neq \pm\nu/2$. Thus, we have

$$\begin{aligned} \|(kt)^p e^{A(k)t}\partial_k^q \hat{U}_0\|_{L^2}^2 &\leq C\nu^{-4}\|\hat{U}_0\|_{C^q}^2 \int |kt|^{2p} e^{-2(\nu k^2+\nu/4)t} dk \\ &\leq C\nu^{-4-p-1/2}\|\hat{U}_0\|_{C^q}^2 t^{p-1/2} e^{-\nu t/2} \\ &\leq C\nu^{-4-2p} e^{-\nu t/4}\|\hat{U}_0\|_{C^q}^2, \end{aligned}$$

which proves the result for $\hat{w}_{high}$. A similar proof works for the $\|\partial_k^j(\psi g e^{\lambda_- t})\|_{L^2}$ term. □

Proof of Proposition 2.2.4

We now derive bounds on the residual term $\psi e^{-\nu_T k^2 t}\bar{w}_{low}^{res}$. Recall the integral formula for the Taylor Remainder:

$$\bar{w}_{low}^{res}(k,t) = \int_0^k \int_0^{k_1} \dots \int_0^{k_N} \partial_{k_{N+1}}^{N+1} \bar{w}(k_{N+1},t) d_{k_{N+1}} d_{k_N} \dots d_{k_1}. \tag{2.31}$$

With this formula in mind, we want to derive bounds on the derivatives of $\bar{w}(k,t)$, but we need only deal with $k \in \Omega^<$, since we are ultimately estimating the size of $\psi e^{-\nu_T k^2 t}\bar{w}_{low}^{res}$.

Recall that

$$\bar{w} = e^{\Lambda(k)t}\left(f_1\hat{w}_0 + f_2\hat{v}_0\right).$$

The functions f_1 and f_2 are smooth in $\Omega^<$, so our estimate will depend on derivatives of the initial data and derivatives of $e^{\Lambda(k)t}$. However, the reader should note that f_1 and f_2 are also dependent on ν, but their derivatives give us inverse powers of ν no worse than any others appearing in this section, so we choose not to explicitly keep track of these powers. The following lemma will be used in estimating these derivatives:

Lemma 2.2.5. *Let* $\Phi(k,t) = k^d e^{-\nu_T k^2 t}$. *Then*

$$\|\Phi(\cdot,t)\|_{L^2} \leq C(d)(\nu_T t)^{-\frac{2d+1}{4}}$$

Proof. Use the fact that $\int_{\mathbb{R}} e^{-x^2/4}dx = 2\sqrt{\pi}$ and change variables. $\square$

With this lemma in mind, we need to keep track of the powers of k and t that appear in $\partial_k^j e^{\Lambda(k)t}$. To see why one would expect the powers of ν and t appearing in Proposition 2.2.4, consider the following formal calculation. Recall from the Taylor expansion of $\Lambda(k)$, we have $\bar{w} \approx e^{-\frac{k^4}{\nu^3}t}$.

We are essentially estimating

$$\|\partial_k^j e^{-\nu_T k^2 t} \bar{w}_{low}^{res}\|_{L^2},$$

with the aid of the estimate

$$\|k^d e^{-\nu_T k^2 t}\|_{L^2} \leq C(d)\,(\nu_T t)^{-\frac{d}{2}-\frac{1}{4}}$$

and the Taylor Remainder formula

$$\bar{w}_{low}^{res}(k,t) = \int_0^k \int_0^{k_1} \cdots \int_0^{k_N} \partial_{k_{N+1}}^{N+1} \bar{w}(k_{N+1},t) d_{k_{N+1}} d_{k_N} \ldots d_{k_1}. \tag{2.32}$$

We'll proceed by finding bounds on $\partial_k^J e^{-\frac{k^4}{\nu^3}t}$, and plug into (2.32) with $J = N+1$. We'll make the following changes of variable: we set

$$\begin{aligned} T &= \frac{t}{\nu^3} \\ x &= T^{1/4}k \end{aligned} \tag{2.33}$$

so that

$$\bar{w} = e^{-x^4}$$

and

$$\partial_k^J \bar{w} = T^{J/4} \partial_x^J \bar{w}. \tag{2.34}$$

Let's proceed by computing $x-$derivatives of $\bar{w}$, only taking into account what powers of x appear at each stage. In the following, a prime means ∂_x. We compute

$$\begin{aligned} \bar{w}' &\sim x^3 e^{-x^4} \\ \bar{w}'' &\sim \left(x^2 + x^6\right) e^{-x^4} \\ \bar{w}''' &\sim \left(x + x^5 + x^9\right) e^{-x^4}. \end{aligned}$$

In particular, notice that the powers of x that appear in the $J-$th derivative can be obtained from the powers of x that appear in the $J-1$st derivative by subtracting one from each power appearing (where only nonnegative powers are permitted), and also adding three to each power appearing:

$$\begin{aligned} \bar{w}^{(4)} &\sim \left(x^0 + x^4 + x^8 + x^{12}\right) e^{-x^4} \\ \bar{w}^{(5)} &\sim \left(x^3 + x^7 + x^{11} + x^{15}\right) e^{-x^4} \\ \bar{w}^{(6)} &\sim \left(x^2 + x^6 + x^{10} + x^{14} + x^{18}\right) e^{-x^4}. \end{aligned}$$

In general, we have

$$\partial_x^J \bar{w} \sim \sum_{l=0}^{J-2} x^{R+4l} e^{-x^4}$$

where $R = (-J) \bmod 4$. In the original variables, we have, using (2.34) and (2.33),

$$\partial_k^J \bar{w} \sim \left(\frac{t}{\nu^3}\right)^{J/4} \sum_{l=0}^{J-2} \left(\left(\frac{t}{\nu^3}\right)^{1/4} k\right)^{R+4l} e^{-\frac{k^4}{\nu^3}t},$$

or more precisely,

$$|\partial_k^J \bar{w}| \leq C(J) \left(\frac{t}{\nu^3}\right)^{J/4} \sum_{l=0}^{J-2} \left(\left(\frac{t}{\nu^3}\right)^{1/4} |k|\right)^{R+4l} e^{-\frac{k^4}{\nu^3}t}.$$

Combining with the Taylor Remainder formula and setting $J = N+1$, we have

$$\|\psi e^{-\nu_T k^2 t}\bar{w}^{res}_{low}\|_{L^2} \le C(N)\sum_{l=0}^{N-1}\frac{t^{(1/4)(R+N+1)+l}}{\nu^{(3/4)(R+N+1)+3l}}\|k^{N+1+R+4l}e^{-\nu_T k^2 t}\|_{L^2}.$$

Using the estimate (2.2.5), we get

$$\begin{aligned}
\|\psi e^{-\nu_T k^2 t}\bar{w}^{res}_{low}\|_{L^2} &\le C(N)\sum_{l=0}^{N-1}\frac{t^{-1/4R-1/4(N+1)-l-1/4}}{\nu^{1/4R+1/4(N+1)+l-1/4}} \\
&= C(N)\,(\nu t)^{-1/4R-1/4(N+1)}\,\frac{t^{-1/4}}{\nu^{-1/4}}\sum_{l=0}^{N-1}\left(\nu^{-1}t^{-1}\right)^l \\
&= C(N)\,(\nu t)^{-1/4R-1/4(N+1)}\,\frac{t^{-1/4}}{\nu^{-1/4}}\left(\frac{1-\left(\nu^{-1}t^{-1}\right)^N}{1-\left(\nu^{-1}t^{-1}\right)}\right).
\end{aligned}$$

Therefore if $t > \frac{2}{\nu}$, we have

$$\|\psi e^{-\nu_T k^2 t}\bar{w}^{res}_{low}\|_{L^2} \le 2C(N)\,(\nu t)^{-1/4R-1/4(N+1)}\,\frac{t^{-1/4}}{\nu^{-1/4}},$$

which implies that

$$\|\psi e^{-\nu_T k^2 t}\bar{w}^{res}_{low}\|_{L^2} \le \frac{C(N)}{\nu^{\frac{N}{4}}t^{\frac{N}{4}+\frac{1}{2}}}$$

as reflected in Proposition 2.2.4. This concludes the formal calculation. We proceed with deriving the precise estimate.

Because k is small in $\Omega^<$, powers of k are helpful, so we only need to record the smallest power of k relative to the largest power of t. We obtain additional powers of t when a derivative falls on the exponential (as opposed to any factors in front of it), which creates not only powers of t but powers of $(\Lambda'(k)t)$. When derivatives fall on factors of $\Lambda'(k)$ in front of the exponential, we obtain fewer powers of k but no additional powers of t. Using (2.28), we see that $\Lambda'(k) \sim k^3/\nu^3$, and so $\partial_k^j e^{\Lambda(k)t}$ will lead to terms of the form

$$\left(\frac{k^3 t}{\nu^3}\right)^q\left(\frac{k^2 t}{\nu^3}\right)^{l_1}\left(\frac{kt}{\nu^3}\right)^{l_2}\left(\frac{t}{\nu^3}\right)^{l_3}e^{\Lambda(k)t}, \qquad q+2l_1+3l_2+4l_3 = j.$$

This implies that

$$|\partial_k^j \bar{w}(k,t)| \leq C(\|\hat{w}_0\|_{C^j} + \|\hat{v}_0\|_{C^j}) \left| \left(\frac{k^3 t}{\nu^3}\right)^q \left(\frac{k^2 t}{\nu^3}\right)^{l_1} \left(\frac{kt}{\nu^3}\right)^{l_2} \left(\frac{t}{\nu^3}\right)^{l_3} e^{\Lambda(k)t} \right|$$

for any $q + 2l_1 + 3l_2 + 4l_3 = j$. Using the fact that, on $\Omega^<$, $|e^{\Lambda(k)t}| \leq 1$, as well as (2.31), we find

$$\|\psi e^{-\nu_T k^2 t} \bar{w}_{low}^{res}\|_{L^2} \leq C(\|\hat{w}_0\|_{C^{N+1}} + \|\hat{v}_0\|_{C^{N+1}}) \left\| \psi e^{-\nu_T k^2 t} \left| \left(\frac{t}{\nu^3}\right)^{q+l_1+l_2+l_3} k^{3q+2l_1+l_2+N+1} \right| \right\|_{L^2}.$$

Note the extra $N+1$ powers of k come from the $N+1$ antiderivatives in the Taylor Remainder formula. We need to estimate

$$\left\| \psi e^{-\nu_T k^2 t} \left(\frac{t}{\nu^3}\right)^{q+l_1+l_2+l_3} k^{3q+2l_1+l_2+N+1} \right\|_{L^2},$$

where

$$q + 2l_1 + 3l_2 + 4l_3 = N+1 \qquad \Rightarrow \qquad \frac{N+1}{4} = \frac{q}{4} + \frac{l_1}{2} + \frac{3l_2}{4} + l_3. \tag{2.35}$$

We being by noting that, since $k \in \Omega^<$,

$$\begin{aligned} \left| \left(\frac{t}{\nu^3}\right)^{q+l_1+l_2+l_3} k^{3q+2l_1+l_2+N+1} \right| &= \left| t^{q+l_1+l_2+l_3} \frac{k^{\frac{5}{2}q+3l_1+\frac{7}{2}l_2+4l_3}}{\nu^{\frac{3}{2}q+2l_1+\frac{5}{2}l_2+3l_3}} \left(\frac{k}{\nu}\right)^{\frac{3}{2}q+l_1+\frac{l_2}{2}} \right| \\ &\leq C \left| t^{q+l_1+l_2+l_3} \frac{k^{\frac{5}{2}q+3l_1+\frac{7}{2}l_2+4l_3}}{\nu^{\frac{3}{2}q+2l_1+\frac{5}{2}l_2+3l_3}} \right|, \end{aligned}$$

where C is independent of ν. Therefore, since $\nu_T \sim \nu^{-1}$,

$$\begin{aligned}
\left\| \psi e^{-\nu_T k^2 t} \left(\frac{t}{\nu^3}\right)^{q+l_1+l_2+l_3} k^{3q+2l_1+l_2+N+1} \right\|_{L^2} &\le C \left\| \psi e^{-\nu_T k^2 t} t^{q+l_1+l_2+l_3} \frac{k^{\frac{5}{2}q+3l_1+\frac{7}{2}l_2+4l_3}}{\nu^{\frac{3}{2}q+2l_1+\frac{5}{2}l_2+3l_3}} \right\|_{L^2} \\
&\le C \frac{t^{q+l_1+l_2+l_3}}{\nu^{\frac{3}{2}q+2l_1+\frac{5}{2}l_2+3l_3}} (\nu_T t)^{-\frac{1}{4}-\frac{1}{2}\left(\frac{5}{2}q+3l_1+\frac{7}{2}l_2+4l_3\right)} \\
&\le C \frac{t^{q+l_1+l_2+l_3-\frac{1}{4}-\frac{1}{2}\left(\frac{5}{2}q+3l_1+\frac{7}{2}l_2+4l_3\right)}}{\nu^{\frac{3}{2}q+2l_1+\frac{5}{2}l_2+3l_3-\frac{1}{4}-\frac{1}{2}\left(\frac{5}{2}q+3l_1+\frac{7}{2}l_2+4l_3\right)}} \\
&= C \frac{t^{-\frac{N+1}{4}-\frac{1}{4}}}{\nu^{\frac{N}{4}}},
\end{aligned}$$

where we used (2.35) in the last equality.

Using a similar calculation, we can bound the L^2 norm of each j^{th} derivative of this remainder term. One can show that for each integer triple $l+s+r=j$, we have

$$\|\partial_k^l \psi \partial_k^s e^{-\nu_T k^2 t} \partial_k^r \bar{w}_{low}^{res}\| \le C(\|\hat{w}_0\|_{C^j} + \|\hat{v}_0\|_{C^j}) \frac{t^{-\frac{N}{4}-\frac{1}{2}}}{\nu^{N/4}} \left(\frac{t}{\nu}\right)^{\frac{s+r}{2}}.$$

The proposition follows from the fact that $s+r \le j$. □

Remark 2.2.6. *The key point is that we can analyze the asymptotic behavior of $\hat{w}$ and $\hat{v}$ to any given order of accuracy $\mathcal{O}(t^{-M})$ (when $t > \mathcal{O}(\frac{1}{\nu})$) by choosing N (and hence m) sufficiently large and studying only the behavior of $e^{-\nu_T k^2 t}\bar{w}_{low}^N$ and $e^{-\nu_T k^2 t}\bar{v}_{low}^N$.*

2.3 Decomposition of Solutions and Proof of the Main Result

In this final section, we state and prove our main result.

Theorem 2.3.1. *Given any $M > 0$, let $N \ge 4M$, and let $m > N + 1/2$. If the initial values $\tilde{w}_0, \tilde{v}_0$ of (2.2) lie in the space $L^2(m)$, then there exists a constant $C = C(m, N, \tilde{w}_0, \tilde{v}_0)$ and approximate solutions w_{app}, v_{app}, computable in terms of the $2N+3$ dimensional system of ODEs (2.7), such that*

$$\|w(\xi,\tau) - w_{app}(\xi,\tau)\|_{L^2(m)} + \|v(\xi,\tau) - v_{app}(\xi,\tau)\|_{L^2(m)} \le \frac{C}{\nu^{\frac{N}{4}+\frac{m}{2}}} e^{-M\tau}$$

for all τ sufficiently large. The approximate solutions w_{app} and v_{app} satisfy equations (2.42) and (2.43) respectively. The functions $\phi_j(\xi)$ are the eigenfunctions of the operator $\mathcal{L}_T$ (corresponding to diffusion with constant $\nu_T = \nu + \frac{1}{\nu}$ in scaling variables) *in the space $L^2(m)$. The quantities $\alpha_k(\tau)$ and $\beta_k(\tau)$ solve system (2.7) and have the following asymptotics, obtainable via a reduction to an $N+2$-dimensional center manifold:*

$$|\alpha_k(\tau)| \leq \begin{cases} \frac{C(N,k)e^{-\frac{k}{4}\tau}}{\nu^{k-1}} & : k = 0 \ mod \ 4 \\ \frac{C(N,k)e^{-\frac{k+1}{4}\tau}}{\nu^{k-1}} & : k = 1 \ mod \ 4 \\ \frac{C(N,k)e^{-\frac{k+2}{4}\tau}}{\nu^{k-1}} & : k = 2 \ mod \ 4 \\ \frac{C(N,k)e^{-\frac{k+3}{4}\tau}}{\nu^{k-1}} & : k = 3 \ mod \ 4. \end{cases} \tag{2.36}$$

$$|\beta_k(\tau)| \leq \begin{cases} \frac{C(N,k)e^{-\frac{k}{4}\tau}}{\nu^{k+1}} & : k = 0 \ mod \ 4 \\ \frac{C(N,k)e^{-\frac{k+1}{4}\tau}}{\nu^{k+1}} & : k = 1 \ mod \ 4 \\ \frac{C(N,k)e^{-\frac{k+2}{4}\tau}}{\nu^{k+1}} & : k = 2 \ mod \ 4 \\ \frac{C(N,k)e^{-\frac{k+3}{4}\tau}}{\nu^{k+1}} & : k = 3 \ mod \ 4. \end{cases} \tag{2.37}$$

Remark 2.3.2. *As we will see in the course of the proof of the theorem, $\tau > \mathcal{O}(\log(\frac{|\log \nu|}{\nu}))$ (or equivalently $t > \mathcal{O}(\log \nu^{-1})$) will suffice for these estimates to hold.*

Proof of Theorem 2.3.1: We first concentrate on defining w_{app} and v_{app} and establishing the error estimates in Theorem 2.3.1; this process will mainly use results from section 3. Recall the decomposition of $\hat{w}$ from section 3:

$$\hat{w}(k,t) = \psi e^{-\nu_T k^2 t} \left(\bar{w}^N_{low} + \bar{w}^{res}_{low}\right) + \psi g e^{\lambda_-(k)t} + \hat{w}_{high}. \tag{2.38}$$

The main results of section 3 essentially said $\hat{w} \approx \psi e^{-\nu_T k^2 t}\bar{w}^N_{low}$, with errors (measured in the $|||\cdot|||$ norm introduced in that section) either algebraically or exponentially decaying. More precisely, using Propositions 2.2.3 and 2.2.4, we obtain

$$|||\hat{w} - \psi e^{-\nu_T k^2 t}\bar{w}^N_{low}||| \leq C\left(\frac{1}{\nu^{\frac{N}{4}+\frac{m}{2}} t^{\frac{N}{4}+\frac{1}{2}}} + \frac{1}{\nu^{m+2}} e^{-\frac{\nu}{8}t}\right).$$

where C is independent of ν. For some t sufficiently large, we can "absorb" the exponentially decaying term into the algebraically decaying term; i.e.

$$\frac{1}{\nu^{m+2}} e^{-\frac{\nu}{8}t} < \frac{1}{\nu^{\frac{N}{4}+\frac{m}{2}} t^{\frac{N}{4}+\frac{1}{2}}}.$$

We want to quantify how large t must be for the above inequality to hold. However, there are several other places in this section where terms of the form $\nu^{-p}e^{-\frac{\nu}{A}t}$ appear, which we wish to absorb into algebraically decaying errors. For this reason, we state and prove the following lemma:

Lemma 2.3.3. *Let $A, M, \ell, p > 0$ with $\nu > 0$ as before. Then there exists a constant $C = C(M, A) > 0$ such that for all $t > \frac{C}{\nu}\log(\nu^{\ell-p-M})$, we have the inequality*

$$\frac{1}{\nu^p}e^{-\frac{\nu}{A}t} < \frac{1}{\nu^\ell t^M}.$$

Remark 2.3.4. *Note in particular that since $\tau = \log(1+t)$, the inequality $t > \frac{C}{\nu}\log(\nu^{\ell-p-M})$ essentially translates to $\tau > \mathcal{O}(\log(\frac{|\log\nu|}{\nu}))$.*

Proof of Lemma 2.3.3: We introduce a few new quantities to simplifiy the notation: we let $d = \nu^{p-\ell}$ and we let $a = \nu/A$. Then the target estimate in the lemma reads

$$t^M e^{-at} < d.$$

Now set $f_\lambda(t) = t^M e^{-a\lambda t}$ where $0 < \lambda < 1$ is fixed. Now, the target estimate in the lemma reads

$$f_\lambda(t)e^{-a(1-\lambda)t} < d.$$

Using basic calculus, we find that the maximum value of f_λ lies at $t = \frac{M}{a\lambda}$, and for $t > \frac{M}{a\lambda}$, we have $f_\lambda(t) < \left(\frac{M}{a\lambda e}\right)^M$. Therefore, if

$$\left(\frac{M}{a\lambda e}\right)^M e^{-a(1-\lambda)t} < d,$$

we have the target estimate. The above inequality holds for

$$t > \frac{-1}{a(1-\lambda)}\log\left(d\left(\frac{a\lambda e}{M}\right)^M\right),$$

or, substituting $a = \nu/A$ and $d = \nu^{p-\ell}$, we have

$$t > \left(\frac{A}{1-\lambda}\right)\frac{1}{\nu}\left(\log(\nu^{\ell-p-M}) + M\left(\log(M) + \log(\frac{A}{\lambda e})\right)\right).$$

The time estimate in the lemma is just a less precise version of this inequality. This concludes the proof of lemma 2.3.3. $\square$

Next, we apply the lemma. Using the definition of $\tilde{w}$ and inverting the Fourier Transform, we obtain, for t sufficiently large,

$$|||\tilde{w}(x,t) - \mathcal{F}^{-1}[\psi(k)e^{-\nu_T k^2 t}\bar{w}_{low}^N](x,t)||| \leq \frac{C}{\nu^{\frac{N}{4}+\frac{m}{2}}}(1+t)^{-N/4}. \tag{2.39}$$

Proceeding, notice we can "drop" the cutoff function ψ in the above estimate with only an exponentially decaying penalty: due to the fact that

$$|\psi(k)e^{-\nu_T k^2 t}\bar{w}_{low}^N - e^{-\nu_T k^2 t}\bar{w}_{low}^N| \quad = \quad |(\psi(k)-1)e^{-\nu_T k^2 t}\bar{w}_{low}^N| = 0$$

for $|k| \leq \frac{\sqrt{15}\nu}{8}$, which implies that

$$\|\partial_k^j \left((\psi(k)-1)e^{-\nu_T k^2 t}\bar{w}_{low}^N \right) \|_{L^2} \leq \frac{C}{\nu^{2j}} e^{-\frac{\nu}{8}t}.$$

From here on out, we will sometimes suppress the ν-dependence of the constants for notational convenience. Proceeding, we define our approximate solution in x and t variables:

$$\mathcal{F}^{-1}[e^{-\nu_T k^2 t}\bar{w}_{low}^N](x,t) \equiv \tilde{w}_{app}(x,t),$$

which gives us the estimate

$$|||\tilde{w}(x,t) - \tilde{w}_{app}(x,t)||| \leq C(1+t)^{-N/4}.$$

This is just estimate (2.39) without the cutoff function; it holds for t sufficiently large as in Lemma 2.3.3. Therefore, using scaling variables and defining

$$\tilde{w}_{app}(x,t) \equiv \frac{1}{\sqrt{1+t}} w_{app}(\xi,\tau),$$

we have the estimate, which holds for $\tau > \mathcal{O}(\log(\frac{|\log \nu|}{\nu}))$,

$$\|w(\xi,\tau) - w_{app}(\xi,\tau)\|_{L^2(m)} \leq \frac{C}{\nu^{\frac{N}{4}+\frac{m}{2}}} e^{-\frac{N}{4}\tau}.$$

(This holds since the $|||\cdot|||$ and $||\cdot||_{L^2(m)}$ norms are equivalent in the way made precise at the beginning of section 3.) Using similar calculations, we have functions $\tilde{v}_{app}(x,t)$

and $v_{app}(\xi,\tau)$ satisfying

$$|||\tilde{v}(x,t) - \tilde{v}_{app}(x,t)||| \leq C(1+t)^{-N/4}$$

and

$$\|v(\xi,\tau) - v_{app}(\xi,\tau)\|_{L^2(m)} \leq \frac{C}{\nu^{\frac{N}{4}+\frac{m}{2}}} e^{-\frac{N}{4}\tau}.$$

This establishes the error estimates in 2.3.1; the remainder of the section is devoted to making more explicit the relationship between our approximate solutions w_{app}, v_{app}, and our center manifold calculations in §2.1.

