1998-02-12

Mathai-Quillen forms and Lefschetz theory

Mathai-Quillen Forms and Lefschetz Theory

Mihail Frumosu[†]
Department of Mathematics, Boston University, Boston, MA 02215
matf@math.bu.edu

Steven Rosenberg[‡]
Department of Mathematics, Boston University, Boston, MA 02215
sr@math.bu.edu

1 Introduction

In [8], Mathai and Quillen introduced a geometric representative, the Mathai-Quillen form, for the Thom class of an oriented Riemannian bundle over an oriented Riemannian manifold. Using a one-parameter family of pullbacks of this form, Mathai and Quillen gave a new proof of both the Hopf index formula and the Chern-Gauss-Bonnet formula for the Euler characteristic.

The Euler characteristic is the Lefschetz number of the identity map of a closed manifold M, and various topological expressions for the Euler characteristic (alternating sum of Betti numbers, self-intersection number of the diagonal in $M \times M$, Hopf index formula) have counterparts in Lefschetz theory for functions $f : M \to M$ (supertrace of f on cohomology, intersection number of f's graph with the diagonal, Lefschetz fixed point formula). However, a Lefschetz counterpart of the integral geometric expression for the Euler characteristic (the Chern-Gauss-Bonnet theorem) has not been known previously. With this motivation, we investigate geometric aspects of Lefschetz theory using Mathai-Quillen forms, and in particular study the analogous one-parameter family of pullback forms.

In §2.2, we give via Poincaré duality an elementary integral formula for the Lefschetz number in terms of the map f and the Mathai-Quillen form of the normal bundle of the diagonal in $M \times M$ (Theorem 2.2). This formula specializes to the Chern-Gauss-Bonnet formula when $f = \mathrm{Id}$. In contrast to the Chern-Gauss-Bonnet integrand, the local expression of the integrand is fairly complicated even for flat manifolds, due to the action of f on the Mathai-Quillen form. In §2.3, we compute the integrand in the flat case (Theorem 2.3) and work a computation on S^1. We give the integrand for general metrics in §2.4 (Theorem 2.4). Here the result is less explicit,

† Current address: Metropolitan College, Boston University, Boston, MA 02215
‡ The second author's research was partially supported by the NSF.

as the integrand depends on solutions of Jacobi fields along geodesics joining x and $f(x)$. These Jacobi fields appear because we must exponentiate the Mathai-Quillen form of the normal bundle to a form on a tubular neighborhood of the diagonal, and the Jacobi fields measure the deviation of the exponential map from the trivial flat case. We are able to make the integrand explicit for constant curvature metrics.

In §§3-4, we investigate the Lefschetz analog of the one-parameter family of pullbacks of Mathai-Quillen forms. In brief, the $t \to \infty$ limit of the pullbacks measures the fixed point set of the map, while the $t \to 0$ limit measures the set of points mapped far from themselves.

More precisely, as $t \to \infty$, we recover in §3 the Lefschetz fixed point formula (Theorem 3.1), and in fact we give a new proof of the more general formula for submanifolds of fixed points. This argument is done both at the topological level using Thom classes, and at the geometric level using Mathai-Quillen forms. The proof can be thought of as a simplified version of the heat equation proof of the Lefschetz formula in [5].

In §4, we consider the $t \to 0$ limit. This limit is trivial in the case considered in [8], but is discontinuous in our case. In fact, the discontinuities occur at the intersection of the graph of f with the boundary of the tubular neighborhood of the diagonal, which in general is quite complicated.

To extract the maximum geometric information, we take the largest possible tubular neighborhood, with boundary the union of the cut loci of the points of M. This choice is motivated by the observation that if $f(x)$ is never in $\mathcal{C}_x$, the cut locus of x, then $L(f) = \chi(M)$. More precisely, there is a singular current supported on $\mathcal{C}(f) = \{x : f(x) \in \mathcal{C}_x\}$, whose singular part measures $L(f) - \chi(M)$ (Theorem 4.1). Assuming that $\mathcal{C}(f)$ is finite and imposing a transversality condition, we find the sharp estimate $|L(f) - \chi(M)| \leq |\mathcal{C}(f)|$ (Theorem 4.2). These assumptions place strong restrictions on the metric on M, and for diffeomorphisms f with $L(f) \neq \chi(M)$, $|\mathcal{C}(f)|$ is infinite for most metrics (Theorem 4.3).

These results give geometric information for Lefschetz theory via Mathai-Quillen forms. In the appendix, we show how Hodge theory techniques give upper bounds for the Lefschetz number in terms of the geometry of M.

2 The basic formula and its local expression

Let $f : M \to M$ be a smooth map of a closed oriented Riemannian manifold M. After a review of Poincaré duality in §2.1, we give in §2.2 an integral formula (Theorem 2.2) for the Lefschetz number of f which reduces to the Chern-Gauss-Bonnet theorem when $f = \mathrm{Id}$. The Mathai-Quillen formalism is also easily extended to odd rank bundles. The local expression for the integrand for flat manifolds is computed in §2.3 (Theorem 2.3), and the integrand for arbitrary metrics is computed in §2.4 (Theorem 2.4). This last formula is then specialized to constant curvature metrics.

2.1 Topological preliminaries

The Lefschetz number of f is

$$L(f) = \sum_q (-1)^q \text{ tr } f^q,$$

where f^q denotes the induced map on the real cohomology group $H^q(M)$. This has a well known Poincaré duality formulation, [1, Ex. 11.26], which we give for completeness.

Lemma 2.1 *Let $f : M \to M$ be a smooth map of a closed oriented manifold. Then*

$$L(f) = \int_\Delta \eta_\Gamma,$$

where η_Γ is the Poincaré dual of the graph Γ of f in $M \times M$ and Δ is the diagonal of M.

Here we do not distinguish between the cohomology class η_Γ and a representative form. Before the proof of the lemma, we collect the basic results about Poincaré duality and Thom classes. Recall that the Poincaré dual η_N of an oriented k-submanifold N of a closed oriented manifold X is the real cohomology class defined by (or characterized by, depending on one's definition)

$$\int_N \omega = \int_X \omega \wedge \eta_N, \tag{2.1}$$

for all closed k-forms ω on X [1, (5.13)].

Theorem 2.1 *(i) Let N' be another closed oriented submanifold of X with transverse intersection with N. Then*

$$\eta_{N \cap N'} = \eta_N \wedge \eta_{N'}.$$

(ii) A closed form $U \in H^k_c(E)$, the compactly supported cohomology of an oriented rank k bundle E over X, represents the Thom class iff the integral of U over each fiber of E is one.

(iii) Identify the total space of ν_N^X, the normal bundle of N in X, with a tubular neighborhood of N in X, so that the Thom class of the normal bundle can be considered as a cohomology class on X. Then the Poincaré dual of N is the same as the Thom class of the normal bundle of N in X.

The proofs of (i)-(iii) are in (6.31), Prop. 6.18 and Prop. 6.24 of [1], respectively. It is pointed out in [8] that the cohomology with compact support in (ii) may be replaced with the cohomology of forms with C^1 exponential decay in the fibers.

Remark: The main technical work in this paper is in making the identification in (iii) explicit. To consider a closed form U on ν_N^X as a closed form on X, we first use a diffeomorphism α from the ϵ-ball $B_\epsilon(0) \subset \mathbf{R}^n$ to $\mathbf{R}^n$ to pull U back to the form $\alpha^* U$ on the ϵ-neighborhood of the zero section in ν_N^X. For ϵ small enough, the exponential map $\exp : \nu_N^X \to X$ is a diffeomorphism onto a tubular neighborhood of N, and so it is really $(\exp^{-1})^* \alpha^* U$ which is a form supported on the tubular neighborhood.

PROOF OF LEMMA 2.1: Let $\{\omega_i\}$ be a basis for $H^*(M)$ and $\{\tau_j\}$ the dual basis under Poincaré duality, i.e. $\int_M \omega_i \wedge \tau_j = \delta_{ij}$. Let $\pi,\ \rho$ be the projections of $M \times M$ onto the first and second factors. $H^*(M \times M)$ has as basis $\{\pi^*\omega_i \wedge \rho^*\tau_j\}$, so $\eta_\Gamma = \sum_{i,j} c_{ij} \pi^*\omega_i \wedge \rho^*\tau_j$ for some $c_{ij} \in \mathbf{R}$. We now determine the c_{ij}.

LEMMA 2.2 $\eta_\Gamma = \sum_{i,j} (-1)^{(\deg\ \omega_i)(\deg\ \omega_j)} \alpha_{ji} \pi^*\omega_i \wedge \rho^*\tau_j$, *with* α_{ij} *defined by* $f^*\omega_i = \alpha_{ij}\omega_j$.

PROOF: We compute $\int_\Gamma \pi^*\tau_k \wedge \rho^*\omega_l$ in two ways. Using the graph map $i : M \longrightarrow \Gamma \subset M \times M$, $i(x) = (x, f(x))$, we obtain

$$\begin{aligned}
\int_\Gamma \pi^*\tau_k \wedge \rho^*\omega_l &= \int_M i^*\pi^*\tau_k \wedge i^*\rho^*\omega_l = \int_M \tau_k \wedge f^*\omega_l \\
&= \int_M \alpha_{lj}\tau_k \wedge \omega_j = \alpha_{lj}(-1)^{(\deg\ \omega_j)(\deg\ \tau_k)}\delta_{kj} \\
&= \alpha_{lk}(-1)^{(\deg\ \omega_k)(\deg\ \tau_k)}.
\end{aligned}$$

By (2.1),

$$\begin{aligned}
\int_\Gamma \pi^*\tau_k \wedge \rho^*\omega_l &= \int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_l \wedge \eta_\Gamma \\
&= \sum_{i,j} c_{ij} \int_{M\times M} \pi^*\tau_k \wedge \rho^*\omega_l \wedge \pi^*\omega_i \wedge \rho^*\tau_j \\
&= \sum_{i,j} c_{ij}(-1)^{(\deg\ \tau_k + \deg\ \omega_l)(\deg\ \omega_i)} \int_{M\times M} \pi^*(\omega_i \wedge \tau_k) \wedge \rho^*(\omega_l \wedge \tau_j) \\
&= (-1)^{(\deg\ \tau_k + \deg\ \omega_l)(deg\ \omega_l)} c_{kl}.
\end{aligned}$$

Thus $c_{kl} = \alpha_{lk}(-1)^{(\deg\ \omega_k)(\deg\ \omega_l)}$. □

For the proof of Lemma 2.1, we have

$$\begin{aligned}
\int_\Delta \eta_\Gamma &= \sum_{i,j} (-1)^{(\deg\ \omega_i)(\deg\ \omega_j)} \alpha_{ji} \int_M i^*\pi^*\omega_i \wedge i^*\rho^*\tau_j \\
&= \sum_{i,j} (-1)^{(\deg\ \omega_i)(\deg\ \omega_j)} \alpha_{ji} \int_M \omega_i \wedge \tau_j = \sum_i (-1)^{(\deg\ \omega_i)} \alpha_{ii} \\
&= \sum_q (-1)^q \ \mathrm{tr}\ f^q = L(f).
\end{aligned}$$

□

If Γ is transversal to Δ in $M \times M$, this lemma leads to a quick proof of the Lefschetz fixed point formula:

$$L(f) = \sum_{p, f(p)=p} \sigma_p,$$

with $\sigma_p = \text{sgn } \det(\text{Id} - (df)_p)$ and $(df)_p : T_pM \to T_pM$ the derivative map. Here $\Gamma \cap \Delta$ is a finite set of points. Theorem 2.1 and Poincaré duality give

$$\begin{aligned} L(f) &= \int_\Delta \eta_\Gamma = \int_{M\times M} \eta_\Gamma \wedge \eta_\Delta = \int_{M\times M} \eta_{\Gamma\cap\Delta} \\ &= \int_{\Gamma\cap\Delta} 1 = \sum_{p, f(p)=p} \pm 1. \end{aligned}$$

Thus $L(f)$ is the sum of the orientations ± 1 of the fixed points p of f. By [GP, p.121], the orientation equals $\text{sgn } \det(\text{Id} - (df)_p)$ in our sign convention. Note also that $L(f) = \int_{\Gamma\cap\Delta} 1$ implies that $L(f) = I(\Delta, \Gamma)$, the intersection number of Δ and Γ, so we have three equivalent definitions of the Lefschetz.