Observe that

$$\begin{aligned}
\tilde{w}_{app}(x,t) &= \sum_{j=0}^{N} \frac{\partial_k^j \bar{w}(0,t)}{j!} \mathcal{F}^{-1}[k^j e^{-\nu_T k^2 t}](x,t) \\
&= \sum_{j=0}^{N} \frac{\partial_k^j \bar{w}(0,t)}{j!(i)^j} \partial_x^j \mathcal{F}^{-1}[e^{-\nu_T k^2 t}](x,t) \\
&= \sum_{j=0}^{N} \frac{\partial_k^j \bar{w}(0,t)}{j!(i)^j} \partial_x^j \left(\frac{1}{\sqrt{4\pi\nu_T t}} e^{-\frac{x^2}{4\nu_T t}} \right).
\end{aligned}$$

Defining new scaling variables

$$\tilde{\xi} := \frac{x}{\sqrt{t}}, \qquad \tilde{\tau} := \log(t),$$

and defining

$$\tilde{w}_{app}(x,t) := \frac{1}{\sqrt{t}} w_{app}(\tilde{\xi},\tilde{\tau}),$$

gives us

$$\begin{aligned}
w_{app}(\tilde{\xi},\tilde{\tau}) &= \sum_{j=0}^{N} \frac{\partial_k^j \bar{w}(0,e^{\tilde{\tau}})}{j!(i)^j} e^{-\frac{j}{2}\tilde{\tau}} \partial_{\tilde{\xi}}^j \left(\phi_0(\tilde{\xi})\right) \\
&= \sum_{j=0}^{N} \frac{\partial_k^j \bar{w}(0,e^{\tilde{\tau}})}{j!(i)^j} e^{-\frac{j}{2}\tilde{\tau}} \phi_j(\tilde{\xi}),
\end{aligned} \tag{2.40}$$

where the $\phi_j(\tilde{\xi})$ above are again the eigenfunctions of the operator $\mathcal{L}_T$ on the space $L^2(m)$.

We now show that the coefficients in (2.40) can be expressed in terms of the functions $\{\alpha_k(\tau), \beta_k(\tau)\}$ from §2.1, demonstrating that the leading order asymptotic behavior of the solution is determined by the center-manifold.

First, recall from §2.2, formulas (2.27) and (2.29) that

$$\bar{w}(k,t) = e^{\nu_T k^2 t}\hat{w}(k,t) + g(k)e^{\lambda_- t}.$$

Differentiating, we have

$$\begin{aligned}
\partial_k^j \bar{w}(k,t) &= \sum_{l=0}^{j} \binom{j}{l} \partial_k^{j-l}(e^{\nu_T k^2 t})\partial_k^l \hat{w}(k,t) + \partial_k^j (g(k)e^{\lambda_- t}) \\
&= \sum_{l=0}^{j} \binom{j}{l} (\nu_T t)^{\frac{j-l}{2}} P^{j-l}(\sqrt{\nu_T t}k) e^{\nu_T k^2 t}\partial_k^l \hat{w}(k,t) + \partial_k^j (g(k)e^{\lambda_- t}),
\end{aligned}$$

where P^{j-l} is a polynomial of degree $j-l$. Setting $k=0$, and substituting $t = e^{\tilde{\tau}}$, we have

$$\partial_k^j \bar{w}(0, e^{\tilde{\tau}}) = \sum_{l=0}^{j} C_{j,l}^{\nu} e^{(\frac{j-l}{2})\tilde{\tau}} \partial_k^l \hat{w}(0, e^{\tilde{\tau}}) + \mathcal{O}(e^{-\frac{\nu}{2}e^{\tilde{\tau}}}), \tag{2.41}$$

where $C_{j,l}^{\nu} = \binom{j}{l}\nu_T^{\frac{j-l}{2}} P^{j-l}(0)$, and $\partial_k^j(g(k)e^{\lambda_- t})|_{k=0}$ is $\mathcal{O}(e^{-\frac{\nu}{2}e^{\tilde{\tau}}})$ since $\lambda_-(0) = -\nu$. We will proceed by computing the derivatives $\partial_k^j \hat{w}(0,t)$ in terms of the α_j from §2.1.

Recall from §2.1, formula (2.6), that we have the decomposition (using the orignal scaling variables ξ and τ)

$$\begin{aligned}
w(\xi,\tau) &= w_c(\xi,\tau) + w_s(\xi,\tau), \qquad w_c(\xi,\tau) = \sum_{j=0}^{N} \alpha_j(\tau)\varphi_j(\xi), \qquad w_s = (w - w_c), \\
v(\xi,\tau) &= v_c(\xi,\tau) + v_s(\xi,\tau), \qquad v_c(\xi,\tau) = \sum_{j=0}^{N} \beta_j(\tau)\varphi_j(\xi), \qquad v_s = (v - v_c),
\end{aligned}$$

and note the following.

Lemma 2.3.5. *$\int \xi^k w_s(\xi,\tau)d\xi = \int \xi^k v_s(\xi,\tau)d\xi = 0$ for all $k \leq N$.*

Proof: We will prove the result for w_s only, as the proof for v_s is analogous. Note that

$$w_s = w - P_n w = w - \sum_{j=0}^{N} \langle H_j, w\rangle \phi_j,$$

and so $\langle H_k, w_s\rangle = 0$ for all $k \leq N$. We'll proceed by induction on k. The $k = 0$ case follows because $\xi^0 = 1 = H_0(\xi)$. Next,

$$0 = \langle (H_{k+1} - H_k), w_s\rangle = c_{k+1}\int \xi^{k+1} w_s(\xi)d\xi + \sum_{j=0}^{k} c_k \int \xi^k w_s(\xi)d\xi = c_{k+1}\int \xi^{k+1} w_s(\xi)d\xi$$

by the inductive assumption. Since $c_{k+1} \neq 0$, the result follows. $\square$

Using this lemma, we can compute

$$\begin{aligned}
\partial_k^l \hat{w}(0,t) &= \left[\partial_k^l \int e^{ikx}\tilde{w}(x,t)dx\right]|_{k=0} \\
&= \left[\partial_k^l \int e^{ik\sqrt{t+1}\xi} w(\xi,\tau)d\xi\right]|_{k=0} \\
&= (i\sqrt{t+1})^l \int \xi^l w(\xi,\tau)d\xi \\
&= (i\sqrt{t+1})^l \int \xi^l w_c(\xi,\tau)d\xi
\end{aligned}$$

for all $l \leq N$, and similarly for $\hat{v}$. As a result, we have a relationship between $\partial_k^l \hat{w}(0,t)$ and the quantities α_r: from §2.1

$$\partial_k^l \hat{w}(0,t) = \frac{1}{\sqrt{1+t}} \sum_{r=0}^{N} \alpha_r(\log(1+t)) \int x^l \phi_r(\frac{x}{\sqrt{1+t}})dx,$$

or equivalently

$$\partial_k^l \hat{w}(0, e^{\tilde{\tau}}) = \sum_{r=0}^{N} \alpha_r(\log(1+e^{\tilde{\tau}}))(1+e^{\tilde{\tau}})^{\frac{l}{2}} \int \xi^l \phi_r(\xi)d\xi.$$

Inserting into (2.41), we obtain

$$\begin{aligned}
\partial_k^j \bar{w}(0, e^{\tilde{\tau}}) &= \sum_{l=0}^{j} C_{j,l}^{\nu} e^{(\frac{j-l}{2})\tilde{\tau}} \sum_{r=0}^{N} \alpha_r(\log(1+e^{\tilde{\tau}}))(1+e^{\tilde{\tau}})^{\frac{l}{2}} \int \xi^l \phi_r(\xi)d\xi + \mathcal{O}(e^{-\frac{\nu}{2}e^{\tilde{\tau}}}) \\
&= e^{\frac{j}{2}\tilde{\tau}} \sum_{r=0}^{N} \alpha_r(\log(1+e^{\tilde{\tau}})) \sum_{l=0}^{N} (1+e^{-\tilde{\tau}})^{\frac{l}{2}} C_{j,l}^{\nu} \int \xi^l \phi_r(\xi)d\xi + \mathcal{O}(e^{-\frac{\nu}{2}e^{\tilde{\tau}}}).
\end{aligned}$$

Therefore we can replace the coefficients in (2.40) and write

$$w_{app}(\tilde{\xi},\tilde{\tau}) = \sum_{j=0}^{N}\left(\frac{1}{j!(i)^j}\sum_{r=0}^{N}\alpha_r(\log(1+e^{\tilde{\tau}}))\sum_{l=0}^{j}(1+e^{-\tilde{\tau}})^{\frac{l}{2}}C^{\nu}_{j,l}\int \xi^l\phi_r(\xi)d\xi\right)\phi_j(\tilde{\xi}) \tag{2.42}$$

where $C^{\nu}_{j,l} \sim \nu^{-j}$ (see the line after (2.41)), and where we also have omitted an error term of $\mathcal{O}(e^{-\frac{\nu}{2}e^{\tilde{\tau}}})$ (This can be absorbed into the estimate in the original definition of w_{app} by applying Lemma 2.3.3 with $\tilde{\tau} = \log(t)$.) Analogous calculations give us a similar result for v:

$$v_{app}(\tilde{\xi},\tilde{\tau}) = \sum_{j=0}^{N}\left(\frac{1}{j!(i)^j}\sum_{r=0}^{N}\beta_r(\log(1+e^{\tilde{\tau}}))\sum_{l=0}^{j}(1+e^{-\tilde{\tau}})^{\frac{l}{2}}D^{\nu}_{j,l}\int \xi^l\phi_r(\xi)d\xi\right)\phi_j(\tilde{\xi}) \tag{2.43}$$

This completes the proof of Theorem 2.3.1.

Remark 2.3.6. *Note that Theorem 2.3.1 is stated in terms of the original scaling variables ξ and τ. Since $\tau = \log(1+e^{\tilde{\tau}})$, errors in $\tilde{\tau}$, for large $\tilde{\tau}$, are equivalent to errors in τ, for large τ.*

Chapter 3

Construction of the Center Manifold and long-time asymptotics of solutions

We now return to the full equations for the Taylor dispersion problem, written in the form of the infinite systems of PDEs (1.16). We will analyze this system using methods similar to those applied to the model problem in section 2.1. We begin by rewriting (1.16) in a way which will simplify the computation of the center manifold.

We set

$$\begin{aligned} w_0(\xi,\tau) &= \sum_{k=0}^{N} \alpha_k(\tau)\phi_k(\xi) + w_0^s(\tau,\xi) \\ u_n(\xi,\tau) &= \sum_{k=0}^{N} \beta_k^n(\tau)\phi_k(\xi) + u_n^s(\tau,\xi) \end{aligned} \tag{3.1}$$

The quantities w_0^s and u_n^s denote the projections orthogonal to the first $N+1$ eigenfunctions of $\mathcal{L}_T$. Plugging into system (1.16) and using the spectral properties mentioned above (specifically the relations $\phi_{k+1} = \partial_\xi \phi_k$ and $\mathcal{L}_T \phi_k = -\frac{k}{2}\phi_k$), we find that the coefficients α_k and β_k^n actually *decouple* from w_0^s and u_n^s, giving an infinite system of ODES for their evolution:

$$
\begin{aligned}
\dot{\alpha_0} &= 0 \\
\dot{\alpha_1} &= -\frac{1}{2}\alpha_1 \\
\dot{\alpha_2} &= -\alpha_2 - \frac{D_T}{\nu}\alpha_0 - A\sum_{m=1}^{\infty}\chi_m\beta_0^m \\
\dot{\alpha_k} &= -\frac{k}{2}\alpha_k - \frac{D_T}{\nu}\alpha_{k-2} - A\sum_{m=1}^{\infty}\chi_m\beta_{k-2}^m \text{ for } 3 \le k \le N \\
\dot{\beta_0^n} &= -e^{\tau}\left(\nu\mu_n\beta_0^n + A\chi_n\alpha_0\right) \\
\dot{\beta_1^n} &= -\frac{1}{2}\beta_1^n - e^{\tau}\left(\nu\mu_n\beta_1^n + A\chi_n\alpha_1\right) - Ae^{\tau/2}\sum_{m=1}^{\infty}\chi_{n,m}\beta_0^m \\
\dot{\beta_k^n} &= -\frac{k}{2}\beta_k^n - \nu_T\beta_{k-2}^n - e^{\tau}\left(\nu\mu_n\beta_k^n + A\chi_n\alpha_k\right) \\
&\quad - Ae^{\tau/2}\sum_{m=1}^{\infty}\chi_{n,m}\beta_{k-1}^m \text{ for } 2 \le k \le N
\end{aligned}
\tag{3.2}
$$

where $n \ge 1$. Notice that both quantities α_0 and β_0^n are $\mathcal{O}(1)$ in τ. If we can show that the coefficients α_k and β_k^n decay exponentially in τ , then the only nondecaying term for the solution $w_0(\xi,\tau)$ is $\alpha_0\phi_0(\xi)$. It is not hard to show that $\Phi(\xi,\tau) = \alpha_0\phi_0(\xi)$ solves $\partial_\tau\Phi = \mathcal{L}_T\Phi$; therefore w_0 will be approximated, up to exponentially in τ decaying errors, by a function that solves the observationally "correct" diffusion equation. (We will have to treat the orthogonal projections w_0^s and u_n^s with a separate analysis). We will actually show that the coefficients α_k and β_k^n decay exponentially in τ with rates that increase with k using a center manifold reduction; hence, we will show that Taylor dispersion can be viewed as the leading order term in a systematic approximation scheme, and one can use the corresponding center manifold to obtain correction terms to the behavior to any order in τ. Hence, the main goal of this chapter is to prove the following Proposition:

Proposition 3.0.1. *Let α_k and β_k^n be the solutions to system* (3.2)*. Then α_k and β_k^n have the following decay rates:*

$$\alpha_k, \|\{\beta_k^n\}\|_{\ell^2} = \begin{cases} \mathcal{O}(e^{-\frac{k}{6}\tau}) & \textit{if } k = 0 \textit{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \textit{if } k = 1 \textit{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \textit{if } k = 2 \textit{ mod } 6 \\ \mathcal{O}(e^{-\frac{k}{6}\tau}) & \textit{if } k = 3 \textit{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \textit{if } k = 4 \textit{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \textit{if } k = 5 \textit{ mod } 6. \end{cases} \tag{3.3}$$

3.1 Transformation of the subsystem

We start by performing some additional setup relevant to the upcoming center manifold calculation. Recall in the formal analysis that in long time limit, $\nu\mu_n w_n + A\chi_n\partial_\xi w_0 = 0$. In system (3.2), the corresponding quantities are $\nu\mu_n\beta_k^n + A\chi_n\alpha_k$; notice they appear with prefactor e^τ. Therefore if we:

- rescale time using $\tau = \log(1+t)$,
- "diagonalize",
- and "autonomize" appropriately,

the long time behavior of (3.2) should appear as the restriction of the transformed system to its center manifold. Indeed, if we introduce new variables

$$\begin{aligned} a_k &= \alpha_k \\ b_k^n &= \beta_k^n + \frac{A\chi_n}{\nu\mu_n}\alpha_k \end{aligned} \tag{3.4}$$

we obtain, for $k \geq 3$,

$$\begin{aligned} \dot{a_k} &= -\frac{k}{2}a_k - A\sum_{m=1}^{\infty}\chi_m b_{k-2}^m \\ \dot{b_k^n} &= -e^\tau\nu\mu_n b_k^n - e^{\tau/2}\left(-A\sum_{m=1}^{\infty}\chi_{n,m}\left(-\frac{A\chi_m}{\mu_m}a_{k-1} + b_{k-1}^m\right)\right) \\ &- \frac{k}{2}b_k^n - \nu_T\left(-\frac{A\chi_n}{\nu\mu_n}a_{k-2} + b_{k-2}^n\right) - \frac{A^2\chi_n}{\nu\mu_n}\sum_{m=1}^{\infty}\chi_m b_{k-2}^m. \end{aligned}$$

Next, we rescale time using $\tau = \log(1+t)$, which divides the right-hand side by e^{τ} (or $1+t$) and replaces $d/d\tau$ with d/dt (which we denote by a prime $'$). Further, if we autonomize by setting $\sigma = (1+t)^{-1/2}$, we obtain

$$
\begin{aligned}
a_0' &= 0 \\
a_1' &= \sigma^2 \left(-\frac{1}{2} a_1\right) \\
a_2' &= \sigma^2 \left(-a_2 - A \sum_{m=1}^{\infty} \chi_m b_0^m\right) \\
a_k' &= \sigma^2 \left(-\frac{k}{2} a_k - A \sum_{m=1}^{\infty} \chi_m b_{k-2}^m\right) \\
b_0^{n\prime} &= -\nu \mu_n b_0^n \\
b_1^{n\prime} &= -\nu \mu_n b_1^n + \sigma \left(-A \sum_{m=1}^{\infty} \chi_{n,m} \left(-\frac{A\chi_m}{\mu_m} a_0 + b_0^m\right) + \right) + \sigma^2 \left(-\frac{1}{2} b_1^n\right) \\
b_2^{n\prime} &= -\nu \mu_n b_2^n + \sigma \left(-A \sum_{m=1}^{\infty} \chi_{n,m} \left(-\frac{A\chi_m}{\mu_m} a_1 + b_1^m\right)\right) \\
&+ \sigma^2 \left(-b_2^n - \nu_T \left(-\frac{A\chi_n}{\nu\mu_n} a_0 + b_0^n\right) - \frac{A^2\chi_n}{\nu\mu_n} \sum_{m=1}^{\infty} \chi_m b_0^m\right) \\
b_k^{n\prime} &= -\nu \mu_n b_k^n + \sigma \left(-A \sum_{m=1}^{\infty} \chi_{n,m} \left(-\frac{A\chi_m}{\nu\mu_m} a_{k-1} + b_{k-1}^m\right)\right) \\
&+ \sigma^2 \left(-\frac{k}{2} b_k^n - \nu_T \left(-\frac{A\chi_n}{\mu_n} a_{k-2} + b_{k-2}^n\right) - \frac{A^2\chi_n}{\nu\mu_n} \sum_{m=1}^{\infty} \chi_m b_{k-2}^m\right) \\
\sigma' &= -\frac{1}{2} \sigma^3
\end{aligned}
\tag{3.5}
$$

where $3 \le k \le N$ and $n \ge 1$.

3.2 Construction of the exact Center Manifolds

Recall system (3.5), which we rewrite here as

$$
\begin{aligned}
a_k' &= \sigma^2 \left(-\frac{k}{2} a_k - A \sum_{m=1}^{\infty} \chi_m b_{k-2}^m \right) \\
b_k^{n\prime} &= -\nu \mu_n b_k^n + \sigma \left(-A \sum_{m=1}^{\infty} \chi_{n,m} \left(-\frac{A\chi_m}{\nu\mu_m} a_{k-1} + b_{k-1}^m \right) \right) \\
&+ \sigma^2 \left(-\frac{k}{2} b_k^n - \nu_T \left(-\frac{A\chi_n}{\nu\mu_n} a_{k-2} + b_{k-2}^n \right) - \frac{A^2\chi_n}{\nu\mu_n} \sum_{m=1}^{\infty} \chi_m b_{k-2}^m \right) \\
\sigma' &= -\frac{1}{2}\sigma^3
\end{aligned}
$$

The goal of this section is to compute exact center manifolds for this system, and then prove that these center manifolds attract all solutions. First, we'll rewrite the system as

$$
\begin{aligned}
a_k' &= \sigma^2 \left(-\frac{k}{2} - A\langle \chi, b_{k-2} \rangle \right) \\
b_k' &= -\nu M b_k + \sigma \left(\left(A^2 \tilde{\chi} * \frac{\chi}{\mu} \right) a_{k-1} - A\tilde{\chi} * b_{k-1} \right) \\
&+ \sigma^2 \left(-\frac{k}{2} b_k + A \left(\nu_T \frac{\chi}{\mu} \right) a_{k-2} + (-\nu_T)\, b_{k-2} \right)
\end{aligned}
\tag{3.6}
$$

Here $b_k = (b_k^0, b_k^1, b_k^2, \ldots)$, $\chi = (\chi_0, \chi_1, \chi_2, \ldots)$, and $\mu = (\mu_0, \mu_1, \mu_2, \ldots)$ are in ℓ^2, and $\tilde{\chi} * \cdot$ is a bounded linear operator on ℓ^2 to itself defined by $(\tilde{\chi} * Y)_n = \sum_m \chi_{n,m} Y_m$, ν_T is a scalar (as defined in the previous section), and $(Mx)_n = \mu_n x_n$ for $x \in \ell^2$. We'll further rewrite the system as:

$$
\begin{aligned}
a_k' &= \sigma^2 \left(-\frac{k}{2} a_k + L_{b,2}^a b_{k-2} \right) \\
b_k' &= -\nu M b_k + \sigma \left(L_{a,1}^b a_{k-1} + L_{b,1}^b b_{k-1} \right) \\
&+ \sigma^2 \left(-\frac{k}{2} b_k + L_{a,2}^b a_{k-2} + L_{b,2}^b b_{k-2} \right) \\
\sigma' &= -\frac{1}{2}\sigma^3.
\end{aligned}
\tag{3.7}
$$

Here $L_{b,j}^a : \ell^2 \to \mathbb{R}$, $L_{a,j}^b : \mathbb{R} \to \ell^2$, $L_{b,j}^b : \ell^2 \to \ell^2$ are bounded linear operators defined by comparing (3.6) and (3.7). Additionally, since $\mu_n \sim n$ for n large (by Weyl's law in dimension 2), we have that $L^b_{a,1}, L^b_{a,2}, L^b_{a,3}$ are "smoothing" in the sense that they map into $h^1 = \{x \in \ell^2 | \, \{nx_n\}_{n=1}^\infty\}$. Let's proceed with the invariant manifold calculation; owing to our intuition from the model problem, we may expect exact center manifolds of the form

$$b_k = \tilde{b}_k(a_0, a_1, \ldots, a_k - 1, \sigma) = \sum_{\ell=1}^{k} C^k_{k-\ell} a_{k-\ell} \sigma^\ell \tag{3.8}$$

where $C^k_{k-\ell}$ are ℓ^2 sequences to be determined. In fact, we can take $\tilde{b}_0 \equiv 0$, since in (3.5), the equation for b_0 decouples from the rest of the system. Proceeding, we find that

$$C^1_0 = (\nu M)^{-1} L^b_{a,1}$$

so that $\tilde{b}_1 = (\nu M)^{-1} L^b_{a,1} a_0 \sigma$. Here, in a slight abuse of notation, we are treating the $L^b_{a,j}$ as either linear operators from $\mathbb{R}$ to ℓ^2, or an element of ℓ^2, depending on context. In this case we consider $L^b_{a,1} \in \ell^2$. Continuing, notice that since $L^b_{a,1}$ is "smoothing" (it is actually an h^1 sequence) and the operator $(\nu M)^{-1}$ is also smoothing, we have that $C^1_0 \in h^2$. Next, we find

$$C^2_1 = (\nu M)^{-1} L^b_{a,1}$$

(which is again in h^2) and

$$C^2_0 = (\nu M)^{-1} \left(L^b_{b,1} C^1_0 + L^b_{a,2} \right)$$

which is again in h^2, since $L^b_{a,2} \in h^1$, $C^1_0 \in h^2$, and $(\nu M)^{-1}$ is smoothing. Continuing, we find

$$\begin{aligned}
C^3_2 &= (\nu M)^{-1} L^b_{a,1} \\
C^3_1 &= (\nu M)^{-1} \left(L^b_{b,1} C^2_1 + L^b_{a,2} \right) \\
C^3_0 &= (\nu M)^{-1} \left(L^b_{b,1} C^2_0 + L^b_{b,2} C^1_0 \right)
\end{aligned}$$

One can check that each $C_j^3 \in h^2$. Proceeding for $k = 4$, we have

$$
\begin{aligned}
C_3^4 &= (\nu M)^{-1} L_{a,1}^b \\
C_2^4 &= (\nu M)^{-1} \left(L_{a,2}^b + L_{b,1}^b C_2^3 \right) \\
C_1^4 &= (\nu M)^{-1} \left(L_{b,1}^b C_1^3 + L_{b,2}^b C_1^2 \right) \\
C_0^4 &= (\nu M)^{-1} \left(L_{b,1}^b C_0^3 + L_{b,2}^b C_0^2 - C_3^4 L_{b,2}^a C_0^1 \right)
\end{aligned}
$$

Again, one can check that each $C_j^4 \in h^2$. Now we proceed by finding formulas for the general coefficents $C_{k-\ell}^k$. Hence, we prove the following

Proposition 3.2.1. *For any $k = 1, 2, 3, \dots$, there exist coefficients $\{C_{k-\ell}^k)\}$ such that the graph of the function*

$$b_k = \tilde{b}_k(a_0, a_1, \dots, a_{k-1}, \sigma) = \sum_{\ell=1}^{k} C_{k-\ell}^k a_{k-\ell} \sigma^\ell$$

gives the invariant manifold for b_k. Furthermore, for any fixed k, the coefficients $\{C_{k-\ell}^k)\}$ can be explicitly determined.

Proof:

The proof is by induction; we have already verified the base cases. Let $k > 0$ and assume for all $j \leq k$ that the graph of the function

$$b_j = \tilde{b}_j(a_0, a_1, \dots, a_{j-1}, \sigma) = \sum_{\ell=1}^{j} C_{j-\ell}^j a_{j-\ell} \sigma^\ell$$

gives the invariant manifold for b_j, and that the coefficients $C_{j-\ell}^j$ have been determined. Proceeding with the induction step, we set

$$\tilde{b}_k = \sum_{\ell=1}^{k} C_{k-\ell}^k a_{k-\ell} \sigma^\ell.$$

Our goal is to show that the coefficients $C_{k-\ell}^k$ can be computed in terms of already determined coefficients, from the induction hypothesis. Proceeding, we plug (3.8) into (3.5), so that

$$\begin{aligned}
\tilde{b}'_k &= \sum_{\ell=1}^{k} C^k_{k-\ell}\left(a'_{k-\ell}\sigma^\ell + \ell a_{k-\ell}\sigma^{\ell-1}(-\frac{1}{2}\sigma^3)\right)\\
&= \sum_{\ell=1}^{k} C^k_{k-\ell}\left(\left(\sigma^{\ell+2}\left(-\frac{k-\ell}{2}a_{k-\ell} + L^a_{b,2}b_{k-\ell-2}\right) - \frac{\ell}{2}a_{k-\ell}\sigma^{\ell+2}\right)\right)\\
&= -\frac{k}{2}\sigma^2\tilde{b}_k + \sum_{\ell=1}^{k}\left(C^k_{k-\ell}\left(\sigma^{\ell+2}L^a_{b,2}\sum_{j=1}^{k-\ell-2} C^{k-\ell-2}_{k-\ell-j-2}a_{k-\ell-j-2}\sigma^j\right)\right).
\end{aligned}$$

Note that for an integer $p \geq 1$,

$$\sigma^{\ell+p}\sum_{j=1}^{k-\ell-p} C^{k-\ell-p}_{k-\ell-j-p}a_{k-\ell-j-p}\sigma^j = \sum_{j=\ell+p}^{k} C^{k-\ell-p}_{k-j}a_{k-j}\sigma^j.$$

Therefore, after applying this identity with $p = 2$, we have

$$\begin{aligned}
\tilde{b}'_k &= -\frac{k}{2}\sigma^2\tilde{b}_k + \sum_{\ell=1}^{k}\left(C^k_{k-\ell}\left(L^a_{b,2}\sum_{j=\ell+3}^{k} C^{k-\ell-2}_{k-j}a_{k-j}\sigma^j\right)\right)\\
&= -\frac{k}{2}\sigma^2\tilde{b}_k + \sum_{j=4}^{k}\left(\sum_{\ell=1}^{j-3} C^k_{k-\ell}L^a_{b,2}C^{k-\ell-2}_{k-j}\right)a_{k-j}\sigma^j,
\end{aligned}$$

where we have exchanged the order of summation. It will also be convenient to exchange the names of the indices of summation:

$$\tilde{b}'_k = -\frac{k}{2}\sigma^2\tilde{b}_k \quad + \textstyle\sum_{\ell=4}^{k}\left(\sum_{j=1}^{\ell-3} C^k_{k-j}L^a_{b,2}C^{k-j-2}_{k-\ell}\right)a_{k-\ell}\sigma^\ell$$

which is also equal to, by the invariance condition,

$$\begin{aligned}
-\nu M\tilde{b}_k + \sigma&\left(L^b_{a,1}a_{k-1} + L^b_{b,1}\sum_{\ell=1}^{k-1} C^{k-1}_{k-\ell-1}a_{k-\ell-1}\sigma^\ell\right)\\
&+ \sigma^2\left(-\frac{k}{2}\tilde{b}_k + L^b_{a,2}a_{k-2} + L^b_{b,2}\sum_{\ell=1}^{k-2} C^{k-2}_{k-\ell-2}a_{k-\ell-2}\sigma^\ell\right).
\end{aligned}$$

Note that, for an integer $p \geq 1$, we have

$$\sigma^p \sum_{\ell=1}^{k-p} C^{k-p}_{k-\ell-p} a_{k-\ell-p} \sigma^\ell = \sum_{\ell=p+1}^{k} C^{k-p}_{k-\ell} a_{k-\ell} \sigma^\ell.$$

Applying this identity and combining the two expressions for $\tilde{b}'_k$, we have

$$\begin{aligned}
& C^k_{k-1} L^a_{b,2} C^{k-3}_{k-4} a_{k-4} \sigma^4 + \sum_{\ell=5}^{k} \left(\sum_{j=1}^{\ell-3} C^k_{k-j} L^a_{b,2} C^{k-j-2}_{k-\ell} \right) a_{k-\ell} \sigma^\ell \\
&= -\nu M \sum_{\ell=1}^{k} C^k_{k-\ell} a_{k-\ell} \sigma^\ell + \sigma L^b_{a,1} a_{k-1} + \sigma^2 L^b_{a,2} a_{k-2} + L^b_{b,1} \sum_{\ell=2}^{k} C^{k-1}_{k-\ell} a_{k-\ell} \sigma^\ell \\
&+ L^b_{b,2} \sum_{\ell=3}^{k} C^{k-2}_{k-\ell} a_{k-\ell} \sigma^\ell.
\end{aligned}$$

Note we arranged / separated out some of the terms to make it easier to match coefficients. For $\ell = 1$, we find

$$C^k_{k-1} = (\nu M)^{-1} L^b_{a,1}$$

which is in h^2,

$$C^k_{k-2} = (\nu M)^{-1} \left(L^b_{a,2} + L^b_{b,1} C^{k-1}_{k-2} \right)$$

which is also in h^2 since $(\nu M)^{-1}$ is smoothing, and the sequences it acts on are in h^1 (notice C^{k-1}_{k-2} is already determined since the superscript is smaller than k, and the difference between superscript and subscript is 1, which corresponds to $\ell = 1$.). Proceeding, we have

$$C^k_{k-3} = (\nu M)^{-1} \left(L^b_{a,2} + L^b_{b,1} C^{k-1}_{k-3} + L^b_{b,2} C^{k-2}_{k-3} \right)$$

(which again is in h^2 as one can check),

$$C^k_{k-4} = (\nu M)^{-1} \left(-(L^a_{b,2} C^{k-3}_{k-4}) C^k_{k-1} + L^b_{b,1} C^{k-1}_{k-4} + L^b_{b,2} C^{k-2}_{k-4} \right)$$

and for general $\ell \geq 5$,

$$C^k_{k-\ell} = (\nu M)^{-1} \left(-\sum_{j=1}^{\ell-3} C^k_{k-j} \left(L^a_{b,2} C^{k-j-2}_{k-\ell} \right) + \sum_{j=1}^{\ell-4} L^b_{b,1} C^{k-1}_{k-\ell} + L^b_{b,2} C^{k-2}_{k-\ell} \right).$$

Notice that all terms on the right hand side are of the form C^m_s where either $m < k$, or $m - s < \ell$, and in both cases, these terms have already been determined, and are in h^2; therefore we can solve for $C^k_{k-\ell}$ and the result is in h^2. This concludes the proof of Proposition 3.2.1.

3.2.1 Showing the Center Manifolds are globally attracting

In this section, we'll show that the exact invariant manifolds previously constructed are globally attracting. Specifically, we set

$$B_k = b_k - \tilde{b}_k, \tag{3.9}$$

and we prove the following Lemma.

Lemma 3.2.2. *For all $t > 0$,*

$$\|B_0(t)\|_{\ell^2} \leq Ce^{-\nu\mu_1 t} \text{ and}$$
$$\|B_k(t)\|_{\ell^2} \leq C(1+t)^{\frac{1}{4}+\frac{k-2}{2}} e^{-\nu\mu_1 t} \text{ for } k \geq 1.$$

The proof is by induction. We first get an equation for the B_k. Since the equations governing a_k and b_k are linear in a_k and b_k, one obtains an equation for B_k independent of the a_k and b_k:

$$B'_k = b'_k - \tilde{b}'_k = -M(B_k + \tilde{b}_k) + \sigma \left(L^b_{a,1} a_{k-1} + L^b_{b,1}(B_{k-1} + \tilde{b}_{k-1}) \right)$$
$$\sigma^2 \left(-\frac{k}{2}(B_k + \tilde{b}_k) + L^b_{a,2} a_{k-2} + L^b_{b,2}(B_{k-2} + \tilde{b}_{k-2}) \right)$$

Due to the invariance condition on the $\tilde{b}_k$, all terms involving the $\tilde{b}_k$ and a_k cancel, leaving

$$B'_k = -\nu M B_k + \sigma L^b_{b,1} B_{k-1} + \sigma^2 (-\frac{k}{2} B_k + L^b_{b,2} B_{k-2})$$

which we rewrite as

$$B_k' = -\nu M B_k - \frac{k}{2}\frac{1}{1+t}B_k + \frac{1}{\sqrt{1+t}}L^b_{b-1}B_{k-1} + \frac{1}{1+t}L^b_{b,2}B_{k-2}.$$

Notice for $k = 0$, we have $\|B_0\| \leq Ce^{-\nu\mu_1 t}$. For $k = 1$, we have

$$B_1' = -\nu M B_1 - \frac{1}{2}\frac{1}{1+t}B_1 + \frac{1}{\sqrt{1+t}}L^b_{b-1}B_0.$$

Solving using Duhamel's formula, we get

$$B_1(t) = e^{-\nu M t}(1+t)^{-\frac{1}{2}}B_1(0) + \int_0^t e^{-\nu M(t-s)}(1+t)^{-\frac{1}{2}}(1+s)^{\frac{1}{2}}L^b_{b-1}B_0(s)ds.$$

Next, we take the $\|\cdot\|_{\ell^2}$ norm and use the fact that $\|B_0\| \leq Ce^{-\nu\mu_1 t}$ to obtain the bound

$$\begin{aligned}\|B_1(t)\|_{\ell^2} &\leq Ce^{-\nu\mu_1 t}(1+t)^{-\frac{1}{2}} + C\int_0^t (1+t)^{-\frac{1}{2}}(1+s)^{-1}e^{-\nu\mu_1(t-s)}e^{-\nu\mu_1 s}ds \\ &\leq Ce^{-\nu\mu_1 t}(1+t)^{-\frac{1}{2}} + C\log(1+t)(1+t)^{-\frac{1}{2}}e^{-\nu\mu_1 t} \\ &\leq C(1+t)^{-\frac{1}{4}}e^{-\nu\mu_1 t}.\end{aligned}$$

The last line in the above string of inequalities establishes the base case for our induction argument. Next is the induction hypothesis. Assume for some $k > 0$ that

$$\|B_j(t)\|_{\ell^2} \leq C(1+t)^{\frac{1}{4}+\frac{j-2}{2}}e^{-\nu\mu_1 t} \tag{3.10}$$

for all $t > 0$ and $j \leq k$. We want to show that $\|B_{k+1}\|_{\ell^2} \leq C(1+t)^{\frac{1}{4}+\frac{(k+1)-2}{2}}e^{-\nu\mu_1 t}$. We solve for $B_{k+1}(t)$ using Duhamel's formula, and obtain the following upper bound after taking the ℓ^2 norm:

$$\begin{aligned}\|B_{k+1}(t)\|_{\ell^2} \leq & \; Ce^{-\nu\mu_1 t}(1+t)^{-\frac{(k+1)}{2}} + C\int_0^t e^{-\nu\mu_1(t-s)}(1+t)^{-\frac{(k+1)}{2}} \times \\ & (1+s)^{\frac{(k+1)}{2}}((1+s)^{-\frac{1}{2}}(1+s)^{\frac{1}{4}+\frac{k-2}{2}}e^{-\nu\mu_1 s} \\ & + (1+s)^{-1}(1+s)^{\frac{1}{4}+\frac{k-1-2}{2}}e^{-\nu\mu_1 s})ds \\ & \leq Ce^{-\nu\mu_1 t}(1+t)^{-\frac{(k+1)}{2}} + Ce^{-\nu\mu_1 t}\int_0^t (1+t)^{-\frac{(k+1)}{2}} \times \\ & (1+s)^{\frac{(k+1)}{2}}\left((1+s)^{-\frac{1}{2}+\frac{1}{4}+\frac{k-2}{2}}\right)ds \\ & \leq C(1+t)^{\frac{k}{2}-\frac{1}{4}}e^{-\nu\mu_1 t}.\end{aligned}$$

Since the exponent $\frac{k}{2}-\frac{1}{4}$ in the last line is equal to $\frac{1}{4}+\frac{(k+1)-2}{2}$, we have proven that

$$\|B_{k+1}\|_{\ell^2} \leq C(1+t)^{\frac{1}{4}+\frac{(k+1)-2}{2}}e^{-\nu\mu_1 t}.$$

Therefore Lemma 3.2.2 is proven. □

3.3 Analysis of the reduced system

The goal of this section is to compute the decay rates of the a_k and b_k, by considering the system (3.5) reduced to its Center Manifold. We will prove the following via induction:

Proposition 3.3.1. *Let a_k be the solutions to system* (3.5) *on its center manifold. Then a_k have the following decay rates:*

$$a_k = \begin{cases} \mathcal{O}(e^{-\frac{k}{6}\tau}) & \text{if } k = 0 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \text{if } k = 1 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \text{if } k = 2 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k}{6}\tau}) & \text{if } k = 3 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \text{if } k = 4 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \text{if } k = 5 \text{ mod } 6. \end{cases} \tag{3.11}$$

By undoing the change of variables (3.4), we find the same decay rates for the α_k and β_k^n in (3.1):

$$\alpha_k, \beta_k^n = \begin{cases} \mathcal{O}(e^{-\frac{k}{6}\tau}) & \text{if } k = 0 \bmod 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \text{if } k = 1 \bmod 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \text{if } k = 2 \bmod 6 \\ \mathcal{O}(e^{-\frac{k}{6}\tau}) & \text{if } k = 3 \bmod 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \text{if } k = 4 \bmod 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \text{if } k = 5 \bmod 6. \end{cases} \tag{3.12}$$

Rescaling time in (3.5) and substituting from the Center Manifolds, we find that the long-τ asymptotics of the a_k will be determined by system

$$\begin{aligned} \dot{a_0} &= 0 \\ \dot{a_1} &= -\frac{1}{2}a_1 \\ \dot{a_2} &= -a_2 \\ \dot{a_3} &= -\frac{3}{2}a_3 + a_0 e^{-\tau/2} \\ \dot{a_4} &= -2a_4 + a_1 e^{-\tau/2} + a_0 e^{-\tau} \\ \dot{a_k} &= -\frac{k}{2}a_k + a_{k-3}e^{-\tau/2} + a_{k-4}e^{-\tau} + a_{k-5}e^{-3\tau/2} \text{ for } 5 \le k \le N \end{aligned} \tag{3.13}$$