2.2 Mathai-Quillen formalism and the integral formula

In [8], Mathai and Quillen obtained a geometric expression for the Thom class of an oriented even dimensional vector bundle. Let E be a rank $n = 2m$ vector bundle over a manifold M, where E has an inner product and a compatible connection θ. Then a geometric representative MQ of the Thom class of E is given by

$$\text{MQ} = \pi^{-m} e^{-x^2} \sum_{I, |I| \text{ even}} \epsilon(I, I') \text{Pf}\Big(\frac{1}{2}\Omega_I\Big)(dx + \theta x)^{I'}, \tag{2.2}$$

where: x is an orthornormal fiber coordinate; Ω is curvature of the connection θ; Ω_I is the submatrix of Ω with respect to the multi-index I with entries in $\{1, 2, ..., n\}$; $\text{Pf}(\frac{1}{2}\Omega_I)$ is the Pfaffian of $\frac{1}{2}\Omega_I$; I' denotes the complement of I in $\{1, 2, ..., n\}$; $\epsilon(I, I')$ is the sign of I, I' considered as a shuffle permutation in the exterior algebra:

$$dx^I \wedge dx^{I'} = \epsilon(I, I') dx^1 \wedge ... \wedge dx^n;$$

and

$$(dx + \theta x)^{I'} = (dx^{i_1} + \theta^{i_1}_{j_1} x^{j_1}) \wedge (dx^{i_2} + \theta^{i_2}_{j2} x^{j_2}) \wedge ... \wedge (dx^{i_q} + \theta^{i_q}_{j_q} x^{j_q}),$$

with $I' = \{i_1, i_2, ..., i_q\}$. In the expression θx, θ denotes the connection one-forms of the connection for the frame $\{x^i\}$. The ordering of the elements of I' in $dx^{I'}$ is unimportant due to the $\epsilon(I, I')$ factor. For computations at a point $x \in M$, we will often assume that $\{x^i\}$ is a synchronous frame centered at x, in which case the connection one-forms θ vanish at x

Unlike the Euler characteristic, the Lefschetz number can be nonzero for odd dimensional manifolds, so we need to check that this formalism extends to bundles of odd rank. Let $n = 2m+1$ and let E_M be an oriented rank n vector bundle over a manifold M with a connection compatible with a metric on E_M, and let E_{S^1} be the trivial bundle with the trivial connection over S^1. Equip $E_M \times E_{S^1}$ over $M \times S^1$ with the product connection. The Mathai-Quillen representative $\mathrm{MQ}_{M\times S^1} \in H^{2m+2}(M \times S^1)$ of the Thom class of $E = E_M \times E_{S^1}$ is given by

$$\mathrm{MQ}_{M\times S^1} = \pi^{-(m+1)} e^{-x^2} \sum_{I,|I| \text{ even}} \epsilon(I, I') \mathrm{Pf}\Big(\frac{1}{2}\Omega_I\Big)(dx + \theta x)^{I'} \ ,$$

where θ is the connection one-form with respect to a product orthonormal frame $\{x^i\}$ of E, and $\Omega = \Omega_{M\times S^1}$ is the curvature of this connection over $M \times S^1$. The curvature matrix for $M \times S^1$ is:

$$\Omega_{M\times S^1} = \begin{pmatrix} \Omega_M & 0 \\ 0 & 0 \end{pmatrix},$$

where Ω_M is the curvature matrix of E_M.

Recall that for an even-dimensional $k \times k$ matrix ω, the Pfaffian is a homogeneous polynomial of degree $k/2$ in the entries of ω characterized up to sign by $\mathrm{Pf}^2(\omega) = \det(\omega)$. If Ω_I is a submatrix of $\Omega_{M\times S^1}$, then $\mathrm{Pf}\big(\frac{1}{2}\Omega_I\big) = 0$, unless Ω_I is a submatrix of Ω_M itself. Thus in the definition of $\mathrm{MQ}_{M\times S^1}$ we may assume that $n+1 \notin I$, where $n+1$ corresponds to the dt variable and t is the coordinate on S^1. Moreover, we have

$$\begin{aligned}(dx + \theta x)^{I'} &= (dx^{i'_1} + \theta^{i'_1}_{j_1} x^{j_1}) \wedge (dx^{i'_2} + \theta^{i'_2}_{j2} x^{j_2}) \wedge \ldots \wedge (dt + \theta^{n+1}_{j_q} x^{j_q}) \\ &= (dx + \theta x)^{I'_M} \wedge dt,\end{aligned}$$

where $I' = I'_M \cup \{n+1\} = \{i'_1, \ldots, i'_{q-1}, n+1\}$ with I'_M the complement of $I - \{n+1\}$ in $\{1, 2, \ldots, n\}$. Hence $\mathrm{MQ}_{M\times S^1}$ decomposes as follows:

$$\begin{aligned}\mathrm{MQ}_{M\times S^1} &= \pi^{-(m+\frac{1}{2})} e^{-x^2} \sum_{I_M,|I_M| \text{ even}} \epsilon(I_M, I'_M) \mathrm{Pf}\Big(\frac{1}{2}\Omega_{I_M}\Big)(dx + \theta x)^{I'_M} \\ &\qquad \wedge (\sqrt{\pi})^{-1} e^{-t^2} dt \\ &= U_M \wedge U_{S^1},\end{aligned}$$

where

$$U_M = \pi^{-\frac{n}{2}} e^{-x^2} \sum_{I_M,|I_M| \text{ even}} \epsilon(I_M, I'_M) \mathrm{Pf}\Big(\frac{1}{2}\Omega_{I_M}\Big)(dx + \theta x)^{I'_M}, \quad U_{S^1} = \pi^{-1/2} e^{-t^2} dt.$$

We want to show that U_M represent the Thom class of E_M. By Theorem 2.1 (iii), U_{S^1} represents the Thom class of E_{S^1}.

Lemma 2.3 *If E is a vector bundle of odd rank n over M, then*

$$U_M = \pi^{-\frac{n}{2}} e^{-x^2} \sum_{I, |I| \text{ even}} \epsilon(I, I') \mathrm{Pf}\Big(\frac{1}{2}\Omega_I\Big)(dx + \theta x)^{I'}$$

is in the Thom class of E_M.

Proof: We know that $U_M \wedge U_{S^1}$ represents the Thom class for $E_M \times E_{S^1}$. Since $dU_{S^1} = 0$, we have $0 = d(U_{M \times S^1}) = dU_M \wedge U_{S^1}$. U_{S^1} is non-zero, so $dU_M = 0$. Moreover, in each fiber $\int_{E_{S^1}} U_{S^1} = 1$, so in each fiber

$$1 = \int_{E_M \times E_{S1}} U_M \wedge U_{S^1} = \Big(\int_{E_M} U_M\Big) \times \Big(\int_{E_{S^1}} U_{S^1}\Big) = \int_{E_M} U_M.$$

Hence U_M represents the Thom class of E_M. □

The following elementary result is the basic integral formula for the Lefschetz number.

Theorem 2.2 *Let $f : M \to M$ be a smooth map of a closed oriented manifold. Let Δ_ϵ be a tubular neighborhood of the diagonal in $M \times M$ of width ϵ, and let $\mathrm{MQ}_{\Delta_\epsilon}$ be the Mathai-Quillen form of the normal bundle to the diagonal, considered as a form supported in Δ_ϵ. Then the Lefschetz number is given by*

$$L(f) = (-1)^{\dim M} \int_M (\mathrm{Id}, f)^* \mathrm{MQ}_{\Delta_\epsilon}, \tag{2.3}$$

where $(\mathrm{Id}, f) : M \to \Gamma \subset M \times M$ is the graph map.

Proof: By Lemma 2.1 and Poincaré duality, we have

$$\begin{aligned} L(f) &= \int_\Delta \eta_\Gamma = \int_{M \times M} \eta_\Gamma \wedge \eta_\Delta \\ &= (-1)^{\dim M} \int_\Gamma \eta_\Delta = (-1)^{\dim M} \int_{(\mathrm{Id}, f)(M)} \mathrm{MQ}_{\Delta_\epsilon} \\ &= (-1)^{\dim M} \int_M (\mathrm{Id}, f)^* \mathrm{MQ}_{\Delta_\epsilon}, \end{aligned}$$

since (Id, f) is an orientation preserving diffeomorphism and hence of degree 1. □

This formula generalizes the Chern-Gauss-Bonnet theorem. The Euler characteristic of an even dimensional Riemannian manifold M is given by

$$\chi(M) = L(\mathrm{Id}) = \int_M (\mathrm{Id}, \mathrm{Id})^* \mathrm{MQ}_{\Delta_\epsilon} = \int_M 0^* \mathrm{MQ}_{TM},$$

since a neighborhood of the zero section in TM is isomorphic to a tubular neighborhood of Δ under an isomorphism taking the zero section 0 to the graph map $(\mathrm{Id}, \mathrm{Id})$ of the identity. For the Levi-Civita connection θ, we have

$$\mathrm{MQ}_{TM} = \pi^{-n/2} \sum_{|I| \text{ even}} \epsilon(I, I')\mathrm{Pf}(\frac{1}{2}\Omega_I)(dx + \theta x)^{I'},$$

and

$$0^*\mathrm{MQ}_{TM} = \pi^{-n/2}\mathrm{Pf}(\frac{1}{2}\Omega)$$

since $x = 0$ on M implies $0^*(dx + \theta x)^{I'} = 0$ if $I' \neq \emptyset$. Thus we obtain the Chern-Gauss-Bonnet theorem

$$\chi(M) = \frac{1}{(2\pi)^{n/2}} \int_M \mathrm{Pf}(\Omega).$$

Similarly, we find $\chi(M) = 0$ if dim M is odd.

It is important to note that the support of the integrand in (2.3) is $\{x \in M : (x, f(x)) \in \Delta_\epsilon\}$.

2.3 Local expressions for flat manifolds

We will check Theorem 2.2 on a simple flat example and derive an integral formula for the Lefschetz number of a general flat manifold.

Example: Let $M = S^1$ and let $f : S^1 \to S^1$ be given by $f(z) = z^n$. Then f has degree n, so $L(f) = 1 - n$.

We now construct the Mathai-Quillen form $\mathrm{MQ}_{\Delta_\epsilon}$ of an ϵ-neighborhood of the diagonal in $S^1 \times S^1$, where we fix $\epsilon = \frac{\pi}{2\sqrt{2}}$ for convenience. Let

$$\mathrm{MQ}_{\nu_\Delta} = \frac{1}{\sqrt{\pi}} e^{-x^2} dx$$

be the Mathai-Quillen form of TS^1, which we identify with $\nu_\Delta^{S^1\times S^1}$, the normal bundle of the diagonal in $S^1 \times S^1$. Let $\alpha : \Delta_\epsilon \to \nu_{\Delta_{S^1}}^{S^1\times S^1}$ be the diffeomorphism

$$\alpha(\theta_1, \theta_2) = \left(\frac{\theta_1 + \theta_2}{2}, \rho\Big(\frac{\theta_1 - \theta_2}{\sqrt{2}}\Big)\right),$$

where (θ_1, θ_2) are the coordinates on $S^1 \times S^1$ and $\rho : \left(-\frac{\pi}{2\sqrt{2}}, \frac{\pi}{2\sqrt{2}}\right) \to (-\infty, \infty)$ is an orientation preserving diffeomorphism given by a fixed odd function ρ. We set $\rho(x) = \infty,\ -\infty$ if $x > \frac{\pi}{2\sqrt{2}},\ x < -\frac{\pi}{2\sqrt{2}}$, respectively.

The condition $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1$ implies

$$\frac{1}{\sqrt{\pi}} \int_{-\frac{\pi}{2\sqrt{2}}}^{\frac{\pi}{2\sqrt{2}}} \rho'(\theta) e^{-\rho^2(\theta)} d\theta = 1, \quad \text{and} \quad \int_0^{\frac{\pi}{2\sqrt{2}}} \rho'(\theta) e^{-\rho^2(\theta)} d\theta = \frac{\sqrt{\pi}}{2},$$

since the integrand is even. Hence we have

$$\mathrm{MQ}_{\Delta_\epsilon} = \alpha^*(\mathrm{MQ}_{\nu_\Delta}) = \frac{1}{\sqrt{2\pi}} e^{-\rho^2(\frac{\theta_2-\theta_1}{\sqrt{2}})}\rho'\Big(\frac{\theta_2-\theta_1}{\sqrt{2}}\Big)(-d\theta_1 + d\theta_2)$$

at (θ_1, θ_2).