Above, the dot denotes $d/d\tau$. We determine the decay rates iteratively, first noting that $a_0 = \mathcal{O}(1)$, $a_1 = \mathcal{O}(e^{-\tau/2})$, and $a_2 = \mathcal{O}(e^{-\tau})$. To determine the decay rate of a_3, we solve via $a_k(\tau)$'s formula, and since a_0 is constant, we find that $a_3 = \mathcal{O}(e^{-\tau/2})$. Similarly, we solve the a_4 equation using Duhamel's formula, using the fact that $a_1 = \mathcal{O}(e^{-\tau/2})$, and we find that both forcing terms (and hence the slowest decaying forcing term) decays like $e^{-\tau}$, so that $a_4 = \mathcal{O}(e^{-\tau})$. We'll proceed as follows; define the decay rates r_k by the relation

$$a_k = \mathcal{O}(e^{-r_k\tau}).$$

By the previous calculations, have already determined that

$$\begin{aligned} r_0 &= 0 \\ r_1 &= \frac{1}{2} \\ r_2 &= 1 \\ r_3 &= \frac{1}{2} \\ r_4 &= 1. \end{aligned}$$

If we solve the a_k equation in (3.13) using Duhamel's formula, we find that

$$r_k = \min \left\{ r_{k-3} + \frac{1}{2}, r_{k-4} + 1, r_{k-5} + \frac{3}{2} \right\}. \tag{3.14}$$

We'll use this inductive formula to compute a number of these decay rates, and then find the pattern. We compute:

$$\begin{aligned} r_5 &= \min \{3/2, 3/2, 3/2\} = 3/2 \\ r_6 &= \min \{1, 2, 2\} = 1 \\ r_7 &= \min \{3/2, 3/2, 5/2\} = 3/2 \\ r_8 &= \min \{2, 2, 2\} = 2 \\ r_9 &= \min \{3/2, 5/2, 5/2\} = 3/2 \\ r_{10} &= \min \{2, 2, 3\} = 2 \\ r_{11} &= \min \{5/2, 5/2, 5/2\} = 5/2 \\ r_{12} &= \min \{2, 3, 3\} = 2 \\ r_{13} &= \min \{5/2, 5/2, 7/2\} = 5/2 \\ r_{14} &= \min \{3, 3, 3\} = 3 \\ r_{15} &= \min \{5/2, 7/2, 7/2\} = 5/2 \\ r_{16} &= \min \{3, 3, 4\} = 3 \\ r_{17} &= \min \{7/2, 7/2, 7/2\} = 7/2 \\ r_{18} &= \min \{3, 4, 4\} = 3 \\ &\vdots \end{aligned} \tag{3.15}$$

To see the pattern, let's look at the even k first. We have

$$\begin{aligned}
r_0 &= 0\\
r_2 &= 1\\
r_4 &= 1\\
r_6 &= 1\\
r_8 &= 2\\
r_{10} &= 2\\
r_{12} &= 2\\
r_{14} &= 3\\
r_{16} &= 3\\
r_{18} &= 3.
\end{aligned} \tag{3.16}$$

Within the even k, the multiples of 6 satisfy $r_k = k/6$. The other r_k can be found by going to the "next" multiple of 6, then dividing by 6; i.e., the pattern is

$$\begin{aligned}
r_k &= \frac{k}{6} \text{ if } k = 0 \bmod 6\\
r_k &= \frac{k+4}{6} \text{ if } k = 2 \bmod 6\\
r_k &= \frac{k+2}{6} \text{ if } k = 4 \bmod 6.
\end{aligned} \tag{3.17}$$

Similarly, a pattern can be found for the odd k:

$$\begin{aligned}
r_k &= \frac{k+2}{6} \text{ if } k = 1 \bmod 6\\
r_k &= \frac{k}{6} \text{ if } k = 3 \bmod 6\\
r_k &= \frac{k+4}{6} \text{ if } k = 5 \bmod 6.
\end{aligned} \tag{3.18}$$

These decay rates can be proved by induction.

Proof of Proposition 3.0.1 We proceed with the induction step. Assume, given $k \geq 0$, that the proposition holds for all $j \leq k$. We begin with the equation

$$\dot{a_k} = -\frac{k}{2}a_k + L^a_{b,2}b_{k-2}.$$

Plugging in from the Center Manifold, we have

$$b_k = \sum_{\ell=1}^{k} C_{k-\ell}^{k} a_{k-\ell} \sigma^{\ell}.$$

Therefore, using Duhamel's formula and the fact that $\sigma = e^{-\tau/2}$, we have that

$$\begin{aligned} |a_k(\tau)| &\leq C \left(e^{-\frac{k}{2}\tau} + \sum_{\ell=1}^{k-2} e^{-\frac{\ell}{2}\tau} |a_{k-2-\ell}(\tau)| \right) \\ &= C \left(e^{-\frac{k}{2}\tau} + e^{-(\frac{k-2}{2})\tau} \sum_{\ell=0}^{k-3} e^{\frac{\ell}{2}\tau} |a_{\ell}(\tau)| \right) \end{aligned}$$

We split the last sum $S(\tau)$ into six separate sums:

$$\begin{aligned} S(\tau) &:= \sum_{\ell=0}^{k-3} e^{\frac{\ell}{2}\tau} |a_{\ell}(\tau)| \\ &= \sum_{j=0}^{s_0} e^{\frac{6j}{2}\tau} |a_{6j}(\tau)| + \sum_{j=0}^{s_1} e^{\frac{6j+1}{2}\tau} |a_{6j+1}(\tau)| + \sum_{j=0}^{s_2} e^{\frac{6j+2}{2}\tau} |a_{6j+2}(\tau)| \\ &+ \sum_{j=0}^{s_3} e^{\frac{6j+3}{2}\tau} |a_{6j+3}(\tau)| + \sum_{j=0}^{s_4} e^{\frac{6j+4}{2}\tau} |a_{6j+4}(\tau)| + \sum_{j=0}^{s_5} e^{\frac{6j+5}{2}\tau} |a_{6j+5}(\tau)|. \end{aligned}$$

Here, the integers s_i are defined as follows. We write $k-3 = 6m + r$ for some integer $m \geq 0$ and integer $0 \leq r \leq 5$. Then $s_i = m$ for $0 \leq i \leq r$ and $s_i = m - 1$ for $i > r$. Proceeding, we bound the sum using the induction hypothesis:

$$\begin{aligned} \frac{1}{C}|S(\tau)| \leq & \sum_{j=0}^{s_0} e^{\frac{6j}{2}\tau} e^{-\frac{6j}{6}\tau} + \sum_{j=0}^{s_1} e^{\frac{6j+1}{2}\tau} e^{-\frac{6j+1+2}{6}\tau} + \sum_{j=0}^{s_2} e^{\frac{6j+2}{2}\tau} e^{-\frac{6j+2+4}{6}\tau} \\ & + \sum_{j=0}^{s_3} e^{\frac{6j+3}{2}\tau} e^{-\frac{6j+3}{6}\tau} + \sum_{j=0}^{s_4} e^{\frac{6j+4}{2}\tau} e^{-\frac{6j+4+2}{6}\tau} + \sum_{j=0}^{s_5} e^{\frac{6j+5}{2}\tau} e^{-\frac{6j+5+4}{6}\tau} \end{aligned}$$

Simplifying, we find the following bound on $|a_k(\tau)|$:

$$\begin{aligned} |a_k(\tau)| \;&\leq\; Ce^{-\frac{k}{2}\tau} \\ &+\; Ce^{-\frac{k-2}{2}\tau} \left(e^{2s_0\tau} + e^{2s_1\tau} + e^{2s_2\tau} + e^{2s_3\tau} e^{\tau} + e^{2s_4\tau} e^{\tau} + e^{2s_5\tau} e^{\tau} \right) \end{aligned}$$

We now proceed by cases. We have six cases depending on the mod-6 parity of k.

First, assume $k-3=6m$. Therefore $s_0=m$ and $s_i=m-1$ for $i\geq 1$. Hence, the most slowly decaying term in the bound for $|a_k(\tau)|$ is

$$e^{-\frac{k-2}{2}\tau}e^{2m\tau}.$$

Simplifying, and using the fact that $m=\frac{k-3}{6}$, this term is equal to

$$e^{-\frac{k}{6}\tau},$$

so that $|a_k(\tau)|\leq Ce^{-\frac{k}{6}\tau}$. Since $k=6m+3$, this is the same decay rate as in the proposition. Continuing, the next case is $k-3=6m+1$. Then $s_0=s_1=m$, and $s_i=m-1$ for $i\geq 2$. Therefore the most slowly decaying term in the bound for $|a_k(\tau)|$ again is

$$e^{-\frac{k-2}{2}\tau}e^{2m\tau}.$$

Simplifying, and using the fact that $m=\frac{k-4}{6}$, this term is equal to

$$e^{-\frac{k+2}{6}\tau},$$

so that $|a_k(\tau)|\leq Ce^{-\frac{k+2}{6}\tau}$. Since $k=6m+4$, this is the same decay rate as in the proposition. Continuing, the next case is $k-3=6m+2$. Then $s_0=s_1=s_2=m$, and $s_i=m-1$ for $i\geq 3$. Therefore the most slowly decaying term in the bound for $|a_k(\tau)|$ again is

$$e^{-\frac{k-2}{2}\tau}e^{2m\tau}.$$

Simplifying, and using the fact that $m=\frac{k-5}{6}$, this term is equal to

$$e^{-\frac{k+4}{6}\tau},$$

so that $|a_k(\tau)|\leq Ce^{-\frac{k+4}{6}\tau}$. Since $k=6m+5$, this is the same decay rate as in the proposition. Continuing, the next case is $k-3=6m+3$. Then $s_0=s_1=s_2=s_3=m$, and $s_i=m-1$ for $i\geq 4$. Therefore the most slowly decaying term in the bound for $|a_k(\tau)|$ is

$$e^{-\frac{k-2}{2}\tau}e^{(2m+1)\tau}.$$

Simplifying, and using the fact that $m = \frac{k-6}{6}$, this term is equal to

$$e^{-\frac{k}{6}\tau},$$

so that $|a_k(\tau)| \leq Ce^{-\frac{k}{6}\tau}$. Since $k = 6m + 6$, this is the same decay rate as in the proposition. Continuing, the next case is $k - 3 = 6m + 4$. Then $s_i = m$ for $0 \leq i \leq 4$, and $s_5 = m - 1$. Therefore the most slowly decaying term in the bound for $|a_k(\tau)|$ again is

$$e^{-\frac{k-2}{2}\tau}e^{(2m+1)\tau}.$$

Simplifying, and using the fact that $m = \frac{k-7}{6}$, this term is equal to

$$e^{-\frac{k+2}{6}\tau},$$

so that $|a_k(\tau)| \leq Ce^{-\frac{k+2}{6}\tau}$. Since $k = 6m + 7$, this is the same decay rate as in the proposition. The last case is $k - 3 = 6m + 5$. Then $s_i = m$ for all i, and the most slowly decaying term in the bound for $|a_k(\tau)|$ again is

$$e^{-\frac{k-2}{2}\tau}e^{(2m+1)\tau}.$$

Simplifying, and using the fact that $m = \frac{k-8}{6}$, this term is equal to

$$e^{-\frac{k+4}{6}\tau},$$

so that $|a_k(\tau)| \leq Ce^{-\frac{k+4}{6}\tau}$. Since $k = 6m + 8$, this is the same decay rate as in Proposition 3.0.1, and hence Proposition 3.0.1 is proved.

3.4 Conclusion of Chapter

If one is willing to discard the "error" terms $w_0^s(\xi, \tau)$, $u_n^s(\xi, \tau)$, we've shown, to lowest order, that

$$\|w_0(\xi, \tau) - \alpha_0\phi_0(\xi)\|_{L^2(m)} + \|\|\{u_n(\xi, \tau) - \beta_0^n(\tau)\phi_0(\xi)\}_{n=1}^{\infty}\|_{\ell^2}\|_{L^2(m)} \leq Ce^{-\frac{\tau}{2}}.$$

To slightly higher order, we have

$$\|w_0(\xi,\tau) - \sum_{k=0}^{6} \alpha_k \phi_k(\xi)\|_{L^2(m)} + \|\|\{u_n(\xi,\tau) - \sum_{k=0}^{6} \beta_k^n(\tau)\phi_k(\xi)\}_{n=1}^{\infty}\|_{\ell^2}\|_{L^2(m)} \le Ce^{-\tau}.$$

More generally, from Proposition 3.0.1, we have the bound

$$\|w_0(\xi,\tau) - \sum_{k=0}^{j} \alpha_k \phi_k(\xi)\|_{L^2(m)} + \|\|\{u_n(\xi,\tau) - \sum_{k=0}^{j} \beta_k^n(\tau)\phi_k(\xi)\}_{n=1}^{\infty}\|_{\ell^2}\|_{L^2(m)} \le Ce^{-\frac{j}{6}\tau}.$$

for $0 \le j \le N$. So, we expect the error terms $w_0^s(\xi,\tau)$, $u_n^s(\xi,\tau)$ to decay about as quickly as $e^{-\frac{N+1}{6}\tau}$, since these error terms are the projections orthogonal to the first $N+1$ eigenfunctions of $\mathcal{L}_T$.

Chapter 4

Fourier Estimates and justification of the splitting

The goal of this section is to estimate the decay rate of the error terms $w_0^s(\xi,\tau)$ and $u_n^s(\xi,\tau)$ from the previous section. We expect the decay rates to be consistent with the highest-indexed terms from the center-manifold calculations; specifically, we expect

$$\|w_0^s(\xi,\tau)\|_{L^2(m)} + \|\|\{u_n^s(\xi,\tau)\}_{n=1}^{\infty}\|_{\ell^2}\|_{L^2(m)} \leq Ce^{-\frac{N+1}{6}\tau}$$

or some similar rate. How can we obtain this decay rate? First, note the equations for w_0^s and u_n^s:

$$\begin{aligned}
\partial_\tau w_0^s &= \mathcal{L}_T w_0^s - \frac{D_T}{\nu}\partial_\xi{}^2 w_0^s - A\sum_{m=1}^{\infty}\chi_m \partial_\xi{}^2 u_m^s \\
&\quad - \frac{D_T}{\nu}\alpha_N(\tau)\phi_{N+2}(\xi) - A\sum_{m=1}^{\infty}\chi_m \beta_N^m(\tau)\phi_{N+2}(\xi) \\
\partial_\tau u_n^s &= \mathcal{L}_T u_n^s - \frac{D_T}{\nu}\partial_\xi{}^2 u_n^s - e^{\tau}\left(\nu\mu_n u_n^s + A\chi_n w_0^s\right) \\
&\quad - Ae^{\tau/2}\sum_{m=1}^{\infty}\chi_{n,m}\partial_\xi u_m^s - \frac{D_T}{\nu}\beta_N^n(\tau)\phi_{N+2}(\xi) \\
&\quad - Ae^{\tau/2}\sum_{m=1}^{\infty}\chi_{n,m}\beta_N^n(\tau)\phi_{N+2}(\xi),
\end{aligned} \tag{4.1}$$

Recall that the original model is dissipative. Therefore we expect the long-time behavior to be dominated by the lowest wavenumbers. Therefore we undo the scaling variables, and take Fourier Transform, giving us

$$\partial_t \hat{V}(k,t) = B(k)\hat{V}(k,t) + \hat{F}(k,t). \tag{4.2}$$

Here $\hat{V}(k,t) = \begin{pmatrix} \hat{u}_0^s(k,t) \\ \{\hat{u}_n^s(k,t)\}_{n=1}^\infty \end{pmatrix}$, and $B(k) = B_0 + kB_1 + k^2B_2$ where

$$\begin{aligned} B_0 &= \begin{pmatrix} 0 & 0 \\ 0 & -\nu M \end{pmatrix} \\ B_1 &= Ai \begin{pmatrix} 0 & \langle \chi, \cdot \rangle \\ \chi & L_\chi \end{pmatrix} \\ B_2 &= -\nu \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{4.3}$$

Here B_0 represents transverse diffusion, and due to the powers of k appearing on B_1 and B_2, B_1 represents longitudinal transport, and B_2 represents longitudinal diffusion. Also, the forcing term $\hat{F}(k,t)$ is a sum of terms of the form

$$(1+t)^p(1+t)^M k^M e^{-\nu_T k^2(1+t)} Y(\begin{pmatrix} \alpha_N(\log(1+t)) \\ \{\beta_N^n(\log(1+t))\}_{n=1}^\infty \end{pmatrix}), \tag{4.4}$$

where Y is a bounded linear operator on $\mathbb{C} \times \ell^2(\mathbb{C})$. Note that the prefactor $(1+t)^{(N+2)/2}k^{N+2}$ results from undoing the scaling variables, and using the fact that the eigenfunctions satisfy $\phi_N(\xi) = \partial_\xi^N \phi_0(\xi)$. Here $p = 0$ or $\frac{1}{2}$, depending on whether or not the term in the sum has the $\tau-$ dependent prefactor $e^{\frac{\tau}{2}}$ in (4.1), and $M = \frac{N+1}{2}$ or $M = \frac{N+2}{2}$ depending on whether or not the term in the sum has ϕ_{N+1} or ϕ_{N+2}. Continuing, we solve (4.2) using Duhamel's formula:

$$\hat{V}(k,t) = e^{B(k)t}\hat{V}(k,0) + \int_0^t e^{B(k)(t-s)}\hat{F}(k,s)ds. \tag{4.5}$$

We will obtain the desired decay rate by giving a detailed analysis of

$$e^{B(k)(t-s)}\hat{G}(k,s), \tag{4.6}$$

where $\hat{G}(k,s)$ will be taken to be $\hat{V}(k,0)$ with $s = 0$, or $\hat{G}(k,s) = \hat{F}(k,s)$. Our estimates will be done in the norm

$$|||\hat{V}(\cdot,t)||| = \sum_{j=0}^m \frac{1}{(1+t)^{j/2}} \|\|\partial_k^j \hat{V}(\cdot,t)\|_Y\|_{L^2}, \tag{4.7}$$

which is equivalent to the $\|\|\cdot\|_{\mathbb{R}\times\ell^2(\mathbb{R})}\|_{L^2(m)}$ used in the Center Manifold section. Specifically, this section will be used to prove the following Proposition:

Proposition 4.0.1. *Let $\hat{V}(k,t)$ be the solution to* (4.2) *with initial condition* $\hat{V}(k,0) =$

$\begin{pmatrix} \hat{u}_0^s(k,0) \\ \{\hat{u}_n^s(k,0)\}_{n=1}^{\infty} \end{pmatrix}$. *Then there exists a constant C depending on the initial data, ν, A, and χ such that*

$$|||\hat{V}(\cdot,t)||| \leq C(1+t)^{-\frac{N}{6}+\frac{5}{4}}.$$

To establish Proposition 4.0.1, we recall that low wavenumbers should dominate the long-time behavior. Therefore we split the semigroup $e^{B(k)(t-s)}\hat{G}(k,s)$ into a low-wavenumber part and a high-wavenumber part. More specifically, we split the semigroup $e^{B(k)(t-s)}$ as follows:

- A high-wavenumber part, which will decay exponentially due to a bound on the spectrum of $B(k)$ for non-small wavenumbers

- A low-wavenumber part, which will be split into a part corresponding to the leading eigenvalue $\lambda_0(k)$ of $B(k)$, and a part corresponding to the rest of the spectrum (which will decay exponentially),

- The low-wavenumber part corresponding to the leading eigenvalue will be Taylor expanded with respect to the wavenumber k around $k = 0$, with remainder decaying at a rate consistent with the Center Manifold analysis, and Taylor polynomial being identically zero for the $\hat{G}(k,s)$ considered above.

The splitting will be done using a smooth bump function $\psi(k)$ equalling 1 for $|k| \leq k_0$ and 0 for $|k| \geq 2k_0$. Here k_0 will be taken to be $\frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$. This choice of k_0 is due to how small $|k|$ must be for the leading eigenvalue to be consistent with Taylor Dispersion. This choice of k_0 also guarantees that the leading eigenvalue stays separated from the rest of the eigenvalues. A similar choice of low- and high-wavenumbers was used in the Fourier estimates for the model problem in Section 2.2. To establish our estimates, we need to establish some facts about the spectrum of the operator $B(k)$, for both small $|k|$ and non-small $|k|$.

4.1 Properties of the operator $B(k)$

4.1.1 Spectral and boundedness properties of B_0 B_1, and B_2

As mentioned above, our estimates depend on knowledge of the spectrum of $B(k)$. We will first prove that, for each k, the operator $B(k)$ has only point eigenvalues on the space $Y = \mathbb{C} \times \ell^2(\mathbb{C})$, by using Weyl's theorem. The fact that $B(k)$ has only point eigenvalues will be useful, because we will prove an estimate on point eigenvalues which holds for all k, and hence can be used in our estimates on high-wavenumbers. Here we note a few facts about the operators B_0 B_1, and B_2:

- B_0 has only point spectrum, and $\sigma(B(0)) = \{0\} \cup \{-\nu\mu_n\}_{n=1}^{\infty}$
- Both B_1 and B_2 are bounded: using Parseval's identity, we find $||B_1\hat{u}|| \leq Asup|\chi|||\hat{u}||$. Similarly $||B_2\hat{u}|| \leq \nu||\hat{u}||$.

4.1.2 Weyl's Theorem: Showing $B(k)$ has only point spectrum

The goal of this section is to show that $B(k)$ has only point spectrum, by establishing the following Lemma.

Lemma 4.1.1. *Let $k \in \mathbb{R}$. Then the essential spectrum of $B(k)$ is empty, i.e.*

$$\sigma_{ess} = \emptyset. \tag{4.8}$$

Hence, $B(k)$ has only point spectrum.

Proof:
We proceed by using Weyl's Theorem. If we show that for each fixed k, $B(k) = B_0 + k(B_1 + kB_2)$ is a relatively compact perturbation of B_0, then $B(k)$ and B_0 have the same (empty) essential spectrum. We want to show that

$$k(B_1 + kB_2)(B_0 + i)^{-1}$$

is a compact operator on $\mathbb{C} \times \ell^2(\mathbb{C})$. By Parseval's identity, this is equivalent to showing that

$$k(-A\chi(y,z) - \nu k)(\nu\Delta + i)^{-1}$$

is a compact operator on $L^2(\Omega)$. We let $\{\hat{u}_n(y,z)\} \subset L^2(\Omega)$ be a bounded sequence: $||\hat{u}_n(y,z)||_{L^2(\Omega)} \leq M$. Then, since i is in the resolvent set of $\nu\Delta$, and $(\nu\Delta + i)^{-1} : L^2(\Omega) \to H^1(\Omega)$ is bounded, it follows that $\{(\nu\Delta + i)^{-1}\hat{u}_n\}$ is a bounded sequence in $H^1(\Omega)$. Therefore

$$\{k(-A\chi(y,z) - \nu k)(\nu\Delta + i)^{-1}\hat{u}_n\}$$

is also a bounded sequence in $H^1(\Omega)$. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, this sequence has an $L^2(\Omega)$ convergent subsequence. Therefore $k(-A\chi(y,z)-\nu k)(\nu\Delta+i)^{-1}$ is compact, and hence $k(B_1+kB_2)(B_0+i)^{-1}$ is compact. Therefore $B(k)=B_0+k(B_1+kB_2)$ is a relatively compact perturbation of B_0, and hence, they have the same (empty) essential spectrum. □

4.1.3 Low wavenumbers: approximating the spectrum of $B(k)$ for small $|k|$

The goal of this section is to show that, for $|k|$ sufficiently small, the eigenvalues of $B(k)$ split into two parts: an eigenvalue near 0, and eigenvalues $\lambda(k)$ satisfying $Re(\lambda(k)) \leq -\nu\mu_1/2$. Therefore, we expect the eigenvalue near 0 to dominate the long-time behavior, and we can obtain estimates on the low-wavenumber part of our solution. In addition, we will show that this leading eigenvalue $\lambda_0(k)$ is approximately $-\nu_T k^2$, so the long-time behavior will correspond with Taylor dispersion. We proceed by establishing a Lemma and its Corollary:

Lemma 4.1.2. *Let $k \in \mathbb{R}$ satisfy $|k| \leq \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$. Then*

$$\sigma(B(k)) = \{\lambda_0(k)\} \cup \sigma_{left} \tag{4.9}$$

where $|\lambda_0(k)| \leq \frac{\sqrt{2}}{2}\nu\mu_1$, and for $\lambda(k) \in \sigma_{left}$,

$$Re(\lambda(k)) \leq -\frac{\nu\mu_1}{2}. \tag{4.10}$$

Corollary 4.1.3. *Let Let $k \in \mathbb{R}$ satisfy $|k| \leq \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$ and let $Q_0(k)$ be the projection orthogonal to the eigenspace for the eigenvalue $\lambda_0(k)$ of $B(k)$. Then, for all $W \in Y = \mathbb{C} \times \ell^2(\mathbb{C})$ and $r > 0$, we have the bound*

$$\|e^{B(k)r}Q_0(k)W\|_Y \leq Ce^{-\frac{\nu\mu_1}{2}r}$$

The proof of Corollary 4.1.3 follows immediately from Lemma 4.1.2.

Proof of Lemma 4.1.2:

Note, this separation between the eigenvalues is true for B_0, since 0 is an eigenvalue of B_0, and all other eigenvalues satisfy $-\nu\mu_n \leq -\nu\mu_1 < 0$. Let k satisfy $|k| \leq \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$. To establish this separation for $B(k)$, we will use Kato's definition of a "gap" between operators (Kato, 1966). Given two operators T and S, and a closed curve $\Gamma \subset \mathbb{C}$ which separates the spectrum of T in two parts (one part inside Γ and one part outside Γ), if the gap $\hat{\delta}(T,S)$ is sufficiently small, the closed curve Γ also separates the spectrum of S. The definition of $\hat{\delta}(T,S)$

$$\hat{\delta}(T,S) := \max\{\sup_{u\in G(T),||u||=1} dist(u, G(S)), \sup_{v\in G(S),||v||=1} dist(v, G(T))\}, \tag{4.11}$$

where $G(L) = \{(u, Lu)|u \in D(L)\}$ is the graph of the operator L with domain $D(L)$, with $||u||$ being the graph norm. If $\hat{\delta}(T,S)$ satisfies

$$\hat{\delta}(T,S) < \min_{z\in\Gamma} \frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+||(T-z)^{-1}||^2}}, \tag{4.12}$$

then the closed curve Γ also separates the spectrum of S (See Kato IV.3.4 Theorem 3.16 (Kato, 1966)).
We will show that the gap $\hat{\delta}(B(k), B_0)$ satisfies

$$\hat{\delta}(B(k), B_0) < \min_{z\in\Gamma_R}\left(\frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+||(B_0-z)^{-1}||^2}}\right)$$

where Γ_R, for $R \geq \nu\mu_1/2$, is the boundary of the rectangle $\{z = x+iy| -\nu\mu_1/2 \leq x \leq R, -R \leq y \leq R\}$. If this holds, for each such R, we have:

- Using $R = \nu\mu_1/2$, $B(k)$ has an eigenvalue $\lambda_0(k)$ near 0, and
- Using larger and larger R-values, the rest of the eigenvalues $\lambda(k)$ of $B(k)$ satisfy $Re(\lambda(k)) \leq -\nu\mu_1/2$.

Let's proceed by computing the gap $\hat{\delta}(B(k), B_0)$. Using the definition (4.11), we first need to bound

$$\sup_{||\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})}+||B(k)\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})}=1} dist\left((\hat{V}, B(k)\hat{V}), G(B_0)\right).$$

Pick a $\hat{V} \in \mathbb{C} \times \ell^2(\mathbb{C})$ with $||\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} + ||B(k)\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} = 1$. Then

$$\begin{aligned} dist((\hat{V}, B(k)\hat{V}), G(B_0)) &\leq ||(\hat{V}, B(k)\hat{V}) - (\hat{V}, B_0\hat{V})|| \\ &= ||(0, B(k)\hat{V} - B_0\hat{V})|| \\ &= ||k(B_1 + kB_2)\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} \\ &\leq |k| \left(A\|\chi\|_{L^\infty_\Omega} + \nu|k|\right) ||\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} \\ &\leq |k| \left(A\|\chi\|_{L^\infty_\Omega} + \nu|k|\right). \end{aligned}$$

Since this holds for all $\hat{V}$ with $\hat{V} \in \mathbb{C} \times \ell^2(\mathbb{C})$ with $||\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} + ||B(k)\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} = 1$, it follows that

$$\sup_{||\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} + ||B(k)\hat{V}||_{\mathbb{C}\times\ell^2(\mathbb{C})} = 1} dist\left((\hat{V}, B(k)\hat{V}), G(B_0)\right) \leq |k| \left(A\|\chi\|_{L^\infty_\Omega} + \nu|k|\right).$$

Next, using the definition of the Kato gap (4.11), we need to bound

$$\sup_{||\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} + ||B(0)\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} = 1} dist\left((\hat{W}, B(0)\hat{W}), G(B(k))\right).$$

Pick a $\hat{W} \in \mathbb{C} \times \ell^2(\mathbb{C})$ with $||\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} + ||B_0\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} = 1$. Then

$$\begin{aligned} dist\left((\hat{W}, B_0\hat{W}), G(B(k))\right) &\leq ||(\hat{W}, B_0\hat{W}) - (\hat{W}, B(k)\hat{W})|| \\ &= ||(0, B_0\hat{W} - B(k)\hat{W})|| \\ &= ||k(B_1 + kB_2)\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} \\ &\leq |k| \left(A\|\chi\|_{L^\infty_\Omega} + \nu|k|\right) ||\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} \\ &\leq |k| \left(A\|\chi\|_{L^\infty_\Omega} + \nu|k|\right). \end{aligned}$$

Since this holds for all $\hat{W} \in \mathbb{C} \times \ell^2(\mathbb{C})$ with $||\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} + ||B_0\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} = 1$, it follows that

$$\sup_{||\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} + ||B(0)\hat{W}||_{\mathbb{C}\times\ell^2(\mathbb{C})} = 1} dist\left((\hat{W}, B(0)\hat{W}), G(B(k))\right) \leq |k| \left(A\|\chi\|_{L^\infty_\Omega} + \nu|k|\right).$$

Therefore

$$\hat{\delta}(B(k), B_0) \leq |k| \left(A\|\chi\|_{L^\infty_\Omega} + \nu|k|\right). \tag{4.13}$$

Next, we bound

$$min_{z\in\Gamma_R}\frac{1}{2}\left(\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+||(B_0-z)^{-1}||^2}}\right).$$

First notice that, for $z\in\Gamma_R$,

$$||(B_0-z)^{-1}||\leq\frac{2}{\nu\mu_1}.$$

This is true since B_0 is self adjoint, and $\frac{\nu\mu_1}{2}$ is the distance from Γ_R to $\sigma(B_0)$ (see Kato V.3.5 formula 3.16 (Kato, 1966)).

Therefore, for all $z\in\Gamma_R$,

$$\frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+(\frac{2}{\nu\mu_1})^2}}\leq\frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+||(B_0-z)^{-1}||^2}}.$$

Hence

$$min_{z\in\Gamma_R}\left(\frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+(\frac{2}{\nu\mu_1})^2}}\right)\leq min_{z\in\Gamma_R}\left(\frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+||(B_0-z)^{-1}||^2}}\right).$$

Also notice that, for $\nu<\frac{2\sqrt{3}}{\mu_1}$,

$$\min_{z\in\Gamma_R}\left(\frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+(\frac{2}{\nu\mu_1})^2}}\right)\geq\frac{1}{2}\frac{1}{\sqrt{1+(\frac{2}{\nu\mu_1})^2}}\geq\frac{\nu\mu_1}{8}.$$

Therefore, using (4.13), we only need k to satisfy

$$|k|\left(A\|\chi\|_{L^\infty_\Omega}+\nu|k|\right)<\frac{\nu\mu_1}{4}.$$

This follows since $|k|\leq\frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$, as long as $\nu<\frac{8A\|\chi\|_{L^\infty(\Omega)}}{\mu_1}$. Therefore

$$\hat{\delta}(B(k),B_0)<\min_{z\in\Gamma_R}\left(\frac{1}{2}\frac{1}{1+|z|^2}\frac{1}{\sqrt{1+||(B_0-z)^{-1}||^2}}\right)$$

as desired. Hence, for all $R\geq\nu\mu_1/2$, the Γ_R separate the spectrum of $B(k)$ into one eigenvalue $\lambda_0(k)$ near 0 (actually $|\lambda_0(k)| \leq \frac{\sqrt{2}}{2}\nu\mu_1$), and all other eigenvalues $\lambda(k)$ satisfying $Re(\lambda(k)) \leq -\nu\mu_1/2$. Therefore Lemma 4.1.2 is proven. $\square$

Next, we actually estimate the eigenvalue $\lambda_0(k)$ for $|k| \leq \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$.

Lemma 4.1.4. *Let $k \in \mathbb{R}$ in satisfy $|k| \leq \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$. Then the leading eigenvalue $\lambda_0(k)$ of $B(k)$ satisfies*

$$\lambda_0(k) = -\nu_T k^2 + \Lambda_0(k), \tag{4.14}$$

where $\Lambda_0(k) = irk^3 + \mathcal{O}(k^4)$ and r is a real number depending on χ, ν, and the numbers $\{\mu_n\}_{n=1}^{\infty}$.

Proof:

Since $\lambda_0(k)$ is a perturbed eigenvalue of the simple eigenvalue 0 of B_0, $\lambda_0(k)$ (and its projection $P_0(k)$) perturbs smoothly in k (Kato, 1966). We write

$$\lambda_0(k) = \lambda_0 + \lambda_1 k + \lambda_2 k^2 + \mathcal{O}(k^3) \tag{4.15}$$

along with its corresponding eigenvector

$$\hat{V}(k) = \hat{V}_0 + \hat{V}_1 k + \hat{V}_2 k^2 + \mathcal{O}(k^3). \tag{4.16}$$

Now the eigenvalue problem reads

$$B(k)\hat{V}(k) = \lambda(k)\hat{V}(k). \tag{4.17}$$

where $\hat{V}(k) = \begin{pmatrix} \hat{u}_0(k) \\ \hat{U}(k) \end{pmatrix}$ and $\hat{V}_j = \begin{pmatrix} \hat{u}_0^j \\ \hat{U}^j \end{pmatrix}$. In this notation, the unperturbed eigenvector corresponding to the simple eigenvalue is $\hat{V}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Plugging (4.16) and (4.15) into (4.17) gives us

$$\begin{gathered} B_0\hat{V}_0 = 0 \cdot \hat{V}_0 \\ B_1\hat{V}_0 + B_0\hat{V}_1 = \lambda_1\hat{V}_0 \\ B_2\hat{V}_0 + B_1\hat{V}_1 + B_0\hat{V}_2 = \lambda_2\hat{V}_0 + \lambda_1\hat{V}_1 \end{gathered} \tag{4.18}$$

and so on. The first equation is just the eigenvalue problem for B_0 restated. We then solve the second equation for λ_1 and $\hat{V}_1$. Proceeding using the definitions of B_0 and

B_1 in (4.3) (and suppressing the subscript on the ℓ^2 inner product), we have

$$-Ai\begin{pmatrix}0 & \langle\chi,\cdot\rangle\\ \chi\cdot & L_\chi\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & -\nu M\end{pmatrix}\begin{pmatrix}\hat{u}_0^1\\ \hat{U}^1\end{pmatrix}=\lambda_1\begin{pmatrix}1\\0\end{pmatrix}.$$

By equating the first component, we get that $\lambda_1 = 0$. By equating the second components, we get

$$-Ai\chi-\nu M\hat{U}^1=0$$

or

$$\hat{U}^1=-Ai(\nu M)^{-1}\chi.$$

The component $\hat{u}_0^1$ is undetermined and we denote $\hat{u}_0^1 = c_1$. However, c_1 can be fixed by normalizing the eigenvectors. Continuing, we have

$$\hat{V}_1=\begin{pmatrix}c_1\\ -Ai(\nu M)^{-1}\chi\end{pmatrix}.$$

Continuing from (4.18) (and using the fact that $\lambda_1 = 0$), we have

$$B_2\hat{V}_0+B_1\hat{V}_1+B_0\hat{V}_2=\lambda_2\hat{V}_0.$$

Inserting the definitions of the B_j and the known quantities, we have

$$\begin{aligned}-\nu\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\; -Ai\begin{pmatrix}0 & \langle\chi,\cdot\rangle\\ \chi\cdot & L_\chi\cdot\end{pmatrix}\begin{pmatrix}c_1\\ -Ai(\nu M)^{-1}\chi\end{pmatrix}&\\ +\begin{pmatrix}0 & 0\\ 0 & -(\nu M)\end{pmatrix}\begin{pmatrix}\hat{u}_0^2\\ \hat{U}^2\end{pmatrix}&=\lambda_2\begin{pmatrix}1\\0\end{pmatrix}.\end{aligned}$$

Again, equating the first component lets us solve for λ_2, and we get

$$\begin{aligned}\lambda_2&=-\nu-Ai\langle\chi,-Ai(\nu M)^{-1}\chi\rangle\\ &=-\left(\nu+\frac{A^2}{\nu}\sum_{m=1}^{\infty}\frac{\chi_m^2}{\mu_m}\right)\\ &=-\nu_T.\end{aligned}$$

Equating the second component gives us

$$-\nu - Aic_1\chi + (Ai)^2 L_\chi(\nu M)^{-1}\chi - (\nu M)\hat{U}^2 = 0,$$

which tells us

$$\hat{U}^2 = -(\nu M)^{-1}\left(\nu + Aic_1\chi + A^2 L_\chi(\nu M)^{-1}\chi\right).$$

Again, $\hat{u}_0^2$ is undetermined and we denote $\hat{u}_0^2 = c_2$. Therefore

$$\hat{V}_2 = \begin{pmatrix} c_2 \\ -(\nu M)^{-1}\left(\nu + Aic_1\chi + A^2 L_\chi(\nu M)^{-1}\chi\right) \end{pmatrix}.$$

Continuing, the next equation in (4.18) will read

$$B_2\hat{V}_1 + B_1\hat{V}_2 + B_0\hat{V}_3 = \lambda_3\hat{V}_0 + \lambda_2\hat{V}_1,$$

which reads

$$\begin{aligned}
&-\nu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ -Ai(\nu M)^{-1}\chi \end{pmatrix} \\
&- Ai\begin{pmatrix} 0 & \langle\chi,\cdot\rangle \\ \chi\cdot & L_\chi\cdot \end{pmatrix}\begin{pmatrix} c_2 \\ -(\nu M)^{-1}\left(\nu + Aic_1\chi + A^2 L_\chi(\nu M)^{-1}\chi\right) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & -(\nu M) \end{pmatrix}\begin{pmatrix} \hat{u}_0^3 \\ \hat{U}^3 \end{pmatrix} = \lambda_3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} c_1 \\ -Ai(\nu M)^{-1}\chi \end{pmatrix}.
\end{aligned}$$

Now again equating the first component gives us

$$\begin{aligned}
&\lambda_3 + c_1\lambda_2 = \\
&- c_1\nu + Ai\langle\nu, (\nu M)^{-1}\chi\rangle \\
&+ Ai\langle\chi, (\nu M)^{-1}\left(\nu + Aic_1\chi + A^2 L_\chi(\nu M)^{-1}\chi\right)\rangle \\
&= Ai(\langle\nu, (\nu M)^{-1}\chi\rangle \\
&+ \langle\chi, (\nu M)^{-1}\left(\nu + A^2 L_\chi(\nu M)^{-1}\chi\right)\rangle) \\
&+ c_1\left(-\nu - A^2\langle\chi, (\nu M)^{-1}\chi\rangle\right).
\end{aligned}$$

Notice the last quantity on the right-hand side is exactly $c_1\lambda_2$, so that the quantities $c_1\lambda_2$ cancel each other out, and we have

$$\lambda_3 = Ai\left(\langle\nu, (\nu M)^{-1}\chi\rangle + \langle\chi, (\nu M)^{-1}\left(\nu + A^2 L_\chi(\nu M)^{-1}\chi\right)\rangle\right).$$

Note in particular that λ_3 is purely imaginary, and therefore

$$\lambda_0(k) = -\nu_T k^2 + irk^3 + \mathcal{O}(k^4)$$

where $r = A\left(\langle \nu, (\nu M)^{-1}\chi\rangle + \langle \chi, (\nu M)^{-1}\left(\nu + A^2 L_\chi (\nu M)^{-1}\chi\right)\rangle\right)$. Finally, we setting $\Lambda_0(k) := \lambda_0(k) - (-\nu_T k^2 + irk^3)$, we have proven Lemma 4.1.4. $\square$

4.1.4 High wavenumbers: symmetry properties of the B_j and bounding the spectrum for large $|k|$

In this section, we show that the spectrum of $B(k)$ sits strictly to the left of the imaginary axis, for non-small wavenumbers $|k|$. In particular we want to establish

Lemma 4.1.5. *Let $\lambda(k)$ be an eigenvalue of $B(k)$ with corresponding unit eigenvector $V(k) \in \mathbb{C} \times \ell^2(\mathbb{C})$. Then for each k,*

$$Re(\lambda(k)) \leq -\nu k^2. \tag{4.19}$$

Although this bound holds for any k, we intend to use it for $|k| \geq k_0$ where $k_0 = |k| \leq \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$. In that case, the bound tells us that $Re(\lambda(k)) \leq -\nu k_0^2 < 0$. Therefore these "high" wavenumbers will decay exponentially, as in the following Corollary.