The graph of f, drawn on $[0, 2\pi] \times [0, 2\pi]$, consists of n line segments $\theta_2 = n\theta_1 - 2(k-1)\pi,\ k = 1, 2, \ldots, n$. Since the upper and lower limits of the tubular neighborhoods are given by $\theta_2 = \theta_1 \pm \frac{\pi}{2}$, it is easy to check that Γ is in the tubular neighborhood iff

$$\frac{(4k-5)\pi}{2(n-1)} \le \theta_1 \le \frac{(4k-3)\pi}{2(n-1)},$$

for $k = 2, ..., n-1$, or

$$0 \le \theta_1 \le \frac{\pi}{2(n-1)}\ , \qquad \frac{(4n-5)\pi}{2(n-1)} \le \theta_1 \le 2\pi,$$

for the first and last segment, respectively. Thus

$$\begin{aligned}(\mathrm{Id}, f)^*\mathrm{MQ}_{\Delta_\epsilon} &= \frac{1}{\sqrt{2\pi}} e^{-\rho^2(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}})}\rho'\Big(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}}\Big) \\ &\quad \cdot(\mathrm{Id}, f)^*(d\theta_2 - d\theta_1) \\ &= \Big(\frac{n-1}{\sqrt{2\pi}}\Big) e^{-\rho^2(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}})}\rho'\Big(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}}\Big)d\theta,\end{aligned}$$

since $f^*d\theta = nd\theta$. This gives

$$\begin{aligned}L(f) &= -\int_{S^1}(\mathrm{Id}, f)^*\mathrm{MQ}_{\Delta_\epsilon} \\ &= -\frac{1}{\sqrt{\pi}}\int_0^{\frac{\pi}{2(n-1)}} e^{-\rho^2(\frac{(n-1)\theta}{\sqrt{2}})}\rho'\Big(\frac{(n-1)\theta}{\sqrt{2}}\Big)d\theta \\ &\quad -\sum_{k=2}^{n-1}\Big[\frac{1}{\sqrt{\pi}}\int_{\frac{(4k-5)\pi}{2(n-1)}}^{\frac{(4k-3)\pi}{2(n-1)}} \Big(\frac{n-1}{\sqrt{2}}\Big) e^{-\rho^2\big(\frac{(n-1)\theta}{\sqrt{2}}-(k-1)\pi\sqrt{2}\big)} \\ &\quad \rho'\Big(\frac{(n-1)\theta-(k-1)2\pi}{\sqrt{2}}\Big)d\theta\Big] \\ &\quad -\frac{1}{\sqrt{\pi}}\int_{\frac{(4n-5)\pi}{2(n-1)}}^{2\pi}\Big(\frac{n-1}{\sqrt{2}}\Big)e^{-\rho^2\big(\frac{(n-1)(\theta-2\pi)}{\sqrt{2}}\big)}\rho'\Big(\frac{(n-1)(\theta-2\pi)}{\sqrt{2}}\Big)d\theta.\end{aligned}$$

Under the change of variables $\lambda = [(n-1)\theta/\sqrt{2}] - (k-1)\pi\sqrt{2},\ k = 1, \ldots, n$, the first and last integrals become $1/2$, and the integrals under the sum become 1. Thus

$$L(f) = -\int_{S^1}(\mathrm{Id}, f)^*\mathrm{MQ}_{\Delta_\epsilon} = -\frac{1}{2} - \sum_{k=2}^{n-1} 1 - \frac{1}{2} = 1 - n,$$

as expected.

The formula for the Lefschetz number for functions on flat manifolds is not much more complicated. Since M is flat, for the Levi-Civita connection θ there exists a local orthornomal frame $\{x^i\}$ for which the connection and the curvature forms vanish. The Mathai-Quillen form for the normal bundle to the diagonal is thus given by

$$\mathrm{MQ}_{\nu_\Delta} = \pi^{-n/2} e^{-x^2} dx^1 \wedge ... \wedge dx^n, \tag{2.4}$$

where x is the fiber coordinate.

We need an explicit isomorphism $\alpha : \Delta_\epsilon \to \nu_\Delta$ between an ϵ-neighborhood of the diagonal and the normal bundle to compute $\mathrm{MQ}_{\Delta_\epsilon} = \alpha^* \mathrm{MQ}_{\nu_\Delta}$. Even though the exponential map is trivial near the diagonal, we will use it initially to avoid confusion between normal vectors and points of $M \times M$.

Fix $(x, y) \in \Delta_\epsilon$. Since the normal bundle consists of vectors of the form $(-v, v)$, there exists $(\bar{x}, \bar{x}) \in \Delta$ such that $(x, y) = \exp_{(\bar{x},\bar{x})}(-v, v) = (\exp_{\bar{x}}(-v), \exp_{\bar{x}} v)$. Thus $\bar{x}$ is the midpoint of the geodesic $\exp_{\bar{x}}(tv)$, $t \in [-1, 1]$ from x to y. This gives an isomorphism $\eta : U \to \Delta_\epsilon$ between a neighborhood U of the zero section in ν_Δ and the ϵ-neighborhood of the diagonal:

$$\eta(v, -v)_{(\bar{x},\bar{x})} = (\exp_{\bar{x}}(v), \exp_{\bar{x}}(-v)).$$

Let $\rho : [0, \epsilon) \to [0, \infty)$ be a fixed diffeomorphism with $\rho(0) = 0$, $\lim_{d\to\epsilon} \rho(d) = \infty$. As before, we extend ρ to take on values ∞ outside of $[0, \epsilon)$.

In the product metric, $d((x, y), (\bar{x}, \bar{x})) = d(x, y)/\sqrt{2}$, so $\Delta_\epsilon = \{(x, y) \in M \times M : d(x, y) < \sqrt{2}\epsilon.\}$. Thus $\beta : U \to \nu_\Delta$ given by

$$\beta(v, -v)_{(\bar{x},\bar{x})} = \begin{cases} \left(\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}, -\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}\right), & v \neq 0, \\ (\bar{x}, \bar{x}), & v = 0, \end{cases}$$

is a diffeomorphism and $\alpha = \beta \circ \eta^{-1} : \Delta_\epsilon \to \nu_\Delta$ is our desired map:

$$\alpha(x, y) = \begin{cases} \left((\bar{x}, \bar{x}), \left(\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}, -\rho\left(\frac{d(x,y)}{\sqrt{2}}\right)\frac{v}{|v|}\right)\right), & x \neq y, \\ ((x, x), 0), & x = y, \end{cases}$$

where $x = \exp_{\bar{x}}(v)$, $y = \exp_{\bar{x}}(-v)$, $|v| = d(x, y)/\sqrt{2}$.

In the flat case, the map η can be treated as identity map, and there exists an isometry between the ϵ-tube Δ_ϵ and U such that the map α reduces to $\beta : \mathbf{R}^n \to \mathbf{R}^n$, with

$$\beta(v) = \begin{cases} \rho(|v|)\frac{v}{|v|}, & v \neq 0, \\ 0, & v = 0, \end{cases}$$

and

$$\mathrm{MQ}_{\Delta_\epsilon} = \alpha^* \mathrm{MQ}_{\nu_\Delta} = \beta^* \mathrm{MQ}_{\nu_\Delta}.$$

By (2.4), computing this last term reduces to calculating $\beta^*\mathrm{dvol}$, which we do in polar coordinates. For $\gamma(t) = v + t\frac{v}{|v|}$, we have

$$\begin{aligned}
\beta_{*v}(\partial_r) &= \frac{d}{dt}\Big|_{t=0}\beta(\gamma(t)) = \frac{d}{dt}\Big|_{t=0}\rho(|\gamma(t)|)\frac{\gamma(t)}{|\gamma(t)|} \\
&= \rho'(|v|)\Big(\frac{d}{dt}\Big|_{t=0}|\gamma(t)|\Big)\frac{\gamma(0)}{|\gamma(0)|} \\
&\qquad +\rho(|v|)\frac{\dot\gamma(0)}{|\gamma(0)|} + \rho(|v|)\Big[\gamma(0)(-\frac{1}{2}\langle\gamma(0),\gamma(0)\rangle^{-3/2}2\langle\dot\gamma(0),\gamma(0)\rangle)\Big] \\
&= \rho'(|v|)\cdot 1\cdot\frac{v}{|v|} + \rho(|v|)\frac{v}{|v|^2} + \rho(|v|)\Big(\frac{-v}{|v|^3}\Big\langle\frac{v}{|v|}, v\Big\rangle\Big) \\
&= \rho'(|v|)\partial_r + \rho(|v|)\Big[\frac{v}{|v|^2} - \frac{v}{|v|^2}\Big] = \rho'(|v|)\partial_r,
\end{aligned}$$

and by a similar calculation using $\gamma(t)$ with $\dot\gamma(0) = (\partial_{\theta^i})_v$, we have

$$\beta_{*v}(\partial_{\theta^i}) = 0 + \rho(|v|)\frac{(\partial_{\theta^i})_v}{|\partial_{\theta^i}|} + \rho(|v|)\frac{v}{|v|^3}\cdot 0 = \rho(|v|)\frac{\partial_{(\theta^i)_v}}{|v|}.$$

Thus

$$\beta^* dr_v = \rho'(|v|)dr_{\beta(v)}. \tag{2.5}$$

Now $(\partial_{\theta^i})_v = (\partial\theta^i\big|_{\frac{v}{|v|}})|v|$, as $\langle\partial_{\theta^i},\partial_{\theta^i}\rangle = r$, and $(\partial_{\theta^i})_{\beta(v)} = (\partial\theta^i\big|_{\frac{v}{|v|}})\rho(|v|)$, as $|\beta(v)| = \rho(|v|)$. Hence $(\partial_{\theta^i})_{\beta(v)} = \frac{\rho(|v|)}{|v|}(\partial_{\theta^i})_v$, so $(\beta_*)_v(\partial_{\theta^i}) = \frac{\rho(|v|)}{|v|}(\partial_{\theta^i})_v$ implies $\beta_{*v}(\partial_{\theta^i}) = (\partial_{\theta^i})_{\beta(v)}$. Thus

$$\beta^* d\theta^i_v = d\theta^i_{\beta(v)}. \tag{2.6}$$

By (2.5), (2.6), we get

$$\begin{aligned}
\mathrm{MQ}_{\Delta_\epsilon} &= \beta^*\mathrm{MQ}_{\nu_\Delta} \\
&= \beta^*(\pi^{-n/2}e^{-x^2}r^{n-1}dr\wedge d\theta^1\wedge\ldots\wedge d\theta^{n-1}) \\
&= \pi^{-n/2}e^{-\rho^2(|v|)}\rho(|v|)^{n-1}\rho'(|v|)\mathrm{dvol}_{\alpha(x,y)} \\
&= \pi^{-n/2}e^{-\rho^2(\frac{d(x,y)}{\sqrt{2}})}\rho'\Big(\frac{d(x,y)}{\sqrt{2}}\Big)\mathrm{dvol}_{\alpha(x,y)}.
\end{aligned}$$

The last step is to calculate

$$\begin{aligned}
(\mathrm{Id},f)^*\mathrm{MQ}_{\Delta\epsilon} &= (\mathrm{Id},f)^*\Big[\pi^{-n/2}e^{-\rho^2(\frac{d(x,y)}{\sqrt{2}})}\rho'\Big(\frac{d(x,y)}{\sqrt{2}}\Big)\mathrm{dvol}_{\alpha(x,y)}\Big] \\
&= \pi^{-n/2}e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}\rho'\Big(\frac{d(x,f(x))}{\sqrt{2}}\Big)(\mathrm{Id},f)^*(\mathrm{dvol}_{\alpha(x,y)}).
\end{aligned}$$

Here $\mathrm{dvol}_{\alpha(x,y)}$ is the volume element on the normal bundle, considered as a form near the diagonal. On a general manifold, calculating $(\mathrm{Id}, f)^*\mathrm{dvol}_{\alpha(x,y)}$ will require introducing coordinates on the tubular neighborhood via the exponential map. Since M is flat, the calculations reduce to the case $M = \mathbf{R}^n$.