Corollary 4.1.6. *Let $k_0 = |k| \leq \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$ and let $\Psi(k)$ be a smooth cutoff function equalling 1 for $|k| \leq k_0$ and 0 for $|k| \geq 2k_0$. Then for any $W \in Y$ and $r > 0$, we have*

$$\|(1-\psi(k))e^{B(k)r}W\|_Y \leq Ce^{-\nu k_0^2 r}$$

The proof of Corollary 4.1.6 follows immediately from Lemma 4.1.5 and the fact that $(1-\psi(k)) = 0$ for $|k| \leq k_0$. We proceed with the proof of Lemma 4.1.5:

Proof of Lemma 4.1.5:

Recall that $B(k) = B_0 + kB_1 + k^2 B_2$, and from (4.3),

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & -\nu M \end{pmatrix}$$
$$B_1 = Ai\begin{pmatrix} 0 & \langle \chi, \cdot \rangle \\ \chi & L_\chi \end{pmatrix}$$
$$B_2 = -\nu\begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}.$$

Note that B_0 and B_2 are diagonal and therefore symmetric, and hence

$$\mathcal{S}(k) := B_0 + k^2 B_2 \tag{4.20}$$

is symmetric. Also note that

$$\mathcal{A}(k) := kB_1 \tag{4.21}$$

is anti-symmetric, due to the following argument. Let $V = \{V_n\}_{n=0}^{\infty} \in \mathbb{C} \times \ell^2(\mathbb{C})$ and let $v(y,z) = V_0 + \sum_{n=1}^{\infty} V_n \psi_n(y,z)$. We compute, using Parseval's identity,

$$\begin{aligned} \langle B_1 V, V \rangle_{\mathbb{C}\times\ell^2(\mathbb{C})} &= \langle Ai\chi(y,z)v(y,z), v(y,z)\rangle_{L^2(\Omega)} \\ &= -\langle v(y,z), Ai\chi(y,z)v(y,z)\rangle_{L^2(\Omega)} \\ &= -\langle V, B_1 V\rangle_{\mathbb{C}\times\ell^2(\mathbb{C})}. \end{aligned}$$

Therefore B_1 is anti-symmetric, and hence $\mathcal{A}(k) = kB_1$ is anti-symmetric.

We proceed by computing a bound on the spectrum of $B(k)$. Notice that

$$B(k) = \mathcal{S}(k) + \mathcal{A}(k), \tag{4.22}$$

where $\mathcal{S}(k)$ is symmetric and $\mathcal{A}(k)$ is antisymmetric.

Since $\lambda(k)$ is an eigenvalue of $B(k)$ with eigenvector $V(k)$, we have

$$B(k)V(k) = \lambda(k)V(k).$$

Taking the $\mathbb{C} \times \ell^2(\mathbb{C})$ inner product of both sides with $V(k)$, we get

$$\begin{aligned} \langle B(k)V(k), V(k)\rangle &= \langle \lambda(k)V(k), V(k)\rangle \\ &= \lambda(k)\langle V(k), V(k)\rangle \\ &= \lambda(k) \end{aligned}$$

where the last line holds since $V(k)$ is a unit vector. Next, using (4.22), we have

$$\begin{aligned} \langle(\mathcal{S}(k)+\mathcal{A}(k))V(k),V(k)\rangle &= \lambda(k) \\ \langle\mathcal{S}(k)V(k),V(k)\rangle+\langle\mathcal{A}(k)V(k),V(k)\rangle &= \lambda(k) \\ \langle V(k),\mathcal{S}(k)V(k)\rangle-\langle V(k),\mathcal{A}(k)V(k)\rangle &= \lambda(k), \end{aligned} \tag{4.23}$$

where the last line is obtained using the symmetry properties of $\mathcal{S}(k)$ and $\mathcal{A}(k)$. Taking the middle line in (4.23) and taking the complex conjugate, we have

$$\langle V(k),\mathcal{S}(k)V(k)\rangle+\langle V(k),\mathcal{A}(k)V(k)\rangle = \overline{\lambda}(k). \tag{4.24}$$

Finally, adding (4.24) and the last line in (4.23) yields

$$Re(\lambda(k)) = \langle V(k),\mathcal{S}(k)V(k)\rangle_{\mathbb{C}\times\ell^2(\mathbb{C})}.$$

Next, we let $v(y,z,k)=V_0(k)+\sum_{n=1}^{\infty}V_n(k)\psi_n(y,z)$ and apply Parseval's identity to get

$$\begin{aligned} Re(\lambda(k)) &= \langle V(k),\mathcal{S}(k)V(k)\rangle_{\mathbb{C}\times\ell^2(\mathbb{C})} \\ &= \frac{1}{area(\Omega)}\int_\Omega v(y,z,k)(\nu\Delta-\nu k^2)\overline{v}(y,z,k)dydz \\ &= -\nu\int_\Omega \nabla v(y,z,k)\cdot\nabla\overline{v}(y,z,k)dydz-\nu k^2\int_\Omega |v(y,z,k)|^2dydz \\ &\le -\nu k^2\int_\Omega |v(y,z,k)|^2dydz \\ &= -\nu k^2. \end{aligned} \tag{4.25}$$

Therefore

$$Re(\lambda(k)) \le -\nu k^2$$

as stated in the Lemma. □

4.2 Splitting of the semigroup $e^{B(k)(t-s)}$ and estimates

The goal of this section is to establish the decay rates on the low and high wavenumber parts of the semigroup $e^{B(k)(t-s)}$ mentioned in the beginning of 4. We split the semigroup $e^{B(k)(t-s)}$ as follows:

- A high-wavenumber part, which will decay exponentially due to Corollary in 4.1.6,
- A low-wavenumber part, which will be split into a part corresponding to the leading eigenvalue $\lambda_0(k)$ of $B(k)$, and a part corresponding to the rest of the spectrum (which will decay exponentially), as in Corollary 4.1.3,
- The low-wavenumber part corresponding to the leading eigenvalue will be Taylor expanded with respect to the wavenumber k around $k = 0$, with remainder decaying at a rate consistent with the Center Manifold analysis, and Taylor polynomial being identically zero for the $\hat{G}(k, s)$ considered above.

The splitting will be done using a smooth bump function $\psi(k)$ equalling 1 for $|k| \leq k_0$ and 0 for $|k| \geq 2k_0$. Here $k_0 = \frac{\nu\mu_1}{8A\|\chi\|_{L^\infty}}$. Again, this choice of k_0 is due to how small $|k|$ must be for the leading eigenvalue to be consistent with Taylor Dispersion. We proceed with our splitting, defining terms along the way.

$$e^{B(k)(t-s)}\hat{G}(k, s) = \psi(k)e^{B(k)(t-s)}\hat{G}(k, s) + (1 - \psi(k))e^{B(k)(t-s)}\hat{G}(k, s)$$

and define

$$exp_{high}(k, t-s)\hat{G}(k, s) = (1 - \psi(k))e^{B(k)(t-s)}\hat{G}(k, s). \tag{4.26}$$

Therefore

$$e^{B(k)(t-s)}\hat{G}(k, s) = \psi(k)e^{B(k)(t-s)}\hat{G}(k, s) + exp_{high}(k, t-s)\hat{G}(k, s).$$

Continuing, we split the above expression according to the projection and orthogonal projection $P_0(k)$ and $Q_0(k)$ respectively, onto the eigenspace for the leading eigenvalue $\lambda_0(k)$ of $B(k)$. Hence, we write

$$\begin{aligned}\psi(k)e^{B(k)(t-s)}\hat{G}(k, s) &= \psi(k)\left(P_0(k) + Q_0(k)\right)e^{B(k)(t-s)}\hat{G}(k, s)\\ &= \psi(k)P_0(k)e^{B(k)(t-s)}\hat{G}(k, s) + \psi(k)Q_0(k)e^{B(k)(t-s)}\hat{G}(k, s).\end{aligned}$$

Next, we define

$$exp_{low}(k, t-s)\hat{G}(k, s) = \psi(k)Q_0(k)e^{B(k)(t-s)}\hat{G}(k, s). \tag{4.27}$$

Next, since $P_0(k)$ is the projection onto the eigenspace for the leading eigenvalue $\lambda_0(k)$ of $B(k)$, we have that

$$P_0(k)e^{B(k)(t-s)} = P_0(k)e^{\lambda_0(k)(t-s)}.$$

Therefore

$$\psi(k)P_0(k)e^{B(k)(t-s)}\hat{G}(k,s) = \psi(k)P_0(k)e^{\lambda_0(k)(t-s)}\hat{G}(k,s).$$

Next, we use the fact that $\lambda_0(k) = -\nu_T k^2 + \Lambda_0(k)$ from Lemma 4.1.2, and obtain

$$\psi(k)P_0(k)e^{B(k)(t-s)}\hat{G}(k,s) = e^{-\nu_T k^2}\psi(k)P_0(k)e^{\Lambda_0(k)(t-s)}\hat{G}(k,s).$$

Continuing, we Taylor expand $\psi(k)P_0(k)e^{\Lambda_0(k)(t-s)}\hat{G}(k,s)$ with respect to k to degree N, and define

$$T_N(k,t-s)\hat{G}(k,s) = e^{-\nu_T k^2}\sum_{\ell=0}^{N}\frac{1}{\ell!}\partial_k^{\ell}\left(\psi(k)P_0(k)e^{\Lambda_0(k)(t-s)}\hat{G}(k,s)\right)|_{k=0}k^{\ell}. \quad (4.28)$$

and define

$$\begin{aligned} Rem_N(k,t-s)\hat{G}(k,s) = e^{-\nu_T k^2}(&\psi(k)P_0(k)e^{\Lambda_0(k)(t-s)}\hat{G}(k,s) \\ &- T_N(k,t-s)\hat{G}(k,s)). \end{aligned} \quad (4.29)$$

Finally, combining definitions (4.26), (4.27), (4.28), and definition (4.29), we have the splitting

$$\begin{aligned} e^{B(k)(t-s)}\hat{G}(k,s) =& exp_{high}(k,t-s)\hat{G}(k,s) + exp_{low}(k,t-s)\hat{G}(k,s) \\ &+ T_N(k,t-s)\hat{G}(k,s) + Rem_N(k,t-s)\hat{G}(k,s). \end{aligned} \quad (4.30)$$

We proceed by establishing lemmas giving us decay rates on each of the terms in the above splitting. We start with the exponentially decaying low wavenumber part.

4.2.1 Exponentially decaying terms - low wavenumbers

The goal of this section is to estimate the exponentially decaying, low-wavenumbe part of the semigroup $e^{B(k)t}$:

$$exp_{low}(k,t-s)\hat{G}(k,s) = \psi(k)e^{B(k)(t-s)}Q_0(k)\hat{G}(k,s). \tag{4.31}$$

Here $Q_0(k) = 1 - P_0(k)$ and $P_0(k)$ is the projection onto the eigenspace of $\lambda_0(k)$. We want to estimate $|||exp_{low}(k,t-s)\hat{G}(k,s)|||$. Specifically, we aim to establish the following two lemmas:

Lemma 4.2.1.

$$|||exp_{low}(\cdot,t)\hat{V}(\cdot,0)||| \leq Ce^{-\frac{\nu\mu_1}{4}t}$$

Lemma 4.2.2.

$$|||\int_0^t exp_{low}(\cdot,t-s)\hat{F}(\cdot,s)ds||| \leq C(1+t)^{-\frac{N+1}{6}}.$$

Recall $exp_{low}(k,t-s)\hat{G}(k,s) = \psi(k)e^{B(k)(t-s)}Q_0(k)\hat{G}(k,s)$, from the splitting of our solution, corresponds to low-wavenumbers, but the eigenvalues away from 0. Therefore we expect the unforced part to decay exponentially, and the forcing term $\int_0^t exp_{low}(k,t-s)\hat{F}(k,s)ds$ to decay at the same rate as $\hat{F}(k,s)$. We proceed with some calculations that are common to the proofs of both Lemmas 4.2.1 and 4.2.2. Recall that the triple norm is a weighted sum of L^2 norms of derivatives with respect to k. Thus, we begin by evaluating the j-th derivative with respect to k:

$$\begin{aligned}&\partial_k^j exp_{low}(k,t-s)\hat{G}(k,s)\\ &\qquad = \sum_{j_1+j_2+j_3=j} C_{j_1,j_2,j_3}\partial_k^{j_1}\psi(k)\partial_k^{j_2}\left(e^{B(k)(t-s)}Q_0(k)\right)\partial_k^{j_3}\hat{G}(k,s).\end{aligned}$$

We can rewrite the middle derivative as

$$\partial_k^{j_2}\left(e^{B(k)(t-s)}Q_0(k)\right) = \sum_{p_1+p_2=j_2} C_{p_1,p_2}\partial_k^{p_1}e^{B(k)(t-s)}\partial_k^{p_2}Q_0(k).$$

We can further rewrite the summand as

$$\left(\partial_k^{p_1} e^{B(k)(t-s)}\right) \partial_k^{p_2} Q_0(k) = \sum_{q=0}^{p_1} (t-s)^q D_q(k) e^{B(k)(t-s)} \partial_k^{p_2} Q_0(k)$$

where the $D_q(k)$ are linear operators which are products of powers of B_1, B_2, and k. We want to use Corollary 4.1.3, which tells us

$$\|e^{B(k)(t-s)} Q_0(k) \hat{W}(k)\|_Y \leq C e^{-\frac{\nu\mu_1}{2}(t-s)} \|\hat{W}(k)\|_Y. \tag{4.32}$$

To use this, we need to rewrite the terms

$$e^{B(k)(t-s)} \partial_k^{p_2} Q_0(k)$$

as a sum of terms, each of which has $e^{B(k)(t-s)} Q_0(k)$ appearing in its expression. Hence, we prove the following lemma:

Lemma 4.2.3. *For each integer $j \geq 0$, the expression for*

$$e^{B(k)(t-s)} \partial_k^j Q_0(k) \tag{4.33}$$

can be written as a sum of terms of the form

$$Left_j(k, t-s) e^{B(k)(t-s)} Q_0(k) Right_j(k, t-s) \tag{4.34}$$

where $Left_j(k, t-s)$ is a bounded, linear operator on Y which is a product of powers of derivatives of $Q_0(k)$, $B(k)$, and $t-s$, and $Right_j(k, t-s)$ is a bounded, linear operator on Y which is a product of powers of derivatives of $Q_0(k)$.

Proof:
We prove this by induction. First, to shorten the notation, we let $Q_0(k) = Q$, $B(k) = B$, and $t - s = r$. The base case we prove as follows. Note that since Q is a projection,

$$QQ = Q.$$

Therefore $\partial_k Q$, which is a bounded operator (since Q is bounded and perturbs smoothly in k), satisfies

$$\partial_k Q = (\partial_k Q)\, Q + Q \partial_k Q,$$

so

$$e^{Br} \partial_k Q = e^{Br}\, (\partial_k Q)\, Q + e^{Br} Q \partial_k Q. \tag{4.35}$$

The second term in this sum is of the form in the lemma. Therefore we need only to deal with the first term. To do so, note that

$$Q e^{Br} = e^{Br} Q. \tag{4.36}$$

Taking ∂_k of both sides, we get

$$(\partial_k Q)\, e^{Br} + Q\, (\partial_k B)\, r e^{Br} = (\partial_k B)\, r e^{Br} Q + e^{Br}\, (\partial_k Q)\,.$$

Solving for $e^{Br} \partial_k Q$, we get

$$e^{Br}\, (\partial_k Q) = (\partial_k Q)\, e^{Br} + Q\, (\partial_k B)\, r e^{Br} - (\partial_k B)\, r e^{Br} Q.$$

Substituting for $e^{Br} \partial_k Q$ appearing in right-hand side of (4.35), we get

$$e^{Br}\, (\partial_k Q) = (\partial_k Q)\, e^{Br} Q + Q\, (\partial_k B)\, e^{Br} Q - (\partial_k B)\, r e^{Br} Q Q + e^{Br} Q\, (\partial_k Q)\,.$$

Therefore, since $\partial_k B = B_1 + 2kB_2$, it follows that $e^{Br} \partial_k Q$ is of the form described in the lemma. Next, we proceed with the induction step. Fix an integer $j > 0$ and assume, for each integer ℓ satisfying $0 \leq \ell \leq j - 1$, that

$$e^{Br} \partial_k^{\ell} Q$$

can be written as a sum of terms of the form

$$Left_{\ell}(k, r) e^{Br} Q Right_{\ell}(k, r)$$

where $Left_{\ell}(k, r)$ is a bounded, linear operator on Y which is a product of powers of derivatives of Q, B, and r, and $Right_{\ell}(k, r)$ is a bounded, linear operator on Y which is a product of powers of derivatives of Q. We proceed by computing $e^{Br} \partial_k^{j} Q$

as follows: Since $QQ = Q$, we have

$$\partial_k^j Q = \sum_{j_1+j_2=j} C_{j_1,j_2} \partial_k^{j_1} Q \partial_k^{j_2} Q.$$

Therefore

$$\begin{aligned} e^{Br}\partial_k^j Q &= \sum_{j_1+j_2=j} C_{j_1,j_2} e^{Br} \partial_k^{j_1} Q \partial_k^{j_2} Q & (4.37) \\ &= e^{Br}\partial_k^j Q Q + \sum_{j_1+j_2=j, j_1 \leq j-1} C_{j_1,j_2} e^{Br} \partial_k^{j_1} Q \partial_k^{j_2} Q. & (4.38) \end{aligned}$$

Notice the sum on the right-hand side contains $e^{Br}\partial_k^{j_1} Q$ where $j_1 \leq j-1$, so by the induction hypothesis, this second sum is of the form described in the lemma. We now deal with the first term, which contains $e^{Br}\partial_k^j Q$. We compute this as follows: recall from (4.36)

$$Qe^{Br} = e^{Br}Q. \tag{4.39}$$

Taking ∂_k^j of both sides, we get

$$\begin{aligned} \sum_{j_1+j_2=j} C_{j_1,j_2} \partial_k^{j_1} Q \partial_k^{j_2} e^{Br} &= \sum_{j_1+j_2=j} C_{j_1,j_2} \partial_k^{j_1} e^{Br} \partial_k^{j_2} Q \\ &= e^{Br}\partial_k^j Q + \sum_{j_1+j_2=j, j_2 \leq j-1} C_{j_1,j_2} \partial_k^{j_1} e^{Br} \partial_k^{j_2} Q. \end{aligned}$$

Solving for $e^{Br}\partial_k^j Q$, we get

$$e^{Br}\partial_k^j Q = \sum_{j_1+j_2=j} C_{j_1,j_2} \partial_k^{j_1} Q \partial_k^{j_2} e^{Br} - \sum_{j_1+j_2=j, j_2 \leq j-1} C_{j_1,j_2} \partial_k^{j_1} e^{Br} \partial_k^{j_2} Q.$$

Notice the summand in the last sum on the right-hand side contains $\partial_k^{j_1} e^{Br} \partial_k^{j_2} Q$. We can rewrite

$$\partial_k^{j_1} e^{Br}$$

as a product of powers of derivatives of B and r times e^{Br}, which is a product of powers of B_1, B_2, k, and r with powers not exceeding j_1. Therefore, since $j_2 \leq j-1$, by the induction hypothesis, $\partial_k^{j_1} e^{Br} \partial_k^{j_2} Q$ can be written as a sum of products of powers of derivatives of B, Q, and r times $e^{Br}Q$ times products of powers of derivatives of Q, as desired. Finally, plugging $e^{Br}\partial_k^j Q$ back into the right-hand side of (4.37) gives us an expression of $e^{Br}\partial_k^j Q$ as described in Lemma 4.2.3. This concludes the proof of Lemma 4.2.3. $\square$

We proceed with calculations for the proofs of Lemmas 4.2.1 and 4.2.2. Due to our lemma, each term in our sum for $\partial_j exp_{low}(k,t-s)\hat{G}(k,s)$ is of the form

$$\partial_k^{j_1}\psi(k)D_q(k)(t-s)^q D_r(k)(t-s)^r e^{B(k)(t-s)}Q_0(k)Right_{p_2}(k)\partial_k^{j_3}\hat{G}(k,s) \tag{4.40}$$

where $D_q(k)$ and $D_r(k)$ are bounded, linear operators which are the products of derivatives of B_1, B_2, Q_0, and k, and the indices satisfy:

- $j_1 + j_2 + j_3 = j$
- $p_1 + p_2 = j_2$
- $q \le p_1$
- $r \le p_2$

We are now in a position to prove Lemma 4.2.1.

Proof of Lemma 4.2.1:

We set $s = 0$ and $\hat{G}(k,0) = \hat{V}(k,0)$ in equation (4.40). Taking the $\|\cdot\|_Y$ norm, we obtain the bound

$$\begin{aligned}\|\partial_j exp_{low}(k,t)\hat{V}(k,0)\|_Y \le C\sum|\partial_k^{j_1}\psi(k)||t|^{q+r}\times\\ \|D_q(k)D_r(k)e^{B(k)(t)}Q_0(k)Right_{p_2}(k)\partial_k^{j_3}\hat{V}(k,0)\|_Y\end{aligned}$$

Where the summation is over the indices in (4.2.1) above. Continuing, we use Corollary 4.1.3 to obtain

$$\|\partial_j exp_{low}(k,t)\hat{V}(k,0)\|_Y \;\; \le \;\; C\sum|\partial_k^{j_1}\psi(k)||t|^{q+r}|k|^{p+r}e^{-\frac{\nu\mu_1}{2}t}\|Right_{p_2}(k)\partial_k^{j_3}\hat{V}(k,0)\|_Y$$

Next, we take the $\|\cdot\|_{L^2(\mathbb{R})}$ and obtain

$$\begin{aligned}\|\|\partial_j exp_{low}(\cdot,t)\hat{V}(\cdot,0)\|_Y\|_{L^2(\mathbb{R})} &\le C|t|^{p+r}e^{-\frac{\nu\mu_1}{2}t}\\ &\le Ce^{-\frac{\nu\mu_1}{4}t}.\end{aligned}$$

Dividing by $(1+t)^{j/2}$ and summing over j, we have

$$|||exp_{low}(\cdot,t)\hat{V}(\cdot,0)||| \leq Ce^{-\frac{\nu\mu_1}{4}t}.$$

This concludes the proof of Lemma 4.2.1. $\square$

Next, we prove Lemma 4.2.2.

Proof of Lemma 4.2.2:

Recall from (4.4) that the forcing function $\hat{F}(k,s)$ is of the form

$$\hat{F}(k,s) = \sum_{i=1}^{5}(1+s)^{M_i/2}k^{M_i}e^{-\nu_T k^2(1+s)}(1+s)^{p_i}D_i\begin{pmatrix}\alpha_J(\log(1+s))\\ \beta_J(\log(1+s))\end{pmatrix},$$

where $M_i = N+1$ or $N+2$, $p_i = -3/2$ or -2, and the D_i are bounded, linear operators on $\mathbb{R}\times\ell^2(\mathbb{R})$. Therefore $\partial_k^{j_3}\hat{F}(k,s)$ is a sum of terms of the form

$$(1+s)^{M_i/2}k^{M_i-\ell_1}\partial_k^{\ell_2}e^{-\nu_T k^2(1+s)}(1+s)^{p_i}D_i\begin{pmatrix}\alpha_J(\log(1+s))\\ \beta_J(\log(1+s))\end{pmatrix}$$

where the indices ℓ_1 and ℓ_2 satisfy $\ell_1+\ell_2 = j_3$. Next, notice that

$$\partial_k^{\ell_2}e^{-\nu_T k^2(1+s)} = (\nu_T(1+s))^{\ell_2/2}H_{\ell_2}(k\sqrt{\nu_T(1+s)})e^{-\nu_T k^2(1+s)},$$

Where H_{ℓ_2} is the Hermite polynomial of order ℓ_2. Therefore $\partial_k^{j_3}\hat{F}(k,s)$ is a sum of terms of the form

$$(1+s)^{M_i/2}k^{M_i-\ell_1}(1+s)^{\ell_2/2}k^{\ell_3}(1+s)^{\ell_3/2}e^{-\nu_T k^2(1+s)}(1+s)^{p_i}D_i\begin{pmatrix}\alpha_J(\log(1+s))\\ \beta_J(\log(1+s))\end{pmatrix}$$

where $\ell_3 \leq \ell_2$. Therefore we have the following bound on the $\|\cdot\|_Y$ norm of $exp_{low}(k,t-s)\hat{F}(k,s)$:

$$\begin{aligned}&\|exp_{low}(k,t-s)\hat{F}(k,s)\|_Y\\ \leq C|\partial_k^{j_1}\psi(k)||(1+t-s)^{j_2}e^{-\frac{\nu\mu_1}{2}(t-s)}&(1+s)^{M_i/2+\ell_2/2+\ell_3/2+p_i}k^{M_i-\ell_1+\ell_3}\\ e^{-\nu_T k^2(1+s)}(|\alpha_J(\log(1+s)|&+\|\beta_J(\log(1+s))\|_{\ell^2(\mathbb{R})}|.\end{aligned}$$

This bound follows from the conditions above on the indices in (4.2.1), and Corollary 4.1.3. Continuing, we compute the $\|\cdot\|_{L^2}$ norm of this bound, using the fact that

$\|k^d e^{-k^2 T}\|_{L^2} \leq CT^{-d/2-1/4}$:

$$\begin{aligned} & \| \|exp_{low}(k, t-s)\hat{F}(k,s)\|_Y\|_{L^2} \\ \leq C(1+t-s)^{j_2} e^{-\frac{\nu\mu_1}{2}(t-s)}(1+s)^{M_i/2+\ell_2/2+\ell_3/2+p_i}&(1+s)^{-\frac{1}{2}(M_i-\ell_1+\ell_3)-1/4}(1+s)^{-J/6}. \end{aligned}$$

Notice the $(1+s)^{-J/6}$ follows from the bounds on α_J and β_J in section 2. Continuing, we simplify and get

$$\| \|exp_{low}(k,t-s)\hat{F}(k,s)\|_Y\|_{L^2} \leq C(1+t-s)^{j_2} e^{-\frac{\nu\mu_1}{2}(t-s)}(1+s)^{j_3/2-J/6+p_i-1/4}.$$

Next, observe that $(1+t-s)^{j_2} e^{-\frac{\nu\mu_1}{2}(t-s)} \leq Ce^{-\frac{\nu\mu_1}{4}(t-s)}$. With this observation, we continue by integrating from $s=0$ to $s=t$:

$$\begin{aligned} \| \|exp_{low}(k,t-s)\hat{F}(k,s)\|_Y\|_{L^2} &\leq C\int_0^t e^{-\frac{\nu\mu_1}{4}(t-s)}(1+s)^{j_3/2-J/6+p_i-1/4} ds \\ &\leq C(1+t)^{j_3/2-J/6+p_i-1/4}. \end{aligned}$$

Finally dividing by $(1+t)^{j/2}$ as in the definition of the triple norm, we have the bound

$$||| \int_0^t exp_{low}(k,t-s)\hat{F}(k,s)ds||| \leq C(1+t)^{-J/6+p_i-1/4}.$$

Since $J \geq N$ and $p_i \leq -3/2$, this bound can be written in terms of N:

$$||| \int_0^t exp_{low}(k,t-s)\hat{F}(k,s)ds||| \leq C(1+t)^{-\frac{N+1}{6}-3/4}.$$

Therefore

$$\begin{aligned} ||| \int_0^t exp_{low}(k,t-s)\hat{F}(k,s)ds||| &\leq \int_0^t |||exp_{low}(k,t-s)\hat{F}(k,s)|||ds \\ &\leq C(1+t)^{-\frac{N+1}{6}-3/4}. \end{aligned}$$

Hence, Lemma 4.2.2 is proven. □

4.2.2 Exponentially decaying terms - high wavenumbers

The goal of this section is to give estimates on

$$exp_{high}(k,t-s)\hat{G}(k,s) = (1-\psi(k))e^{B(k)(t-s)}\hat{G}(k,s) \tag{4.41}$$

in the triple norm $|||\cdot|||$. Specifically, we aim to establish the following two lemmas:

Lemma 4.2.4.

$$|||exp_{high}(\cdot,t)\hat{V}(\cdot,0)||| \leq Ce^{-\frac{\nu}{4}k_0^2 t}$$

Lemma 4.2.5.

$$|||\int_0^t exp_{high}(\cdot,t-s)\hat{F}(\cdot,s)ds||| \leq Ce^{-\frac{\nu}{4}k_0^2 t}$$

Recall that this term, in the splitting of our solution, comes from separating the low- and high-wavenumbers, and from the high-wavenumber bound on the spectrum $Re(\lambda(k)) \leq -\nu k_0^2 < 0$, we expect this term to decay exponentially. We proceed with a calculation used in the proofs of both Lemmas 4.2.4 and 4.2.5. Using the definition of $|||\cdot|||$, we have to:

- compute ∂_k^j,
- compute the $\|\cdot\|_Y$ norm,
- compute the $\|\cdot\|L^2$ norm,
- and finally divide by $(1+t)^{j/2}$ and sum from $j=0$ to $j=m$.

First computing ∂_k^j of both sides of (4.41), we have

$$\partial_k^j exp_{high}(k,t-s)\hat{G}(k,s) = \sum C_{j_1,j_2,j_3}\partial_k^{j_1}(1-\psi(k))\partial_k^{j_2}(e^{B(k)(t-s)})\partial_k^{j_3}\hat{G}(k,s),$$

where the sum is taken over indices j_1, j_2, and j_3 satisfying $j_1+j_2+j_3=j$. Proceeding, we substitute for $\partial_k^{j_2}e^{B(k)(t-s)}$:

$$\partial_k^{j_2}e^{B(k)(t-s)} = \sum_{q\leq j_2} D_q(k)(t-s)^q e^{B(k)(t-s)}$$

where, for each k, the $D_q(k)$ are bounded, linear operators satisfying $\|D_q(k)\| \leq$

$C(1+|k|)^{j_2}$ for all $q \le j_2$. Next, we bound the $\|\cdot\|_Y$ norm:

$$\begin{aligned}
&\|\partial_k^j exp_{high}(k,t-s)\hat{G}(k,s)\|_Y \le \\
&C\sum(t-s)^q\|D_q(k)\partial_k^{j_1}(1-\psi(k))e^{B(k)(t-s)}\partial_k^{j_3}\hat{G}(k,s)\|_Y \\
&\le C\sum(t-s)^q(1+|k|)^{j_2}\|\partial_k^{j_1}(1-\psi(k))e^{B(k)(t-s)}\partial_k^{j_3}\hat{G}(k,s)\|_Y \\
&\le C\sum(t-s)^{j_2}(1+|k|)^{j_2}|\partial_k^{j_1}(1-\psi(k))|e^{-\nu k^2(t-s)}\|\partial_k^{j_3}\hat{G}(k,s)\|_Y,
\end{aligned}$$

where we have used Corollary 4.1.6. Therefore

$$\begin{aligned}
\|\partial_k^j exp_{high}(k,t-s)\hat{G}(k,s)\|_Y \le C\sum(t-s)^{j_2}(1+|k|)^{j_2}\times \\
|\partial_k^{j_1}(1-\psi(k))|e^{-\nu k^2(t-s)}\|\partial_k^{j_3}\hat{G}(k,s)\|_Y.
\end{aligned} \tag{4.42}$$

We are now in a position to prove Lemma 4.2.4.

Proof of Lemma 4.2.4: We set $s=0$ and $\hat{G}(k,s)=\hat{V}(k,0)$ in (4.42) and bound the $L^2(\mathbb{R})$ norm:

$$\begin{aligned}
&\|\|\partial_k^j exp_{high}(k,t-s)\hat{G}(k,s)\|_Y\|_{L^2} \\
&\le C(1+t)^j\|(1+|k|)^j e^{-\frac{\nu}{2}k^2t}\partial_k^{j_1}(1-\psi(k))\|_{L^2}\sup_{|k|\ge k_0} e^{-\frac{\nu}{2}k^2t} \\
&\le C(1+t)^j e^{-\frac{\nu}{2}k_0^2t}.
\end{aligned}$$

Hence, after dividing by $(1+t)^{j/2}$ and summing over j as in the definition of $|||\cdot|||$, we have

$$|||exp_{high}(k,t)\hat{V}(k,0)||| \le C(1+t)^{j/2}e^{-\frac{\nu}{2}k_0^2t} \le Ce^{-\frac{\nu}{4}k_0^2t}.$$

Therefore Lemma 4.2.4 is proven. □

We are now in a position to prove Lemma 4.2.5.

Proof of Lemma 4.2.5:

Let $s \ge 0$ and take $\hat{G}(k,s)=\hat{F}(k,s)$ in (4.42). Recall from (4.4) that

$$\hat{F}(k,s) = \sum_{i=1}^{5}(1+s)^{M_i/2}k^{M_i}e^{-\nu_T k^2(1+s)}(1+s)^{p_i}D_i\begin{pmatrix}\alpha_{J_i}(\log(1+s)) \\ \beta_{J_i}(\log(1+s)).\end{pmatrix} \tag{4.43}$$

Therefore we can compute $\partial_k^{j_3}\hat{F}(k,s)$:

$$\begin{aligned}\partial_k^{j_3}\hat{F}(k,s) \qquad\qquad &= \textstyle\sum_{i=1}^5 C_{\ell_1,\ell_2}(1+s)^{M_i/2}k^{M_i-\ell_1}\times \\ &\partial_k^{\ell_2}\left(e^{-\nu_T k^2(1+s)}\right)(1+s)^{p_i}D_i\begin{pmatrix}\alpha_{J_i}(\log(1+s)) \\ \beta_{J_i}(\log(1+s))\end{pmatrix},\end{aligned}$$

where the indices ℓ_1 and ℓ_2 satisfy $\ell_1+\ell_2=j_3$. We can further substitute for $\partial_k^{\ell_2}\left(e^{-\nu_T k^2(1+s)}\right)$:

$$\partial_k^{\ell_2}\left(e^{-\nu_T k^2(1+s)}\right)=\sum_{\ell_3\le\ell_2}C_{\ell_3}(1+s)^{\ell_3/2}k^{\ell_3}e^{-\nu_T k^2(1+s)}.$$

Therefore

$$\begin{gathered}\|\partial_k^j exp_{high}(k,t-s)\hat{F}(k,s)\|_Y \\ \le C\sum|\partial_k^{j_1}(1-\psi(k))|(1+t-s)^{j_2}(1+|k|)^{j_2}e^{-\nu k^2(t-s)} \\ \times(1+s)^{M_i/2+\ell_3/2}k^{M_i-\ell_1+\ell_3}e^{-\nu_T k^2(1+s)} \\ \times\left(|\alpha_{J_i}(\log(1+s))|+\|B_{J_i}\log(1+s)\|\right).\end{gathered}$$

Observe that $e^{-\nu_T k^2(1+s)}\le e^{-\nu k^2(1+s)}$, so that we can bound the exponential terms by $e^{-\nu k^2(1+t)}$. We can then further rewrite $e^{-\nu k^2(1+t)}=e^{-\frac{\nu}{2}k^2(1+t)}e^{-\frac{\nu}{2}k^2(1+t)}$, and then bound the L^2 norm:

$$\begin{aligned}\|\|exp_{high}(k,t-s)\hat{F}(k,s)\|_Y\|_{L^2} \;\;&\le\;\; C(1+t-s)^{j_2}(1+t)^{-\frac{1}{2}(M_i/2-\ell_1/2+\ell_3/2)-1/4} \\ &\times\;\; (1+s)^{M_i/2+p_i+\ell_3/2-J_i/6}\sup_{|k|\ge k_0}e^{-\frac{\nu}{2}k^2(1+t)}.\end{aligned}$$

Integrating from $s=0$ to $s=t$, we obtain

$$|||\int_0^t exp_{high}(\cdot,t-s)\hat{F}(\cdot,s)ds|||\le Ce^{-\frac{\nu}{4}k_0^2(1+t)}.$$

Therefore

$$\begin{aligned}\int_0^t |||exp_{high}(\cdot,t-s)\hat{F}(\cdot,s)|||ds &\le |||\int_0^t exp_{high}(\cdot,t-s)\hat{F}(\cdot,s)ds||| \\ &\le Ce^{-\frac{\nu}{4}k_0^2(1+t)}.\end{aligned}$$

This concludes the proof of Lemma 4.2.5.

4.2.3 Algebraically decaying remainder terms

The goal of this section is to give estimates on the Remainder terms

$$\begin{aligned} & Rem(k,t-s)\hat{G}(k,s) = \\ & e^{-\nu_T k^2(t-s)}\times \\ & \int_0^k \int_0^{k1} \cdots \int_0^{k_N} \partial_{k_{N+1}}^{N+1}(e^{\Lambda_0(k_{N+1})(t-s)}P_0(k_{N+1})\hat{G}(k_{N+1},s)\psi(k))dk_{N+1}\dots dk_1. \end{aligned} \tag{4.44}$$

in the triple norm $|||\cdot|||$. Specifically, we establish the two lemmas:

Lemma 4.2.6. *Let $Rem(k,t-s)\hat{G}(k,s)$ be defined as in (4.29). Let $s=0$ and take $\hat{G}(k,s) = \hat{V}(k,0)$. Then*

$$|||Rem(\cdot,t)\hat{V}(\cdot,0)||| \leq C(1+t)^{\frac{-N+1}{6}-\frac{1}{4}} \tag{4.45}$$

for all $t>0$.

Lemma 4.2.7. *Let $Rem(k,t-s)\hat{G}(k,s)$ be defined as in (4.29). Let $s \geq 0$ and take $\hat{G}(k,s) = \hat{F}(k,s)$. Then*

$$|||\int_0^t Rem(\cdot,t-s)\hat{F}(\cdot,s)ds||| \leq C(1+t)^{\frac{-N}{6}+\frac{5}{4}} \tag{4.46}$$

for all $t>0$.

We start by introducing some notation to shorten the involved expressions, and then we will proceed with some calculations common to the proofs of both of these lemmas. First, we denote

$$\mathcal{I}\left(H(\cdot,r)\right)(k) = \int_0^k H(\tilde{k},r)d\tilde{k}.$$

where $H(k,r) \in Y$, is smooth and compactly supported in k. With a slight abuse of notation we write

$$\mathcal{I}\left(H(\cdot,r)\right)(k) = \mathcal{I}\left(H(k,r)\right).$$

In this notation,

$$\begin{aligned}
&Rem(k, t-s)\hat{G}(k,s) = \\
&e^{-\nu_T k^2(t-s)} \times \\
&\mathcal{I}^{N+1}\left(\partial_k^{N+1}(e^{\Lambda_0(k)(t-s)}P_0(k)\hat{G}(k,s)\psi(k))\right)
\end{aligned}$$

To estimate in the norm $|||\cdot|||$, we have to:

- compute ∂_k^j,
- compute the $\|\cdot\|_Y$ norm,
- compute the $\|\cdot\|L^2$ norm,
- and finally divide by $(1+t)^{j/2}$ and sum from $j=0$ to $j=m$.

We start by computing. $\partial_k^j Rem(k,t-s)\hat{G}(k,s)$, From the product rule, $\partial_k^j Rem(k,t-s)\hat{G}(k,s)$ is a sum of terms of the form

$$\partial_k^{j_1} e^{-\nu_T k^2(t-s)} \mathcal{I}^{N+1-j_2} \partial_k^{N+1}(e^{\Lambda_0(k)(t-s)}P_0(k)\hat{G}(k,s)\psi(k))$$

where $j_1+j_2=j$. Next, note that $\partial_k^{j_1} e^{-\nu_T k^2(t-s)}$ is a sum of terms of the form

$$k^p(t-s)^{\frac{p}{2}} e^{-\nu_T k^2(t-s)}$$

where $0 \le p \le j_1$. Also note that, by the product rule, $\partial_k^{N+1}(e^{\Lambda_0(k)(t-s)}P_0(k)\hat{G}(k,s)\psi(k))$ is a sum of terms of the form

$$(\partial_k^{m_1} e^{\Lambda_0(k)(t-s)})(\partial_k^{m_2}P_0(k))(\partial_k^{m_3}\hat{G}(k,s))(\partial_k^{m_4}\psi(k))$$

where $m_1+m_2+m_3+m_4 = N+1$. Therefore $\partial_k^j Rem(k,t-s)\hat{G}(k,s)$ is a sum of terms of the form

$$k^p(t-s)^{\frac{p}{2}} e^{-\nu_T k^2(t-s)} \mathcal{I}^{N+1-j_2}\left(\partial_k^{m_1}(e^{\Lambda_0(k)(t-s)})\partial_k^{m_2}(P_0(k))\partial_k^{m_3}(\hat{G}(k,s))\partial_k^{m_4}(\psi(k))\right) \tag{4.47}$$

where the indices satisfy

- $j_1+j_2=j$,

- $0 \leq p \leq j_1$,
- and $m_1 + m_2 + m_3 + m_4 = N + 1$.

To establish Lemmas 4.2.6 and 4.2.7, for appropriate choices of $\hat{G}(k,s)$, two of the required steps are to take the $\|\cdot\|_Y$ norm, of $\partial_k^j Rem(k,t-s)\hat{G}(k,s)$ and then take the $\|\cdot\|_{L^2}$ norm. We will use the estimate

$$\|k^M e^{-ak^2T}\|_{L^2} \leq C(aT)^{-\frac{M}{2}-\frac{1}{4}}. \tag{4.48}$$

This estimate tells us that powers of k times Gaussians decay faster with higher power. We eventually need to take the L^2 norm of the expression in (4.47), which has the Gaussian factor $e^{-\nu_T k^2(t-s)}$ and some powers of k. However, some additional powers of k and also some additonal powers of $t-s$ will appear when computing $\partial_k^{m_1}(e^{\Lambda_0(k)(t-s)})$. This is due to the fact that Lemma 4.1.2 tells us $\Lambda_0(k)(t-s) = irk^3 + \mathcal{O}(k^4)$. Therefore, if we are bounding the L^2 norm of the expression in (4.47), we need to know what the powers of k are relative to the powers of $t-s$. In fact, $\partial_k^{m_1}(e^{\Lambda_0(k)(t-s)})$ is a sum of terms of the form

$$(k^2(t-s))^{\rho_1}(k(t-s))^{\rho_2}(t-s)^{\rho_3}e^{\Lambda_0(k)(t-s)} \tag{4.49}$$

where $\rho_1 + 2\rho_2 + 3\rho_3 = m_1$. Therefore we note the following Lemma:

Lemma 4.2.8. *Let $Rem(k,t-s)\hat{G}(k,s)$ be as defined in* (4.29)*, and let the indices j_1, j_2, p, m_1, m_2, m_3, m_4 satisfy the conditions in* (4.47)*, and let $\rho_1+2\rho_2+3\rho_3 = m_1$. Then $\|\partial_k^j Rem(k,t-s)\hat{G}(k,s)\|_Y$ is bounded above by a sum of terms of the form*

$$|k|^p(t-s)^{\frac{p}{2}}e^{-\nu_T k^2(t-s)}\||k|^{N+1-j_2}\|h(k,t-s)(\partial_k^{m_3}\hat{G}(k,s))\|_Y,$$

where $h(k,t-s)$ takes values in Y, is smooth and compactly supported in k, and satisfies

$$\|h(k,t-s)\|_Y \leq (k^2(t-s))^{\rho_1}(k(t-s))^{\rho_2}(t-s)^{\rho_3}. \tag{4.50}$$

Proof:

The power of $|k|$ comes from bounding $\mathcal{I}^{N+1-j_2}$. Each $h(k,t-s)$ in the sum is $(\partial_k^{m_2}P_0(k))(\partial_k^{m_4}\psi(k))$ multiplied by a term from the derivative of $e^{\Lambda_0(k)(t-s)}$. Hence, each $h(k,t-s)$ is smooth and compactly supported in k. Furthermore, the bound on

$\|h(k,t-s)\|_Y$ comes from the form of the terms in the derivatives of $e^{\Lambda_0(k)(t-s)}$ noted before the Lemma. $\square$

We are now in a position to prove Lemma 4.2.6.