For this let (x^i) be flat coordinates near x, and let (y^i) be flat coordinates at y given by parallel translating the ∂_{x^i} along the geodesic through $\bar{x}$. (Here we are assuming that $(x, f(x))$ is in the tubular neighborhood, as $(\mathrm{Id}, f)^*\mathrm{MQ}_{\Delta_\epsilon}$ vanishes otherwise.) Then

$$\mathrm{dvol}_{\alpha(x,y)} = \bigwedge_{i=1}^{n} \Big(\frac{-dx^i + dy^i}{\sqrt{2}} \Big),$$

since the normal fiber $\nu_{(\bar{x},\bar{x})}$ at $(\bar{x},\bar{x})$ consists of vectors of the form $(-v, v)$. In the (x^i), (y^i) coordinates, we may write $f = (f^1, ..., f^n)$. Then

$$\begin{aligned}
(\mathrm{Id}, f)^*\mathrm{dvol}_{\alpha(x,y)} &= (\mathrm{Id}, f)^* \bigwedge_{i=1}^{n} \Big(\frac{-dx^i + dy^i}{\sqrt{2}} \Big) = 2^{-n/2} \bigwedge_{i=1}^{n} (-dx^i + df^i) \\
&= 2^{-n/2} \bigwedge_{i=1}^{n} \Big(-dx^i + \frac{\partial f^i}{\partial x^j} dx^j \Big) \\
&= 2^{-n/2} \bigwedge_{i=1}^{n} \Big[\Big(\frac{\partial f^i}{\partial x^i} - 1 \Big) dx^i + \sum_{i \neq j} \frac{\partial f^i}{\partial x^j} dx^j \Big].
\end{aligned}$$

Let A be the matrix:

$$a_{ij} = \begin{cases} (\partial f_i/\partial x^i) - 1, & i = j, \\ \partial f_i/\partial x^j, & i \neq j, \end{cases}$$

i.e. $A = df \circ \| - \mathrm{Id}$, where $\|$ denotes parallel translation from $f(x)$ to x along their geodesic. Then

$$\begin{aligned}
(\mathrm{Id}, f)^*\mathrm{dvol}_{\alpha(x,y)} &= 2^{-n/2} \bigwedge_{i=1}^{n} \sum_{j=1}^{n} a_{ij} dx^j = 2^{-n/2} \det(A) \mathrm{dvol}_M \\
&= 2^{-n/2} \det(df \circ \| - \mathrm{Id}) \mathrm{dvol}_M,
\end{aligned}$$

and so

$$\begin{aligned}
(\mathrm{Id}, f)^*\mathrm{MQ}_{\Delta_\epsilon} &= (\mathrm{Id}, f)^*\alpha^*\mathrm{MQ}_{\nu_\Delta} \qquad (2.7)\\
&= (2\pi)^{-n/2} e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})} \rho' \Big(\frac{d(x, f(x))}{\sqrt{2}} \Big) \det(df \circ \| - \mathrm{Id}) \mathrm{dvol}_M.
\end{aligned}$$

Since $(-1)^n \det(df \circ \| - \mathrm{Id}) = \det(\mathrm{Id} - df \circ \|)$, Theorem 2.2 and (2.7) yield:

THEOREM 2.3 *Let $f : M \to M$ be a smooth map of a closed, oriented, flat n-manifold M. Let $\rho : [0, \epsilon) \to [0, \infty)$ be an orientation preserving diffeomorphism and set $\rho(t) = \infty$ for $t \geq \epsilon$. Then the Lefschetz number of f is given by*

$$L(f) = \frac{1}{(2\pi)^{n/2}} \int_M e^{-\rho^2\left(\frac{d(x,f(x))}{\sqrt{2}}\right)} \rho'\left(\frac{d(x,f(x))}{\sqrt{2}}\right) \det\left(\mathrm{Id} - df \circ \|\right) \mathrm{dvol}_M.$$

Note that when f is the identity map, $\det(\mathrm{Id} - df \circ \|)$ vanishes and we get $\chi(M) = L(\mathrm{Id}) = 0$, which reflects the fact that the Euler characteristic of a flat manifold is zero. Of course, the $\sqrt{2}$ factor in the integrand can be incorporated into the diffeomorphism ρ.

We give some easy applications of our techniques. Let A_f be the portion of the graph of f inside the tubular neighborhood:

$$A_f = \{x \in M | (x, f(x)) \in \Delta_\epsilon^M\}.$$

COROLLARY 2.1 *(i) Let $f : M \to M$ be a smooth map of a closed, oriented, flat manifold M such that*

a. For some $\epsilon > 0$, A_f is connected, and

b. $(\mathrm{Id} - df \circ \|)$ is invertible on A_f.

Then f has a fixed point.

(ii) If g is C^0 close to the identity map, then there exists $x \in M$ and $v \in T_xM$, $v \neq 0$, such that $(dg \circ \|)(v) = v$.

PROOF: (i) Since $\mathrm{Id} - df \circ \|$ is invertible on A_f, sgn $\det(\mathrm{Id} - df \circ \|)$ is constant on the open set A_f. The integrand in Theorem 2.3 thus has constant sign on its support, which is contained in A_f. Thus $L(f) \neq 0$, and so f has a fixed point.

(ii) Since g is close to the identity, $A_g = M$. If $(dg \circ \|)(v) \neq v$ for all $0 \neq v \in TM$, then $\mathrm{Id} - dg \circ \|$ is invertible. This implies $L(g) \neq 0$, which contradicts $L(g) = L(\mathrm{Id}) = \chi(M) = 0$. □

Set $\|df\| = \sup_{x \in M} \|df_x\|$.

PROPOSITION 2.1 *Let $f : M \to M$ be a smooth map of a closed, oriented, flat n-manifold M. Then*

$$|L(f)| \leq \frac{C}{(2\pi)^{n/2}} \mathrm{vol}(M)(\|df\| + 1)^n$$

for some constant $C > 0$.

PROOF: We may choose ρ so that $\lim_{z \to \epsilon} e^{-\rho^2(z)} \rho'(z) = 0$. Hence there exists $C > 0$ such that $0 \leq e^{-\rho^2(z)} \rho'(z) \leq C$. Note also that for $v \in T_{f(x)}M$,

$$|(\mathrm{Id} - df \circ \|)(v)| \leq (\|df| \ + 1)|v|,$$

since parallel translation in an isometry. Thus $|\det(\mathrm{Id} - df \circ \|)| \leq (\|df\| + 1)^n$. By Theorem 2.3, we have

$$\begin{aligned} |L(f)| &\leq \frac{1}{(2\pi)^{n/2}} \int_M \left| e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})} \rho'\Big(\frac{d(x,f(x))}{\sqrt{2}}\Big) \right| \cdot \left| \det(\mathrm{Id} - df \circ \|) \right| \mathrm{dvol} \\ &\leq \frac{C}{(2\pi)^{n/2}} \mathrm{vol}(M)(\|df\| + 1)^n. \end{aligned}$$

□

The appendix contains a similar result for arbitrary manifolds via Hodge theory.

2.4 Local expressions for arbitrary metrics

In this subsection we calculate the local expression for the integrand in Theorem 2.2 for an arbitrary Riemanninan metric. This is more involved than in the flat case because the exponential map is nontrivial. The Jacobi fields which measure the deviation of the exponential map from the identity enter the computations.

A tubular neighborhood Δ_ϵ of the diagonal Δ in $M \times M$ is diffeomorphic to a neighborhood of the zero section in ν_M, which in turn is diffeomorphic to a neighborhood of zero in TM. The Levi-Civita connection on M determines the space H_M of horizontal vectors on TTM, while the space V_M of vertical vectors is independent of the connection. The Mathai-Quillan form MQ_{TM} is written in terms of horizontal and vertical vectors, so we have to identify the corresponding horizontal and vertical vectors in the tube in order to compute $\mathrm{MQ}_{\Delta_\epsilon}$.

Let α be the isomorphism from the neighborhood in ν_M to the tube: for $\nu_M = \{(v, -v) : v \in TM\}$, we have $\alpha(v, -v) = (\exp_{\bar{x}}(v), \exp_{\bar{x}}(-v))$ at $(\bar{x}, \bar{x}) \in \Delta$. As before, we take the radius of the tube small enough so that there exists a unique minimal geodesic between x and y whenever (x, y) is in the tube. If $(x, f(x)) \in \Delta_\epsilon$, we know that $\bar{x}$ is the midpoint of the unique minimal geodesic γ from x to $f(x)$ and $|v| = d(x, f(x))/2$.

Pick an orthonormal frame $\{Y_i\}$ at $\bar{x}$. Let $\beta : TM \to \nu_M$ be the bundle isomorphism $\beta(v_x) = (v_x, -v_x)$. The horizontal space H in the tube is defined to be $d(\alpha\beta)(H_M)$, and the vertical space V in the tube is $d(\alpha\beta)(V_M)$. Define vectors $X_i, \widetilde{X}_i$ at x by

$$\begin{aligned} X_i &= d(\exp_{\bar{x}})_v(Y_i), \\ \widetilde{X}_i &= d(\exp.\, \|v)_{\bar{x}}(Y_i), \end{aligned} \tag{2.8}$$

where in the first line Y_i is trivially translated to a vector in $T_v T_{\bar{x}} M$, and in the second line $\|v$ denotes the parallel translation of v along a curve in M with tangent vector Y_i. Similarly define vectors $Z_i, \widetilde{Z}_i$ at $f(x)$ by replacing v in (2.8) with $-v$. If we parametrize γ from $\bar{x}$ to x as $\gamma(t)$, then $\widetilde{X}_i$ is the endpoint of a Jacobi field J with $J(0) = Y_i$ – i.e. J is the variation vector field of the family of geodesics $\gamma_s(t) = \exp_{\eta(s)}(t\|v)$, where $\dot{\eta}(0) = Y_i$ and $t \in [0,1]$. Similarly, X_i is the endpoint of a Jacobi field J, the variation vector field of the family of geodesics $\gamma_s(t) = \exp_{\bar{x}}(t(v + sY_i))$, which has $(\nabla J)(0) = Y_i$ (cf. [4, Cor. 3.46]). Similar remarks apply to $Z_i, \widetilde{Z}_i$.

Lemma 2.4 *The vertical space V at $(x, f(x))$ is spanned by $\{(-X_i, Z_i)\}$ and the horizontal space H is spanned by $\{(\widetilde{X}_i, \widetilde{Z}_i)\}$.*

Proof: Set $\delta = \alpha\beta$. A vertical vector at $v \in T_{\bar{x}}M$ is a tangent vector Y to a curve $\eta(t) \subset T_xM$ with $\eta(0) = v, \dot{\eta}(0) = Y$. Then

$$\begin{aligned} d\delta_v(Y) &= \left.\frac{d}{dt}\right|_{t=0} (\exp_{\bar{x}} \eta(t),\ \exp_{\bar{x}}(-\eta(t))) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \exp_{\bar{x}} \eta(t), \left.\frac{d}{dt}\right|_{t=0} \exp_{\bar{x}}(-\eta(t)) \right) \\ &= (d(\exp_{\bar{x}})_v Y,\ d(\exp_{\bar{x}})_v(-Y)). \end{aligned}$$

Thus the vertical space at $(x, f(x)) = (\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$ is spanned by $\{(d(\exp_{\bar{x}})_v(Y_i), d(\exp_{\bar{x}})_{-v}(Y_i))\}$.

Let $\|v = \|_y v$ denote the parallel translation of v along radial geodesics centered at $\bar{x}$. Then $\|v$ is parallel at $\bar{x}$, and the horizontal vectors at v are spanned by

$$\left.\frac{d}{dt}\right|_{t=0} \|_{\exp_{\bar{x}}(tY_i)} v.$$

Thus the horizontal vectors at $(x, f(x))$ are spanned by

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \delta(\|_{\exp_{\bar{x}}(tY_i)} v) &= \left.\frac{d}{dt}\right|_{t=0} (\exp_{\exp_{\bar{x}}(tY_i)} \|_{\exp_{\bar{x}}(tY_i)} v, \exp_{\exp_{\bar{x}}(tY_i)} \|_{\exp_{\bar{x}}(tY_i)}(-v)) \\ &= (\widetilde{X}_i, \widetilde{Z}_i). \end{aligned}$$

□

Remarks: 1) The lemma shows that at $(x, f(x))$,

$$\begin{aligned} V &= (-d(\exp_{\bar{x}})_v, d(\exp_{\bar{x}})_{-v}) V_M, \\ H &= (d(\exp \|v)_{\bar{x}}, d(\exp -\|v)_{\bar{x}}) H_M. \end{aligned}$$

2) It is easy to check that $X_i, \widetilde{X}_i, Z_i, \widetilde{Z}_i$ are just parallel translations of Y_i if M is flat.

3) Vertical vectors at $(x, f(x))$ are those pairs of vectors in $T_xM \times T_{f(x)}M$ which are endpoints of a Jacobi field along γ which vanishes at $\bar{x}$. Horizontal vectors are pairs of vectors which are endpoints of a Jacobi field along γ which is parallel at $\bar{x}$.

Lemma 2.5 *Let* $(X,Z) \in T_{(x,f(x))}(M \times M)$. *Take the unique Jacobi field* Y *along* γ *with* $Y(x) = X$, $Y(f(x)) = Z$. *Let* X_1, Z_1 *be the values at* $x, f(x)$ *of the Jacobi field* Y_1 *along* γ *given by* $Y_1(\bar{x}) = 0$, $\frac{DY_1}{dt}(\bar{x}) = \frac{DY}{dt}(\bar{x})$. *Let* $\widetilde{X}_1, \widetilde{Z}_1$ *be the values at* $x, f(x)$ *of the Jacobi field* Y_2 *along* γ *given by* $Y_2(\bar{x}) = Y(\bar{x})$, $\frac{DY_2}{dt}(\bar{x}) = 0$. *Then* $(X,Z) = (X_1, Z_1) + (\widetilde{X}_1, \widetilde{Z}_1)$ *is the decomposition of* (X,Z) *into vertical and horizontal vectors.*

Proof: By Remark 3, $(X_1, Z_1), (\widetilde{X}_1, \widetilde{Z}_1)$ are vertical and horizontal vectors respectively. Since the Jacobi equation is linear, the endpoints of the Jacobi field $Y_1 + Y_2$ are $X_1 + \widetilde{X}_1, Z_1 + \widetilde{Z}_1$. Since $Y_1 + Y_2$ has the same position and velocity vectors as Y as $\bar{x}$, we must have $X = X_1 + \widetilde{X}_1, Z = Z_1 + \widetilde{Z}_1$. □

Let $\rho : [0,\epsilon) \to [0,\infty)$ be a diffeomorphism fixing zero, and which extends smoothly to an even function on $(-\epsilon,\epsilon)$. Let MQ_ν be the Mathai-Quillen form of the normal bundle $\nu = \nu_\Delta$ and let $\mathrm{MQ}_{\Delta_\epsilon} = (\exp^{-1})^*\rho^*\mathrm{MQ}_\nu$ be the corresponding Mathai-Quillen form on $M \times M$. Here we abbreviate $(\exp^{-1}, \exp^{-1})$ to just $\exp^{-1}$.

In (2.2), the vertical coordinates are denoted by x^i and the horizontal coordinates are hidden in Ω_I. For the calculations on TM, we need to make the horizontal coordinates explicit and take care not to confuse them with the vertical coordinates. So let $\{x^i\}$ be a synchronous orthonormal frame centered at $\bar{x}$. In each fiber of ν, we take the orthonormal polar coordinate frame $\{(-x^i, x^i)\}$, with $\{x^i\} = \{x^1 = \partial_r, r^{-1}\partial_{\theta^i}\}$, away from the origin. These frames do not agree at the origin in each fiber, but the formulas below will be smooth at the origin. We know $(\rho^* dx^i)_{v=0} = 0$ since $\rho'(0) = 0$, and for $v \neq 0$,

$$\begin{aligned} \rho^* dx^1_v &= \rho^* dr_v = \rho'(|v|) dr_{\rho(v)}, \\ \rho^* d\theta^i_v &= d\theta^i_{\rho(v)}. \end{aligned}$$

Thus

$$[(\exp^{-1})^*\rho^* dx^i](\alpha) = \begin{cases} \rho'(|(\exp^{-1})_*\alpha|)(\exp^{-1})^* dx^i(\alpha), & \text{if } i = 1, \\ \\ (\exp^{-1})^* dx^i(\alpha), & \text{if } i \neq 1. \end{cases}$$

If $\{y^i\}$ is another synchronous frame centered at $\bar{x}$ (possibly equal to $\{x^i\}$), then the horizontal lifts of y^i into $T\nu$ are orthonormal in the metric on $T\nu$ induced by the metric on M. Since $\rho_*(y^i) = y^i$, we have

$$[(\exp^{-1})^*\rho^* dy^i](\alpha) = (\widetilde{X}_i, \widetilde{Z}_i)^\#(\alpha),$$

where $(\widetilde{X}_i, \widetilde{Z}_i)^\#$ is the cotangent vector dual to $(\widetilde{X}_i, \widetilde{Z}_i)$. If $(\mathrm{Id}, f)^*(\mathrm{MQ}_{\Delta_\epsilon})_{(x,x)} = D\ \mathrm{dvol}_{(x,x)}$, then

$$D = (\mathrm{Id}, f)^*(\mathrm{MQ}_{\Delta_\epsilon})_{(x,x)}\left(\frac{(y^1, y^1)}{\sqrt{2}}, \ldots, \frac{(y^n, y^n)}{\sqrt{2}}\right).$$

Thus

$$D = \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} \epsilon(I,I')[(\exp_{\bar{x}}^{-1})^*\rho^*]\mathrm{Pf}(\Omega_I) \wedge dx^{I'}((y^1, f_*y^1), \ldots, (y^n, f_*y^n)).$$

Write

$$(y^i, f_*y^i) = P_V^i + P_H^i \tag{2.9}$$

for the decomposition of (y^i, f_*y^i) into vertical and horizontal vectors as in the lemma. Let Σ_n be the permutation group on $\{1, \ldots, n\}$. Then

$$D = \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} \frac{\epsilon(I,I')}{|I|!\,|I'|!} \sum_{\mu\in\Sigma_n} (\exp_{\bar{x}}^{-1})^*\mathrm{Pf}(\Omega_I)(P_H^{\mu_1}, \ldots, P_H^{\mu_{|I|}}) \cdot\widetilde{\rho}'_{I'}\left(\frac{d(x,f(x))}{\sqrt{2}}\right) d\widetilde{x}^{I'}\left(P_V^{\mu_{|I|+1}}, \ldots, P_V^{\mu_n}\right),$$

where

$$\widetilde{\rho}'_{I'}\left(\frac{d(x,f(x))}{\sqrt{2}}\right) = \begin{cases} \rho'(\frac{d(x,f(x))}{\sqrt{2}}), & \text{if } i'_1 = 1, \\ 1, & \text{if } i'_1 \neq 1. \end{cases}$$

Here $d\widetilde{x}^{I'} = (\exp_{\bar{x}}^{-1})^* dx^i$, and we have used $\rho^*\ \mathrm{Pf}(\Omega_I) = \mathrm{Pf}(\Omega_I)$, since this Pfaffian is a horizontal form.

Define $n \times n$ matrices $A = A_x$, $B = B_x$ by

$$(\exp_{\bar{x}}^{-1})_* P_H^i = A_j^i y^j, \quad (\exp_{\bar{x}}^{-1})_* P_V^i = B_j^i x^j, \tag{2.10}$$

where strictly speaking the last term is $(-B_j^i x^j, B_j^i x^j)$. For example, at a fixed point $x = f(x)$, the decomposition of (q, f_*q) into vertical and horizontal components is given by

$$(q, f_*q) = \left(\frac{q - f_*q}{2}, \frac{-q + f_*q}{2}\right) + \left(\frac{q + f_*q}{2}, \frac{q + f_*q}{2}\right),$$

since $(\exp^{-1})_* = \mathrm{Id}$ at a fixed point. Since vertical (resp. horizontal) vectors on the diagonal are of the form $(-v, v)$ (resp. (v, v)), A is the matrix of $\frac{1}{2}(df + \mathrm{Id})$ and B is the matrix of $\frac{1}{2}(df - \mathrm{Id})$. It follows easily from Remark 2 that on a flat manifold, $A = \frac{1}{2}(\| \circ df + \mathrm{Id})$, $B = \frac{1}{2}(\| \circ df - \mathrm{Id})$ for arbitrary x.

Then

$$D = \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} c_{|I|} \frac{\epsilon(I,I')}{|I|!\,|I'|!} \sum_{\mu\in\Sigma_n} (\mathrm{sgn}\ \mu) A_{j_1}^{\mu(1)} \cdot \ldots \cdot A_{j_{|I|}}^{\mu(|I|)} \cdot\mathrm{Pf}(\Omega_I)_{\bar{x}}(y^{j_1}, \ldots, y^{j_{|I|}}) \cdot\widetilde{\rho}'_{I'}\left(\frac{d(x,f(x))}{\sqrt{2}}\right) B_{k_1}^{\mu(|I|+1)} \cdot \ldots \cdot B_{k_{|I'|}}^{\mu(n)} dx^{I'}(x^{k_1}, \ldots, x^{k_{|I'|}}). \tag{2.11}$$

(We use summation convention for the j and k indices.) The right hand side of (2.11) vanishes unless $I' = \{k_1, \ldots, k_{|I'|}\} \equiv K$, and

$$\sum_{K, K=I'} B_{k_1}^{\mu(|I|+1)} \cdot \ldots \cdot B_{k_{|I'|}}^{\mu(n)} dx^{I'}(x^{k_1}, \ldots, x^{k_{|I'|}}) = \det(B_{i'_s}^{\mu(|I|+q)}),$$

for $q, s = 1, \ldots, |I'|$. Here $I' = \{i'_1, \ldots, i'_{|I'|}\}$, with $i'_1 < \ldots < i'_{|I'}$. We denote this determinant by $\det(B_{I'}^{\mu})$. For $I = \{i_1, \ldots, i_{|I|}\}$, with $i_1 < \ldots < i_{|I|}$, we have

$$\begin{aligned} \mathrm{Pf}(\Omega_I) &= c_{|I|} \sum_{\sigma,\tau \in \sum_{|I|}} (\mathrm{sgn}\ \sigma)(\mathrm{sgn}\ \tau) R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}} \\ &\quad \cdot dy^1 \wedge \ldots \wedge dy^{|I|}, \end{aligned}$$

with $c_{|I|} = (-1)^{|I|/2}[2^{|I|}(|I|/2)!]^{-1}$ [8, (1.3)]. (The $(-1)^{|I|/2}$ reflects our sign convention on curvature.) As above, the right hand side of (2.11) vanishes unless $I = \{j_1, \ldots, j_{|I|}\} \equiv J$. Summing over such J produces the term $\det(A_{i_k}^{\mu(t)}) \equiv \det(A_I^{\mu})$, for $t, k = 1, \ldots, |I|$.

Thus

$$\begin{aligned} D &= \frac{e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}}{(2\pi)^{n/2}} \sum_{I,I'} \sum_{\mu \in \Sigma_n} c_{|I|} \frac{\epsilon(I,I')}{|I|!\ |I'|!} (\mathrm{sgn}\ \mu) \det(A_I^{\mu}) \det(B_{I'}^{\mu}) \widetilde{\rho}'_{I'} \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \\ &\quad \cdot \sum_{\sigma,\tau \in \Sigma_{|I|}} (\mathrm{sgn}\ \sigma)(\mathrm{sgn}\ \tau) R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}}. \end{aligned}$$

Since $\int_M dy^1 \wedge \ldots \wedge dy^n = 2^{-n/2} \int_\Delta d(y^1, y^1) \wedge \ldots \wedge d(y^n, y^n)$, we have

THEOREM 2.4 *Let $f : M \to M$ and fix $\epsilon > 0$ such that $(x, y) \in \Delta_M^{\epsilon}$ implies the existence of a unique minimal geodesic between x and y. For $x \in M$, define matrices A, B by (2.9), (2.10), provided $(x, f(x))$ is in the ϵ-neighborhood of the diagonal; otherwise set $A = B = 0$. Then*

$$\begin{aligned} L(f) &= \frac{(-1)^{\dim\ M}}{(2\pi)^{n/2}} \int_M e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})} \sum_{I,I'} c_{|I|} \frac{\epsilon(I,I')}{|I|!\ |I'|!} \sum_{\mu \in \Sigma_n} (\mathrm{sgn}\ \mu) \det(A_I^{\mu}) \det(B_{I'}^{\mu}) \\ &\quad \cdot \widetilde{\rho}'_{I'} \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \cdot \sum_{\sigma,\tau \in \Sigma_{|I|}} R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}} \\ &\quad \cdot \mathrm{dvol}. \end{aligned}$$

The hypothesis on ϵ is satisfied once ϵ is less than half the injectivity radius of M. If the injectivity radius i is known, we can set $\rho(x) = \sec(\pi x/2i) - 1$, for example.

As a check, we consider the case where $f = \mathrm{Id}$. Then $A = \mathrm{Id}$ and $B = 0$, so if $n = \dim M$ is odd, the integrand vanishes and we get $\chi(M) = L(\mathrm{Id}) = 0$. If n is even, the only contribution to the integrand occurs when $I = \{1, \ldots, n\},\ I' = \emptyset$. For this I, we have $(\text{sgn } \mu) \det(A_I^\mu) = 1$. The theorem becomes

$$\begin{aligned}\chi(M) = L(\text{Id}) &= \frac{(-1)^{n/2}}{(8\pi)^{n/2}(n/2)!} \int_M \sum_{\tau,\sigma \in \Sigma_n} (\text{sgn } \sigma)(\text{sgn } \tau) e^0 R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \\ &\quad \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}} \text{dvol},\end{aligned}$$

the Chern-Gauss-Bonnet theorem.

The other extremal case occurs when M is flat. Because $R_{ijkl} = 0$, the only contribution to the integrand occurs when $I = \emptyset$. Then $c_0 = 1$ and $(\text{sgn } \mu) \det(B_{I'}^\mu) = \det(\| \circ df - \text{Id})$, so the integrand becomes

$$\frac{1}{(2\pi)^{n/2}} e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})} \rho' \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \det(\text{Id} - \| \circ df),$$

which agrees with the flat case formula, since parallel translation is an isometry.

Note that at a fixed point, the integrand becomes

$$\sum_{\substack{I,I' \\ 1 \notin I'}} c_{|I|} \frac{\epsilon(I, I')}{|I|!\ |I'|!} \sum_{\mu \in \Sigma_n} (\text{sgn } \mu) \det\Big(\frac{1}{2}(df - \text{Id})_{I'}^\mu \Big) \det\Big(\frac{1}{2}(df + \text{Id})_I^\mu \Big)$$

$$\cdot \rho' \left(\frac{d(x, f(x))}{\sqrt{2}} \right) \sum_{\tau,\sigma \in \Sigma_{|I|}} (\text{sgn } \sigma)(\text{sgn } \tau) R_{i_{\sigma(1)} i_{\sigma(2)} i_{\tau(1)} i_{\tau(2)}} \cdot \ldots \cdot R_{i_{\sigma(|I|-1)} i_{\sigma(|I|)} i_{\tau(|I|-1)} i_{\tau(|I|)}}.$$

Example: We determine the integrand in the Lefschetz formula for M an oriented surface of constant sectional/Gaussian curvature -1. At the end we indicate the changes for constant curvature manifolds in general.