Proof of Lemma 4.2.6:

Let $s=0$ and take $\hat{G}(k,s)=\hat{V}(k,0)$. From Lemma 4.2.8, we have that $\|\partial_k^j Rem(k,t)\hat{V}(k,0)\|_Y$ is bounded above by a sum of terms of the form

$$|k|^p(1+t)^{\frac{p}{2}}e^{-\nu_T k^2(t)}|k|^{N+1-j_2}\|h(k,t)\partial_k^{m_3}\hat{V}(k,0)\|_Y.$$

Therefore $\|\|\partial_k^j Rem(k,t)\hat{V}(k,0)\|_Y\|_{L^2}$ is bounded above by a sum of terms of the form

$$(1+t)^{\frac{p}{2}}(t)^{\rho_1+\rho_2+\rho_3}\ \|\,|k|^{p+N+1-j_2+2\rho_1+\rho_2}e^{-\nu_T k^2(t)}\|_{L^2}.$$

We now apply the estimate (4.48) and find that $\|\|\partial_k^j Rem(k,t)\hat{V}(k,0)\|_Y\|_{L^2}$ is bounded above by a sum of terms of the form

$$(1+t)^{pow}$$

where

$$\begin{aligned} pow &= \frac{p}{2}+\rho_1+\rho_2+\rho_3-\frac{1}{2}\left(p+N+1-j_2+2\rho_1+\rho_2\right) \\ pow &= \frac{\rho_2}{2}+\rho_3-\frac{N+1}{2}+\frac{j_2}{2}-\frac{1}{4}. \end{aligned}$$

Recall that $\rho_1+2\rho_2+3\rho_3=m_1$. Therefore $2\rho_2+3\rho_3\le m_1$, and since

$$\frac{\rho_2}{2}+\rho_3=\frac{1}{3}\left(\frac{3}{2}\rho_2+3\rho_3\right)\le 2\rho_2+3\rho_3,$$

we have that $\frac{\rho_2}{2}+\rho_3\le m_1$. Hence,

$$pow\le\frac{m_1}{3}-\frac{N+1}{2}+\frac{j_2}{2}-\frac{1}{4}.$$

Next, since $m_1 \leq N+1$, we have

$$pow \leq \frac{N+1}{3} - \frac{N+1}{2} + \frac{j_2}{2} - \frac{1}{4},$$

from which we conclude

$$pow \leq -\frac{N+1}{6} + \frac{j_2}{2} - \frac{1}{4}.$$

Therefore

$$\|\|\partial_k^j Rem(k,t)\hat{V}(k,0)\|_Y\|_{L^2} \leq C(1+t)^{-\frac{N+1}{6}+\frac{j_2}{2}-\frac{1}{4}}.$$

Next, we divide by $(1+t)^{\frac{j}{2}}$ and use the fact that $j_2 \leq j$ to obtain

$$\frac{1}{(1+t)^{\frac{j}{2}}}\|\|\partial_k^j Rem(k,t)\hat{V}(k,0)\|_Y\|_{L^2} \leq C(1+t)^{-\frac{N+1}{6}-\frac{1}{4}}.$$

Finally, summing from $j=0$ to $j=m$, we conclude that

$$|||Rem(\cdot,t)\hat{V}(\cdot,0)||| \leq C(1+t)^{-\frac{N+1}{6}-\frac{1}{4}}.$$

Therefore Lemma 4.2.6 is proven. □

Next, we prove Lemma 4.2.7.

Proof of Lemma 4.2.7:

Let $s \geq 0$ and set $\hat{G}(k,s) = \hat{F}(k,s)$. From Lemma 4.2.8, $\|\partial_k^j Rem(k,t-s)\hat{F}(k,s)\|_Y$ is bounded by a sum of terms of the form

$$|k|^p(t-s)^{\frac{p}{2}}e^{-\nu_T k^2(t-s)}\||k|^{N+1-j_2}\|h(k,t-s)\partial_k^{m_3}(\hat{G}(k,s)\|_Y,$$

Recall from (4.4) that $\hat{F}(k,s)$ is a sum of terms of the form

$$(1+s)^p(1+s)^{(N+2)/2}k^{N+2}e^{-\nu_T k^2(1+s)}Y(\begin{pmatrix}\alpha_N(\log(1+s))\\ \{\beta_N^n(\log(1+s))\}_{n=1}^{\infty}\end{pmatrix}).$$

where $0 \leq p \leq \frac{1}{2}$ and Y is a bounded linear operator. Therefore $\partial_k^{m_3}Rem(k,t-s)\hat{F}(k,s)$ is a sum of terms of the form

$$(1+s)^{\frac{1}{2}}(1+s)^{(N+2)/2}k^{N+2-q_1}k^d(1+s)^{\frac{d}{2}}e^{-\nu_T k^2(1+s)}Y(\begin{pmatrix}\alpha_N(\log(1+s))\\ \{\beta_N^n(\log(1+s))\}_{n=1}^{\infty}\end{pmatrix}).$$

where $m_3 = q_1 + q_2$, and $d \le q_2$. Therefore $\|\partial_k^{m_3} \hat{F}(k,s)\|_Y$ is bounded by a sum of terms of the form

$$(1+s)^{\frac{1}{2}}(1+s)^{(N+2)/2+\frac{d}{2}}|k|^{N+2-q_1+d}e^{-\nu_T k^2(1+s)}(1+s)^{-\frac{N}{6}},$$

where we have used the results of Chapter 3 to bound $|\alpha_N(\log(1+s))|$ and $\|\{\beta_N^n(\log(1+s))\}_{n=1}^{\infty}\|_{\ell^2}$. Combining this bound with the bound from Lemma 4.2.8, we have that $\|\|\partial_k^j Rem(k,t-s)\hat{F}(k,s)\|_Y\|_{L^2}$ is bounded by a sum of terms of the form

$$(t-s)^{\frac{p}{2}}(1+s)^{\frac{1}{2}+\frac{N+2}{2}+\frac{d}{2}-\frac{N}{6}}(t-s)^{\rho_1+\rho_2+\rho_3}\||k|^{p+N+1-j_2+2\rho_2+\rho_1+N+2-q_1+d}e^{-\nu_T k^2(t-s)}\|_{L^2}.$$

Employing the estimate (4.48), we have that $\|\|\partial_k^j Rem(k,t-s)\hat{F}(k,s)\|_Y\|_{L^2}$ is bounded by a sum of terms of the form

$$(t-s)^{\frac{p}{2}}(1+s)^{\frac{1}{2}+\frac{N+2}{2}+\frac{d}{2}-\frac{N}{6}}(t-s)^{\rho_1+\rho_2+\rho_3}(t-s)^{-\frac{1}{2}(p+N+1-j_2+2\rho_2+\rho_1+N+2-q_1+d)-\frac{1}{4}}.$$

Integrating from $s=0$ to $s=t$, we find that

$$\int_0^t \|\|\partial_k^j Rem(k,t-s)\hat{F}(k,s)\|_Y\|_{L^2} ds \le C(1+t)^{pow}$$

where pow satisfies

$$\begin{aligned} pow &= \frac{1}{2} + \frac{p}{2} + 1 + \frac{N+2}{2} + \frac{d}{2} - \frac{N}{6} + \rho_1 + \rho_2 + \rho_3 \\ &\quad - \frac{1}{2}(p + N + 1 - j_2 + 2\rho_2 + \rho_1 + N + 2 - q_1 + d) - \frac{1}{4} \\ &= -\frac{N}{6} - \frac{N+1}{2} + \frac{1}{2}\rho_2 + \rho_3 + \frac{q_1}{2} + \frac{j_2}{2} - \frac{1}{4} + 1 + \frac{1}{2}. \end{aligned}$$

Recall that from the proof of Lemma 4.2.6 that $\frac{1}{2}\rho_2 + \rho_3 \le \frac{m_1}{3}$. Therefore

$$pow \le -\frac{N}{6} - \frac{N+1}{2} + \frac{m_1}{3} + \frac{q_1}{2} + \frac{j_2}{2} - \frac{1}{4} + 1 + \frac{1}{2}.$$

Recall that $q_1 + q_2 = m_3$. Therefore $q_1 \leq m_3$, and hence

$$\begin{aligned}\frac{m_1}{3} + \frac{q_1}{2} &\leq \frac{m_1}{3} + \frac{m_3}{2}\\ &\leq \frac{1}{2}(m_1 + m_3)\\ &\leq \frac{N+1}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}pow &\leq -\frac{N}{6} - \frac{N+1}{2} + \frac{N+1}{2} + \frac{j_2}{2} - \frac{1}{4} + 1 + \frac{1}{2}.\\ &= -\frac{N}{6} + \frac{j_2}{2} - \frac{1}{4} + 1 + \frac{1}{2}.\end{aligned}$$

Therefore, dividing by $(1+t)^{\frac{j}{2}}$ and summing from $j = 0$ to $j = m$, we have that

$$\sum_{j=0}^{m} \frac{1}{(1+t)^{\frac{j}{2}}} \int_0^t \| \|\partial_k^j Rem(k, t-s)\hat{F}(k,s)\|_Y \|_{L^2} ds \leq C(1+t)^{-\frac{N}{6}+\frac{5}{4}}.$$

Therefore

$$\begin{aligned}||| \int_0^t Rem(\cdot, t-s)\hat{F}(\cdot, s) ds||| &\leq \int_0^t |||Rem(\cdot, t-s)\hat{F}(\cdot, s)||| ds\\ &\leq C(1+t)^{-\frac{N}{6}+\frac{5}{4}}\end{aligned}$$

and hence Lemma 4.2.7 is proven.

4.2.4 Taylor polynomial terms

In this section we show that the Taylor polynomial terms are actually zero. We claim the following Lemma:

Lemma 4.2.9. *Let $T_N(k, t-s)\hat{G}(k,s)$ be defined as in (4.28). Then if $s = 0$ and $\hat{G}(k,s) = \hat{V}(k,0)$, or if $s \geq 0$ and $\hat{G}(k,s) = \hat{F}(k,s)$, we have that*

$$T_N(k, t-s)\hat{G}(k,s) = 0$$

for all k.

Proof:

Recall that

$$T_N(k,t-s)\hat{G}(k,s) = \sum_{\ell=0}^{N} \frac{1}{\ell!}\partial_k^\ell \left(\psi(k)e^{\Lambda_0(k)(t-s)}P_0(k)\hat{G}(k,s)\right)|_{k=0}k^\ell. \tag{4.51}$$

In this expression, some derivatives fall on $\hat{G}(k,s)$, but the order of these derivatives does not exceed N. We will show these derivatives are zero, and hence the entire expression is zero. There are two cases for $\hat{G}(k,s)$: either $s=0$ and $\hat{G}(k,s) = \hat{V}(k,0)$, or $\hat{G}(k,s) = \hat{F}(k,s)$. In the first case, the components of $\hat{V}(k,0)$ are the Fourier transforms of the quantities $u_n^s(\xi,0) \in RanQ_N$ from 3. It is a property of the projection Q_N that

$$\int \xi^\ell u_n^s(\xi,0)d\xi = 0$$

for all $0 \le \ell \le N$. Taking the Fourier transform gives us

$$\partial_k^\ell \hat{V}_n(k,0)|_{k=0} = 0$$

for $0 \le \ell \le N$ as desired. In the second case, note that

$$\hat{F}(k,s) = k^{N+1}\hat{H}(k,s)$$

where $\hat{H}(k,s)$ is a smooth, bounded function in k and s. This fact can be seen from equation (4.4). Therefore

$$\partial_k^\ell \hat{G}(k,s)|_{k=0} = 0$$

for $0 \le \ell \le N$, so in both cases, the Taylor polynomial term is exactly zero. This concludes the proof of Lemma 4.2.9. □

4.2.5 Conclusion of Chapter: Proof of Proposition 4.0.1

Recall that the goal of this chapter is to prove Proposition 4.0.1. Hence, we want to establish the estimate

$$|||\hat{V}(\cdot,t)||| \le C(1+t)^{-\frac{N}{6}+\frac{5}{4}}.$$

Proof of Proposition 4.0.1:
Recall from (4.5) that

$$\hat{V}(k,t) = e^{B(k)t}\hat{V}(k,0) + \int_0^t e^{B(k)(t-s)}\hat{F}(k,s)ds.$$

Computing the triple norm $|||\cdot|||$ of both sides, we have the bound

$$|||\hat{V}(k,t)||| \le |||e^{B(k)t}\hat{V}(k,0)||| + |||\int_0^t e^{B(k)(t-s)}\hat{F}(k,s)ds|||$$

Recall from the splitting (4.30) that

$$\begin{aligned} e^{B(k)(t-s)}\hat{G}(k,s) =& exp_{high}(k,t-s)\hat{G}(k,s) + exp_{low}(k,t-s)\hat{G}(k,s) \\ &+ T_N(k,t-s)\hat{G}(k,s) + Rem_N(k,t-s)\hat{G}(k,s). \end{aligned}$$

Therefore, for $s=0$ and $\hat{G}(k,s) = \hat{V}(k,0)$, we have

$$\begin{aligned} |||e^{B(k)t}\hat{V}(k,0)||| \le& |||exp_{high}(k,t)\hat{V}(k,0)||| + |||exp_{low}(k,t)\hat{V}(k,0)||| \\ &+ |||T_N(k,t)\hat{V}(k,0)||| + |||Rem_N(k,t)\hat{V}(k,0)|||. \end{aligned}$$

Applying Lemmas 4.2.4, 4.2.1, 4.2.9, and 4.2.6, we have

$$\begin{aligned} |||e^{B(k)t}\hat{V}(k,0)||| \le C(&e^{-\frac{\nu}{4}k_0^2 t} + e^{-\frac{\nu\mu_1}{4}k_0^2 t} \\ &+ (1+t)^{-\frac{N+1}{6}}). \end{aligned}$$

Therefore

$$|||e^{B(k)t}\hat{V}(k,0)||| \le C(1+t)^{-\frac{N+1}{6}}. \tag{4.52}$$

Next, we bound the Duhamel term:

$$\begin{aligned}|||\int_0^t e^{B(k)(t-s)}\hat{F}(k,s)ds||| \leq &||| \int_0^t exp_{high}(k,t-s)\hat{F}(k,s)ds|||\\ &+ |||\int_0^t exp_{low}(k,t-s)\hat{F}(k,s)ds|||\\ &+ |||\int_0^t T_N(k,t-s)\hat{F}(k,s)ds|||\\ &+ |||\int_0^t Rem_N(k,t-s)\hat{F}(k,s)ds|||\end{aligned}$$

Applying Lemmas 4.2.5, 4.2.2, 4.2.9, and 4.2.7, we have

$$|||\int_0^t e^{B(k)(t-s)}\hat{F}(k,s)ds||| \leq C(e^{-\frac{\nu}{4}k_0^2 t} + (1+t)^{-\frac{N+1}{6}-\frac{3}{4}} + (1+t)^{-\frac{N}{6}+\frac{5}{4}}).$$

Therefore

$$|||\int_0^t e^{B(k)(t-s)}\hat{F}(k,s)ds||| \leq C(1+t)^{-\frac{N}{6}+\frac{5}{4}}. \tag{4.53}$$

Finally, combining the bounds (4.52) and (4.53), we have

$$\begin{aligned}|||\hat{V}(\cdot,t)||| &\leq C\left((1+t)^{-\frac{N+1}{6}} + (1+t)^{-\frac{N}{6}+\frac{5}{4}}\right)\\ &\leq C(1+t)^{-\frac{N}{6}+\frac{5}{4}}.\end{aligned}$$

Therefore Proposition 4.0.1 is proven. By undoing the Fourier Transform, and reapplying scaling variables, we obtain the following Corollary:

Corollary 4.2.10. *Let* $(w_0^s(\xi,\tau), \{u_n^s(\xi,\tau)\}_{n=1}^\infty)$ *be the solutions to* (4.1)*. Then*

$$\|w_0^s(\xi,\tau)\|_{L^2(m)} + \|\|\{u_n^s(\xi,\tau)\}_{n=1}^\infty\|_{\ell^2}\|_{L^2(m)} \leq Ce^{(-\frac{N}{6}+1+\frac{1}{2})\tau}$$

In the final chapter we combine the results of Chapters 3 and 4 to prove the main theorem.

Chapter 5

Proof of Main Theorem

In this final chapter we state and prove the main theorem:

Theorem 5.0.1. *Given any $M > 0$, let $N \geq 6M + 9$, and let $m > N + 1/2$. Suppose the initial value $u(x, y, z, 0)$ of (1.3) satisfies $\| \| u(\cdot, 0) \|_{L^2(\Omega)} \|_{L^2(m)} < \infty$. Further assume that for $n \geq 1$, the coefficients $u_n(x, 0)$ in the eigenfunction expansion (1.4) satisfy the zero average condition $\int u_n(x, 0) dx = 0$. Furthermore, suppose $\nu < min\{\frac{8A\|\chi\|_{L^\infty(\Omega)}}{\mu_1}, 1\}$. Then there exists a constant $C = C(N, m, \nu, A, \chi)$ such that*

$$\|w_0(\xi, \tau) - \sum_{k=0}^{N} \alpha_k \phi_k(\xi)\|_{L^2(m)} + \| \| \{u_n(\xi, \tau) - \sum_{k=0}^{N} \beta_k^n(\tau) \phi_k(\xi)\}_{n=1}^{\infty} \|_{\ell^2} \|_{L^2(m)} \leq C e^{(-\frac{N}{6}+1+\frac{1}{2})\tau}$$

for all τ sufficiently large. The functions $\phi_k(\xi)$ are the eigenfunctions of the operator $\mathcal{L}_T$ (corresponding to diffusion with constant $\nu_T = \nu + \frac{A^2}{\nu} \sum_{m=1}^{\infty} \frac{\chi_m^2}{\mu_m}$ in scaling variables) in the space $L^2(m)$. The quantities $\alpha_k(\tau)$ and $\beta_k^n(\tau)$ solve system (3.2) and have the following asymptotics, obtainable via a reduction to an $N + 2$-dimensional center manifold:

$$\alpha_k, \|\{\beta_k^n\}_{n=1}^{\infty}\|_{\ell^2} = \begin{cases} \mathcal{O}(e^{-\frac{k}{6}\tau}) & \text{if } k = 0 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \text{if } k = 1 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \text{if } k = 2 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k}{6}\tau}) & \text{if } k = 3 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+2}{6}\tau}) & \text{if } k = 4 \text{ mod } 6 \\ \mathcal{O}(e^{-\frac{k+4}{6}\tau}) & \text{if } k = 5 \text{ mod } 6. \end{cases} \tag{5.1}$$

Remark 5.0.2. *The smallness condition on ν above is required to make the error estimates in Chapter 4 hold.*

Proof:

Recall the splitting 3.1 of solutions to (1.16):

$$\begin{aligned} w_0(\xi,\tau) &= \sum_{k=0}^{N} \alpha_k(\tau)\phi_k(\xi) + w_0^s(\tau,\xi) \\ u_n(\xi,\tau) &= \sum_{k=0}^{N} \beta_k^n(\tau)\phi_k(\xi) + u_n^s(\tau,\xi). \end{aligned}$$

Therefore

$$\begin{aligned} w_0(\xi,\tau) - \sum_{k=0}^{N} \alpha_k(\tau)\phi_k(\xi) &= w_0^s(\tau,\xi) \\ u_n(\xi,\tau) - \sum_{k=0}^{N} \beta_k^n(\tau)\phi_k(\xi) &= u_n^s(\tau,\xi). \end{aligned}$$

We now apply Corollary 4.2.10 and obtain

$$\|w_0(\xi,\tau) - \sum_{k=0}^{N} \alpha_k\phi_k(\xi)\|_{L^2(m)} + \|\|\{u_n(\xi,\tau) - \sum_{k=0}^{N} \beta_k^n(\tau)\phi_k(\xi)\}_{n=1}^{\infty}\|_{\ell^2}\|_{L^2(m)} \leq$$
$$\|w_0^s(\xi,\tau)\|_{L^2(m)} + \|\|\{u_n^s(\xi,\tau)\}_{n=1}^{\infty}\|_{\ell^2}\|_{L^2(m)} \leq Ce^{(-\frac{N}{6}+1+\frac{1}{2})\tau}.$$

Therefore

$$\begin{aligned} \|w_0(\xi,\tau) - \sum_{k=0}^{N} \alpha_k\phi_k(\xi)\|_{L^2(m)} + \|\|\{u_n(\xi,\tau) - \sum_{k=0}^{N} \beta_k^n(\tau)\phi_k(\xi)\}_{n=1}^{\infty}\|_{\ell^2}\|_{L^2(m)} \\ \leq Ce^{(-\frac{N}{6}+1+\frac{1}{2})\tau}, \end{aligned}$$

which concludes the proof of Theorem 5.0.1. □

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