We first determine the horizontal and vertical components of a vector $(q, f_* q) \in T_{(x,f(x))}(M \times M)$. Assume there exists a unique minimal geodesic γ joining x to $f(x)$ with midpoint $\bar{x}$. Let $|\dot\gamma| = 1$, and let α be the unit normal to γ determined by the orientation. Set $d = d(x, f(x))$.

Let J be a Jacobi field along γ. Plugging $J(t) = a(t)\dot\gamma + b(t)\alpha$ into the Jacobi equation $D^2 J/dt^2 + R(\dot\gamma, J)\dot\gamma = 0$ and using $\langle R(\dot\gamma, J)\dot\gamma, \dot\gamma \rangle = 0,\ \langle R(\dot\gamma, \alpha)\dot\gamma, \alpha \rangle = -1$ yields $\ddot a = 0,\ \ddot b - b = 0$. Thus $J(t) = (c_0 + c_1 t)\dot\gamma + (d_1 \sinh(t) + d_2 \cosh(t))\alpha$. Imposing the boundary conditions $J(0) = q,\ J(d) = f_* q$ gives

$$J(t) = \left(q_1 + \left(\frac{w_1 - q_1}{d} \right) t \right) \dot\gamma + \left[\left(\frac{w_2 - q_2 \cosh(d)}{\sinh(d)} \right) \sinh(t) + q_2 \cosh(t) \right] \alpha,$$

where $q = q_1 \dot\gamma + q_2 \alpha,\ f_* q = w_1 \dot\gamma + w_2 \alpha$. In particular

$$J\left(\frac{d}{2} \right) = \left(\frac{q_1 + w_1}{2} \right) \dot\gamma + \left[\left(\frac{w_2 - q_2 \cosh(d)}{\sinh(d)} \right) \sinh\left(\frac{d}{2} \right) + q_2 \cosh\left(\frac{d}{2} \right) \right] \alpha$$

$$
\begin{aligned}
&= \left(\frac{q_1+w_1}{2}\right)\dot\gamma + \left(\frac{w_2+q_2}{2\cosh(\frac{d}{2})}\right)\alpha,\\
\frac{DJ}{dt}\left(\frac{d}{2}\right) &= \left(\frac{w_1-q_1}{d}\right)\dot\gamma + \left(\frac{w_2-q_2}{2\sinh(\frac{d}{2})}\right)\alpha,
\end{aligned}
$$

The Jacobi fields J_1, J_2 determined by $J_1(d/2) = 0$, $(DJ_1/dt)(d/2) = (DJ/dt)(d/2)$ and $J_2(d/2) = J(d/2)$, $(DJ_2/dt)(d/2) = 0$ are given by

$$
\begin{aligned}
J_1(s) &= \left(\frac{w_1-q_1}{d}\right)s\dot\gamma + \left(\frac{w_2-q_2}{2\sinh(d/2)}\right)\sinh(s)\alpha,\\
J_2(s) &= \left(\frac{q_1+w_1}{2}\right)\dot\gamma + \left(\frac{w_2+q_2}{2\cosh(d/2)}\right)\cosh(s)\alpha,
\end{aligned}
$$

where $s = 0$ corresponds to $\bar x$. Evaluating J_1, J_2 at $s = \pm d/2$ gives the decomposition of (q, f_*q) into vertical and horizontal components:

$$
\begin{aligned}
(q, f_*q)_{\mathrm{vert}} &= \left(\left(\frac{-w_1+q_1}{2}\right)\dot\gamma - \left(\frac{w_2-q_2}{2}\right)\alpha, \left(\frac{w_1-q_1}{2}\right)\dot\gamma + \left(\frac{w_2-q_2}{2}\right)\alpha\right),\\
(q, f_*q)_{\mathrm{hor}} &= \left(\left(\frac{q_1+w_1}{2}\right)\dot\gamma + \left(\frac{w_2+q_2}{2}\right)\alpha, \left(\frac{q_1+w_1}{2}\right)\dot\gamma + \left(\frac{w_2+q_2}{2}\right)\alpha\right).
\end{aligned}
$$

Let $(x, f(x)) = (\exp_{\bar x} v, \exp_{\bar x}(-v))$, so $v = -(d/2)\dot\gamma$ at $\bar x$. We now determine $(\exp_{\bar x}^{-1})_*^{(2)}\colon T_{(x,f(x))}(M\times M) \to T_{(v,-v)}T_{(\bar x,\bar x)}(M\times M)$, where $(\exp_{\bar x}^{-1})_*^{(2)}$ is shorthand for $(-(\exp_{\bar x}^{-1})_*, (\exp_{\bar x}^{-1})_*)$. For a vertical vector $\beta = \beta_1\dot\gamma + \beta_2\alpha \in T_vT_{\bar x}M$, with $\dot\gamma, \alpha$ trivially parallel translated to v,

$$
(\exp_{\bar x})_{*,v}(\beta) = \left.\frac{d}{ds}\right|_{s=0} \exp_{\bar x}(v + s\beta),
$$

which is the value at x of the Jacobi field J along γ with $J(\bar x) = 0$, $(DJ/dt)(\bar x) = 2\beta/d$, since $|v| = d/2$. Solving for J as above, we get

$$
(\exp_{\bar x})_{*,v}(\beta) = \beta_1\dot\gamma + \frac{2}{d}\beta_2\sinh\left(\frac{d}{2}\right)\alpha.
$$

Thus

$$
(\exp_{\bar x}^{-1})_*^{(2)}(q, f_*q)_{\mathrm{vert}} =
$$

$$
\left(\left(\frac{-w_1+q_1}{2}\right)\dot\gamma - \frac{d(w_2-q_2)}{4\sinh(\frac{d}{2})}\alpha, \left(\frac{-w_1+q_1}{2}\right)\dot\gamma - \frac{d(w_2-q_2)}{r\sinh(\frac{d}{2})}\alpha\right).
$$

Similarly, for a horizontal vector $\delta = \delta_1\dot\gamma + \delta_2\alpha$, where $\dot\gamma, \alpha$ now denote the horizontal lifts of $\dot\gamma, \alpha$ to $T_vT_{\bar x}M$, we have

$$
(\exp_{\bar x})_{*,v}(\delta) = \delta_1\dot\gamma + \delta_2\cosh\left(\frac{d}{2}\right)\alpha,
$$

so

$$(\exp_{\bar{x}}^{-1})_*^{(2)}(q, f_*q)_{\rm hor} =$$
$$\left(\left(\frac{q_1+w_1}{2}\right)\dot\gamma + \left(\frac{w_2+q_2}{2\cosh(\frac{d}{2})}\right)\alpha, \left(\frac{q_1+w_1}{2}\right)\dot\gamma + \left(\frac{w_2+q_2}{2\cosh(\frac{d}{2})}\right)\alpha\right).$$

To determine the matrix B, we have to express $(\exp_{\bar{x}}^{-1})_*^{(2)}(q, f_*, q)$, for $q = \dot\gamma, \alpha$, in polar coordinates at $(v, -v)$ in $\nu_{(\bar{x},\bar{x})}$. The radial vector at $(v, -v)$ is

$$r = \left(\frac{v}{|v|\sqrt{2}}, \frac{-v}{|v|\sqrt{2}}\right) = \left(\frac{v\sqrt{2}}{d}, \frac{v\sqrt{2}}{d}\right),$$

and the unit angular vector "$r^{-1}\partial_\theta$" is $(-\alpha/\sqrt{2}, \alpha/\sqrt{2})$. Note that $(\dot\gamma, -\dot\gamma) = -\sqrt{2}r$. For $f_*\dot\gamma = w_{11}\dot\gamma + w_{12}\alpha$, $f_*\alpha = w_{21}\dot\gamma + w_{22}\alpha$, we get

$$(B^i_j) = \begin{pmatrix} -\frac{\sqrt{2}}{2}(-w_{11}+1) & \frac{\sqrt{2}dw_{12}}{4\sinh(d/2)} \\ \frac{\sqrt{2}}{2}w_{21} & \frac{\sqrt{2}d(w_{22}-1)}{4\sinh(d/2)} \end{pmatrix}.$$

Thus

$$\begin{aligned} \det B &= \frac{d}{4\sinh(\frac{d}{2})}((w_{11}-1)(w_{22}-1) - w_{12}w_{21}) \\ &= \frac{d}{\sinh(\frac{d}{2})}\det\left(\frac{1}{2}(\| \circ df - \mathrm{Id})\right). \end{aligned}$$

Similarly,

$$(A^i_j) = \begin{pmatrix} -\frac{w_{11}+1}{\sqrt{2}} & \frac{-\sqrt{2}w_{12}}{2\cosh(d/2)} \\ \frac{-w_{21}}{\sqrt{2}} & \frac{-\sqrt{2}(w_{22}+1)}{2\cosh(\frac{d}{2})} \end{pmatrix},$$

and

$$\det A = \frac{2}{\cosh(\frac{d}{2})}\det\left(\frac{1}{2}(\| \circ df + \mathrm{Id})\right).$$

We now plug this information into the Lefschetz formula. Note that $I = \{1, 2\}$ or $I = \emptyset$ and that $R_{1212} = 1$ in our convention. We obtain

$$\begin{aligned} L(f) &= \frac{1}{2\pi}\int_M e^{-\rho^2(\frac{d}{\sqrt{2}})}\left[\left(\frac{2}{\cosh(\frac{d}{2})}\right)\frac{(-1)\cdot 4}{4\cdot 2!}\det\left(\frac{1}{2}(\| \circ df + \mathrm{Id})\right)\right. \\ &\qquad \left. + \rho'\left(\frac{d}{\sqrt{2}}\right)\left(\frac{d}{\sinh(\frac{d}{2})}\right)\cdot\frac{1}{2!}\det\left(\frac{1}{2}(\| \circ df - \mathrm{Id})\right)\right] dA. \end{aligned}$$

In the first line, there are factors of $c_{\{1,2\}} = -1/4$, $|I|! = 2$, and $\sum_{\sigma,\tau\in\Sigma_2} R_{\sigma(1)\sigma(2)\tau(1)\tau(2)} = 4$. In the second line, $c_\emptyset = 1$ and $|I'|! = 2$. Thus we obtain

Proposition 2.2 *Let M be an oriented surface of constant curvature -1. Then*

$$\begin{aligned} L(f) &= \frac{1}{2\pi}\int_M e^{-\rho^2(\frac{d(x,f(x))}{\sqrt{2}})}\left[\frac{-\det(\frac{1}{2}(\| \circ df_x + \mathrm{Id}))}{\cosh(\frac{d(x,f(x))}{2})}\right. \\ &\qquad \left. + \rho'\left(\frac{d(x,f(x))}{\sqrt{2}}\right)\left(\frac{d(x,f(x))}{2\sinh(\frac{d(x,f(x))}{2})}\right)\det\left(\frac{1}{2}(\| \circ df_x - \mathrm{Id})\right)\right] dA. \end{aligned}$$

It is straightforward to extend this result to higher dimensional constant curvature spaces. The integrand in Proposition 2.2 now involves a sum over I, I'. The general term inside the bracket is $c_{|I|}\epsilon(I,I')/[|I|!|I'|!]$ times

$$\sum_{\mu\in\Sigma_n}(\mathrm{sgn}\ \mu)\frac{2^{|I|/2}\det(\frac{1}{2}(\| \circ df_x + \mathrm{Id}))_I \det(\frac{1}{2}(\| \circ df_x - \mathrm{Id}))_{I'}\tilde{\rho}'(d(x,f(x))/\sqrt{2})}{[\cosh(d(x,f(x))/2)]^{|I|-1}\cdot[\sinh(d(x,f(x))/2)]^{|I'|-1}}$$

for negative curvature -1. For constant curvature 1, cosh, sinh are replaced by cos, sin, and there is an extra factor of $(-1)^{|I|}$ due to $R_{ijij} = -1$.

3 Large parameter behavior–topological methods

In [8, §7], a one-parameter family of pullbacks of the Mathai-Quillen form is constructed which interpolates between the Hopf index formula (as $t \to \infty$) and the Chern-Gauss-Bonnet theorem (at $t = 0$). In this section, we show by topological arguments that the corresponding large parameter behavior for Lefschetz theory is the Hopf fixed point/submanifold formula. At the end, we indicate a geometric refinement.

For motivation, we sketch the argument in [8]. Let $s : M \to TM$ be a vector field on M transverse to the zero section, and let MQ be the Mathai-Quillen form on TM for the Levi-Civita connection for a fixed Riemannian metric. For the zero section 0 of TM, we have $\mathrm{Pf}(\Omega) = 0^*\mathrm{MQ}$. For any $t \geq 0$, $\int_M (ts)^*\mathrm{MQ}$ is independent of t, since the integrands are cohomologous. As $t \to \infty$, the integrand decays exponentially away from the fixed point set, and the contribution from a fixed point p becomes $\pm\int_{T_pM}\mathrm{MQ} = 1$. Identifying the sign with the index $\mathrm{ind}(p)$ of s at p gives

$$\int_M \mathrm{Pf}(\Omega) = \sum_{p,f(p)=p}\mathrm{ind}(p) \tag{3.1}$$

These expressions equal $\chi(M)$ by either Chern-Gauss-Bonnet or the Hopf index formula, but the derivation of (3.1) is new.

This argument still has content at the topological level: if we replace MQ by any representative Φ of the Thom class of TM, we obtain at $t = 0$ the well known expression $\int_M 0^*\Phi$ for $\chi(M)$, and as $t \to \infty$ we still obtain the fixed point sum. Thus

Mathai and Quillen have produced a proof of the Hopf index formula. Note that the choice of section s is irrelevant. In particular, the integral $\int_M s^*\mathrm{MQ}$ at $t = 1$ has no particular significance.

To apply this method to the basic formula $L(f) = \int_M (\mathrm{Id}, f)^*\mathrm{MQ}$, we wish to replace f by tf. Since a tubular neighborhood of the diagonal in $M \times M$ is diffeomorphic to TM, we can consider f as a section of TM whenever the graph of f lies in the tube. In this case, we can define tf as above. Moreover, $(\mathrm{Id}, f)^*\mathrm{MQ}$ vanishes whenever the graph lies outside the tube, so the action of t might as well be trivial on this set. This scaling of f is done in detail below; for technical reasons we deform the graph of f in directions normal to the diagonal rather than vertically. Note that the case $t = 1$ now has particular significance: it is Theorem 2.2. In summary, we can think of f as a section of TM with possible blow up on part (or even all) of M.

From this point of view, we can derive the Lefschetz formula for submanifolds of fixed points (Theorem 3.1) as $t \to \infty$. This proof is an elementary version of the geometric stationary phase proof in [5, Ch. 4]. To be completely honest, there is a little geometry in the proof, but no more than in the usual proof of the tubular neighborhood theorem; even this geometry can be eliminated by a jet bundle argument.

In this section, ν_X^Y denotes the normal bundle of X in Y.

To state the theorem, let $f : M \to M$ be a smooth map of a closed oriented m-manifold M, and assume that the fixed point set of f consists of the disjoint union of smooth submanifolds N_j of dimension n_j. Let N be one such component, and let ν be the quotient bundle $\nu = TM/TN$ over N. Since df preserves the subbundle TN, it induces a map df_ν on ν.

We assume the non-degeneracy condition $\det(\mathrm{Id} - df_\nu) \neq 0$ (also known as clean intersection), i.e. f leaves infinitesimally fixed only directions tangent to N.

If we put df_n, $n \in N$, in Jordan canonical form, TN will be the span of eigenvectors with eigenvalues 1, and ν is isomorphic to the span of the generalized eigenvectors for the remaining eigenvalues. This induces a natural splitting of $TM|_N \simeq TN \oplus \nu$. A choice of Riemannian metric on M gives an identification of ν with ν_N^M, and df_ν with a map on ν_N^M.

Theorem 3.1 *Let $f : M \to M$ be a smooth non-degenerate map of a closed oriented m-manifold M, whose fixed point set consists of the disjoint union of submanifolds $N_1, N_2, ..., N_r$. Then*

$$L(f) = \sum_{j=1}^{r} \mathrm{sgn}(\det(\mathrm{Id} - df_\nu))\chi(N_j).$$

To begin the proof, we may assume that the fixed point set of f consists of a single submanifold N of dimension n. Let Δ_M^ϵ be an ϵ-neighborhood of Δ_M, the diagonal of M in $M \times M$. Choose $\epsilon > 0$ small enough so there exists a unique minimal geodesic from x to y, for all $(x, y) \in \Delta_M^\epsilon$.

We construct a family of diffeomorphisms $F_t : M \times M \to M \times M$, for $t > 0$, with $F_1 = \mathrm{Id}$, which pushes out fibers of $\nu_{\Delta_M}^{M\times M}$, while fixing Δ_M and $M \times M - \Delta_M^\epsilon$. Let $(x, y) \in \Delta_M^\epsilon$ and consider the geodesic γ in $M \times M$ from $(\bar{x}, \bar{x}) \in \Delta_M$ to (x, y), where $\bar{x}$ is the midpoint of the geodesic α between x and y in M. Setting $\dot{\alpha}(\bar{x}) = v = v(x, y) \in T_{\bar{x}}M$, we have $\dot{\gamma}(\bar{x}, \bar{x}) = (-v, v)$. For $v \neq 0$, define a diffeomorphism $\lambda_v(t) : [0, \infty) \to [0, \frac{\epsilon}{|v|})$ with $\lambda_v(1) = 1$, which is smooth in v, and set $\lambda_0(t) = 0$. Define $F_t : M \times M \to M \times M$ by:

$$F_t(x, y) = \begin{cases} (x, y), & (x, y) \notin \Delta_M^\epsilon, \\ \exp_{(\bar{x},\bar{x})}(\lambda_{v(x,y)}(t) \cdot \exp^{-1}_{(\bar{x},\bar{x})}(x, y)), & (x, y) \in \Delta_M^\epsilon. \end{cases}$$

F_t is the desired map. As in §2, we have

$$L(f) = I(\Delta, \Gamma) = (-1)^{\dim\ M} I(\Gamma, \Delta) = (-1)^{\dim\ M} \int_\Gamma \eta_{\Delta_M}^{M\times M},$$

where $\eta_{\Delta_M}^{M\times M}$ is the Poincaré dual of Δ_M in $M \times M$. Since F_t is homotopic to the identity we have

$$\begin{aligned} (-1)^{\dim\ M} L(f) &= \int_\Gamma \eta_{\Delta_M}^{M\times M} = \lim_{t\to\infty} \int_\Gamma F_t^* \eta_{\Delta_M}^{M\times M} = \lim_{t\to\infty} \int_{(\mathrm{Id},f)(\Delta_M)} F_t^* \eta_{\Delta_M}^{M\times M} \\ &= \lim_{t\to\infty} \int_{(\mathrm{Id},f)(\Delta_N^\delta)} F_t^* \eta_{\Delta_M}^{M\times M} = \lim_{t\to\infty} \int_{F_t\circ(\mathrm{Id},f)(\Delta_N^\delta)} \eta_{\Delta_M}^{M\times M}, \end{aligned} \tag{3.2}$$

where Δ_N^δ is a δ-neighborhood of Δ_N in Δ_M, for δ small enough. This uses

$$\lim_{t\to\infty} \int_{(\mathrm{Id},f)(\Delta_M \setminus \Delta_N^\delta)} F_t^* \eta_{\Delta_M}^{M\times M} = 0,$$

as $F_t^* \eta_{\Delta_M}^{M\times M}$ decays uniformly as $t \to \infty$ on $\Delta_M \setminus \Delta_N^\delta$, since $d(x, f(x))$ and hence $|v|$ has positive minimum on M minus a δ-neighborhood of N.

Let $\pi : \Delta_N^\delta \to \Delta_N$ be the projection given by the identification of Δ_N^δ with a δ-neighborhood of the zero section of $\nu_{\Delta_N}^{\Delta_M}$.

The next lemma uses the non-degeneracy hypothesis.

Lemma 3.1 *If* $df_\nu - \mathrm{Id}$ *is invertible at* $n \in N$, *then* $T_{(n,n)}\Big[F_t \circ (\mathrm{Id}, f)(\pi^{-1}(n, n))\Big] \cap T_{(n,n)}\Delta_N = \{0\}$.

Proof: If $T_{(n,n)}\Big[F_t \circ (\mathrm{Id}, f)(\pi^{-1}(n, n))\Big] \cap T_{(n,n)}\Delta_N \neq 0$, there exists $0 \neq (q, q) \in T_{(n,n)}\pi^{-1}(n, n)$ with $q \perp N$ such that $dF_t(q, df_\nu q) = dF_t \circ (\mathrm{Id}, df)(q, q) \in T_{(n,n)}\Delta_N$.

We split $(q, df_\nu q) \in T_{(n,n)}M \times M$ into its components in $T\Delta_M$ and in $\nu_{\Delta_M}^{M\times M}$. Since dF_t leaves vectors in $T\Delta_M$ unchanged and stretches vectors in the normal bundle by a λ factor, we get

$$dF_t(q, df_\nu q) = dF_t\Big[\Big(\frac{q + df_\nu q}{2}, \frac{q + df_\nu q}{2}\Big) + \Big(\frac{q - df_\nu q}{2}, \frac{-q + df_\nu q}{2}\Big)\Big]$$

$$
\begin{aligned}
&= \Big(\frac{q+df_\nu q}{2}, \frac{q+df_\nu q}{2}\Big) + \lambda(t)\Big(\frac{q-df_\nu q}{2}, \frac{-q+df_\nu q}{2}\Big)\\
&= \Big(\frac{1+\lambda(t)}{2}q + \frac{1-\lambda(t)}{2}df_\nu q, \frac{1-\lambda(t)}{2}q + \frac{1+\lambda(t)}{2}df_\nu q\Big),
\end{aligned}
$$

for $\lambda(t) = \lambda_w(t)$ with $w = (q - df_\nu q)/2$. Note that by hypothesis, $w \neq 0$ and so $\lambda(t) \neq 0$.

We have $dF_t \circ (\mathrm{Id}, df)(q, q) = (v, v)$ for some $v \in T_n N$, so

$$
\Big(\frac{1+\lambda(t)}{2}q + \frac{1-\lambda(t)}{2}df_\nu q\Big) - \Big(\frac{1-\lambda(t)}{2}q + \frac{1+\lambda(t)}{2}df_\nu q\Big) = v - v = 0.
$$

This implies $\lambda(t)(\mathrm{Id} - df_\nu)q = 0$. Since $q \neq 0$, $\lambda(t) \neq 0$, this contradicts that $\mathrm{Id} - df_\nu$ is invertible. □

Define $E_{n,t} \subset T_{(n,n)}(M \times M)$ by

$$
E_{n,t} = T_{(n,n)}\Big[F_t \circ (\mathrm{Id}, f)(\pi^{-1}(n, n))\Big],
$$

and note the decomposition

$$
T(M \times M)\Big|_{\Delta_M} \simeq T\Delta_M \oplus \nu_{\Delta_M}^{M\times M} \simeq T\nu_{\Delta_M}^{M\times M}.
$$

Let

$$
\tilde\pi : T(M \times M)\Big|_{\Delta_M} \to \nu_{\Delta_M}^{M\times M}
$$

be the projection to $\nu_{\Delta_M}^{M\times M}$. By Lemma 2.1, $\tilde\pi$ has no kernel on $E_{n,t}$ and hence is an isomorphism of $E_{n,t}$ to a vector subspace $H_{n,t} \subset \nu_{\Delta_M}^{M\times M}$. Let

$$
\beta_{n,t} : E_{n,t} \to F_t \circ (\mathrm{Id}, f)(\pi^{-1}(n, n))
$$

be the diffeomorphism given by the exponential map. Actually, $\beta_{n,t}$ is a diffeomorphism on a neighborhood of 0 in $E_{n,t}$, whose radius goes to infinity as $t \to \infty$.

Thus, $\tilde\pi \circ \beta_{n,t}^{-1} : F_t \circ (\mathrm{Id}, f)(\pi^{-1}(n, n)) \to \tilde H_{n,t} \subset H_{n,t}$ is a diffeomorphism onto its image $\tilde H_{n,t}$, where $\tilde H_{n,t}$ is an arbitrarily large ball in $H_{n,t}$, for large t. Then

$$
\begin{aligned}
(-1)^{\dim M} L(f) &= \lim_{t\to\infty} \int_{F_t\circ(\mathrm{Id},f)(\Delta_N^\delta)} \eta_{\Delta_M}^{M\times M}\\
&= \lim_{t\to\infty} [\deg(\tilde\pi \circ \beta_t^{-1})^{-1}]^{-1} \int_{(\tilde\pi\circ\beta_t^{-1})F_t\circ(\mathrm{Id},f)(\Delta_N^\delta)} ((\tilde\pi \circ \beta_t^{-1})^{-1})^* \eta_{\Delta_M}^{M\times M}\\
&= \lim_{t\to\infty} \deg(\tilde\pi) \int_{\cup_n \tilde H_{n,t}} ((\tilde\pi \circ \beta_t^{-1})^{-1})^* \eta_{\Delta_M}^{M\times M},
\end{aligned}
$$

where $\beta_t : \cup_n E_{n,t} \to F_t \circ (\mathrm{Id}, f)(\Delta_N^\delta)$ is given by $\beta_{n,t}$ on each $E_{n,t}$. This uses $\deg(\tilde\pi \circ \beta_t^{-1})^{-1} = (\deg\tilde\pi)^{-1} = \deg\tilde\pi$, as β_t is an orientation preserving diffeomorhism and $\tilde\pi$ is an isomorphism.

Let $H_t = \cup_n H_{n,t}$ be the subbundle of $\nu_{\Delta_M}^{M\times M}$ over Δ_N with fiber $H_{n,t}$ over (n,n). We obtain

$$(-1)^{\dim\, M} L(f) = \lim_{t\to\infty} \deg(\tilde{\pi}) \int_{H_t} ((\tilde{\pi}\circ\beta_t^{-1})^{-1})^* \eta_{\Delta_M}^{M\times M}.$$

Since the $E_{n,t}$ are getting more "vertical" as $t\to\infty$, $\tilde{\pi}\circ\beta_t^{-1}\to \pm\mathrm{Id}$ and $H_{n,t}\to H_{n,\infty}$, where $H_{n,\infty}$ is the vector subspace of $(\nu_{\Delta_M}^{M\times M})_{(n,n)}$ spanned by the projection of vectors in $H_{n,t}$ into $\nu_{\Delta_M}^{M\times M}$, for any t. Set $H_\infty = \cup_n H_{n,\infty}$ with projection map $p : H_\infty \to \Delta_N$. Then H_∞ is also a subbundle of $\nu_{\Delta_M}^{M\times M}\to\Delta_N$, and

$$\begin{aligned}(-1)^{\dim\, M} L(f) &= \lim_{t\to\infty} \deg(\tilde{\pi}) \int_{H_t} ((\tilde{\pi}\circ\beta_t^{-1})^{-1})^* \eta_{\Delta M}^{M\times M} \\ &= \deg(\tilde{\pi}) \int_{H_\infty} \eta_{\Delta_M}^{M\times M}. \end{aligned} \tag{3.3}$$

By Theorem 2.2, $\eta_{\Delta_M}^{M\times M}$ and $\Phi(\nu_{\Delta_M}^{M\times M})$, the Thom class of $\nu_{\Delta_M}^{M\times M}$ considered as a form on $M\times M$, can be represented by the same form. Since we do not distinguish between the integral of a cohomology class and the integral of a representative form, we have

$$\begin{aligned}\int_{H_\infty} \eta_{\Delta_M}^{M\times M} &= \int_{H_\infty} \Phi(\nu_{\Delta_M}^{M\times M}) = \int_{H_\infty} \Phi(\nu_{\Delta_M}^{M\times M}) \wedge p^*1 \\ &= \int_{\Delta_N} p_*\Phi(\nu_{\Delta_M}^{M\times M}) \wedge 1 = \int_{\Delta_N} p_*\Phi(\nu_{\Delta_M}^{M\times M}), \end{aligned} \tag{3.4}$$

where the push forward formula for integration over the fiber [1, Prop. 6.15] is used between the third and fourth terms.

Let $H_\infty^\perp$ be the orthogonal (or any) complement of H_∞ in $\nu_{\Delta_M}^{M\times M}$. By [1, Prop. 6.19], we have $\Phi(\nu_{\Delta_M}^{M\times M}) = \Phi(H_\infty)\wedge\Phi(H_\infty^\perp)$. It is easy to check that

$$p_*\Phi(\nu_{\Delta_M}^{M\times M}) = p_*(\Phi(H_\infty)\wedge\Phi(H_\infty^\perp)) = p_*(\Phi(H_\infty))\wedge\Phi(H_\infty^\perp),$$

since $\Phi(H_\infty^\perp)$ vanishes in H_∞ directions. Thus (3.4) becomes

$$\int_{\Delta_N} p_*\Phi(\nu_{\Delta_M}^{M\times M}) = \int_{\Delta_N} p_*(\Phi(H_\infty))\wedge\Phi(H_\infty^\perp) = \int_{\Delta_N} \Phi(H_\infty^\perp), \tag{3.5}$$

as $p_*\Phi(H_\infty) = 1$ since $\Phi(H_\infty)$ integrates to one in each fiber.

We claim that

$$\nu_{\Delta_M}^{M\times M}\Big|_{\Delta N} \simeq \nu_{\Delta_N}^{N\times N} \oplus \nu_{\Delta_N}^{\Delta_M}.$$

Indeed, the metric on M is chosen so that

$$TM|_N = TN \oplus \nu_N^M. \tag{3.6}$$

$\nu_{\Delta_N}^{N\times N}$ is isomorphic to TN by the map $(v,-v)\mapsto v$. Similarly, $\nu_{\Delta_M}^{M\times M}\simeq T\Delta_M\simeq TM$. Finally, we trivially have $\nu_{\Delta_N}^{\Delta_M}\simeq \nu_N^M$. Plugging these terms into (3.6) gives the claim.

Thus we have the bundle isomorphisms $\nu_{\Delta_N}^{\Delta_M} \simeq E_t \simeq H_t \simeq H_\infty$, and by the claim we have $H_\infty^\perp \simeq \nu_{\Delta_N}^{N\times N}$. By (3.4), (3.5), we have

$$\begin{aligned} \int_{H_\infty} \eta_{\Delta_M}^{M\times M} &= \int_{\Delta_N} \Phi(H_\infty^\perp) = \int_{\Delta_N} \Phi(\nu_{\Delta_N}^{N\times N}) = \int_{\Delta_N} \eta_{\Delta_N}^{N\times N} \\ &= I(\Delta_N, \Delta_N) = \chi(N), \end{aligned} \tag{3.7}$$

where the self-intersection number of Δ_N appears as in §2.1. Combining (3.3), (3.7) gives the Lefschetz formula up to sign:

$$L(f) = (-1)^{\dim\ M}\deg(\tilde{\pi})\chi(N). \tag{3.8}$$

To compute the degree of $\tilde{\pi} : E_{n,t} \to H_{n,t} \simeq \nu_{\Delta_N}^{\Delta M}$, we pick θ and α, positively oriented bases for $E_{n,t} \subset T_{(n,n)}\Gamma$ and $H_{n,t} \simeq (\nu_{\Delta_N}^{\Delta M})_{(n,n)}$ respectively, and compute the sign of the determinant of the matrix of $\tilde{\pi}$ with respect to θ, α.

There exists a positively oriented basis for T_nM, $(v_1, ..., v_n, w_{n+1}, ..., w_m)$, with $v_i \perp w_j$, such that $v_1, ..., v_n \in T_nN$, $df_nv = v$ and $w_1, ..., w_{m-n} \in \nu_N^M$, $df_nw = df_{\nu_n}w$. A positively oriented basis for $T_{(n,n)}\Gamma$ is then

$$\{(v_1, v_1), ..., (v_n, v_n), (w_1, df_\nu w_1), ..., ..., (w_m, df_\nu w_{m-n})\},$$

and a positively oriented basis for $E_{n,t}$ is

$$\theta = \{(w_1, df_\nu w_1), ..., ..., (w_m, df_\nu w_{m-n})\},$$

since $E_{n,t} \simeq (\nu_{\Delta_N}^{\Delta_M})_{(n,n)}$. A positively oriented basis for $H_{n,t} \simeq (\nu_{\Delta_N}^{\Delta_M})_{(n,n)}$ is

$$\alpha = \{(-w_1, w_1), ..., (-w_{m-n}, w_{m-n})\}.$$

As in Lemma 2.1, the vectors in θ decompose into

$$(w_i, df_\nu w_i) = \Big(\frac{w_i + df_\nu w_i}{2}, \frac{w_i + df_\nu w_i}{2}\Big) + \Big(\frac{w_i - df_\nu w_i}{2}, \frac{-w_i + df_\nu w_i}{2}\Big).$$

Hence

$$\begin{aligned} \deg\ \tilde{\pi} &= \text{sgn}\ \det\Big\{(-w_i, w_i) \mapsto \Big(\frac{w_i - df_\nu w_i}{2}, \frac{-w_i + df_\nu w_i}{2}\Big)\Big\} \\ &= \text{sgn}\ \det\Big\{(-w_i, w_i) \mapsto (df_\nu - \text{Id})(-w_i, w_i)\Big\} \\ &= \text{sgn}\ \det(df_\nu - \text{Id}). \end{aligned}$$

Since the right hand side of (3.8) vanishes if dim N is odd, we assume dim N is even. (3.8) becomes

$$\begin{aligned} L(f) &= (-1)^{\dim\ M}\text{sgn}(\det(df_\nu - \text{Id}))\chi(N) \\ &= (-1)^{\dim\ M}(-1)^{\dim\ M - \dim\ N}\text{sgn}(\det(\text{Id} - df_\nu))\chi(N) \\ &= \text{sgn}(\det(\text{Id} - df_\nu))\chi(N), \end{aligned}$$

which concludes the proof of Theorem 3.1.

If the graph of f is transversal to the diagonal, the fixed point set reduces to a finite number of isolated fixed points $n_1, n_2, ..., n_r$, and the Lefschetz fixed point formula is easily recovered. For let n be one such isolated fixed point. Then H_∞ reduces to $H_{n,\infty}$, the fiber over (n,n) in Δ_N and $\int_{H_{n,\infty}} \Phi(\nu_{\Delta_M}^{M\times M}) = 1$. df_ν is just df_n and $\deg(\tilde{\pi}) = \text{sgn } \det(df_n - \text{Id})$. So (3.3), (3.4) give the fixed point formula

$$L(f) = (-1)^{\dim M} \sum_{i=1}^{r} \deg(\tilde{\pi}_i) \int_{H_{n_i,\infty}} \Phi(\nu_{\Delta_M}^{M\times M}) = \sum_{i=1}^{r} \text{sgn } \det(\text{Id} - df_{n_i}).$$

Remark: We sketch the corresponding geometric proof of the fixed submanifold formula based on Theorem 2.4. Assume that the metric on M is a product near a fixed point submanifold N. If the submanifold is given by $\{x^{k+1} = \ldots = x^n = 0\}$ in local coordinates, then as $t \to \infty$, the integrand for $L(f)$ concentrates on a tubular neighborhood of the fixed point, and the only contribution to the integrand comes from $I = \{1, \ldots, k\}$, since the curvature term vanishes otherwise due to the product metric. Converting back to rectangular coordinates in the normal fiber as in the topological proof eliminates the $\tilde{\rho}'$ factor and introduces a factor of sgn $\det(df_\nu - \text{Id})$. Since $f = \text{Id}$ in submanifold directions, $\det(\frac{1}{2}(d(tf) + \text{Id})_I^\mu) = 1$. Thus the integral splits into the curvature integral over N, yielding $\chi(N)$, and a normal integral, which gives sgn $\det(d(tf)_\nu - \text{Id})$. In the $t \to \infty$ limit, $d(tf)_\nu - \text{Id}$ in the normal fiber goes to the identity map, so its determinant becomes one. Plugging these terms into the integrand in Theorem 2.4 gives the Lefschetz fixed submanifold formula.

4 Small parameter behavior–the role of the cut locus

For a section s of TM, the behavior of ts as $t \to 0$ is trivial. In contrast, the function tf mentioned in the beginning of §3 (or more precisely, the diffeomorphism F_t) becomes discontinuous at $t = 0$ at those $x \in M$ for which $(x, f(x))$ is in the boundary of the tubular neighborhood of the diagonal. If the graph lies entirely within the tube, then tf is well defined, $\lim_{t\to 0} tf = \text{Id}$, and $L(f) = \chi(M)$.

Thus we expect the difference between $L(f)$ and $\chi(M)$ to be concentrated on the intersection of the graph with the tube boundary. Since this intersection is quite bad in general, the difference will be given by a current supported on the intersection.

The maximum amount of information is obtained when the tube is as large as possible. As explained below, this occurs when the boundary of the vertical fiber of the tube at (x, x) is $\mathcal{C}_x$, the cut locus of x. Recall that on a closed manifold, a geodesic $\gamma(t)$ is the minimal length curve joining $x = \gamma(0)$ to $\gamma(t)$ for $t \in [0, T]$ for some finite time T. The point $y = \gamma(T)$ is by definition in $\mathcal{C}_x$.

A point $y \in \mathcal{C}_x$ is characterized by: either there exists more than one minimal length geodesic from x to y, or $d\exp_x$ is singular at the preimage of y [2, Lemma

5.2]. In particular, if the graph of a smooth function $f : M \to M$ has the property that $f(x)$ is never on $\mathcal{C}_x$, then there is a unique minimal geodesic joining x to $f(x)$. Shrinking this geodesic gives a homotopy from f to the identity, and so the Lefschetz number satisfies $L(f) = \chi(M)$. Thus the difference $L(f) - \chi(M)$ is controlled by the *cut locus of f in M*

$$\mathcal{C}(f) = \{x : f(x) \in \mathcal{C}_x\}. \tag{4.1}$$

In the first subsection, we will make this geometric statement more precise by finding a singular current supported on the cut locus of f whose singular part evaluated at the function 1 gives $L(f) - \chi(M)$. The main idea is to define the function tf and to let $t \to 0$. In the second subsection, we assume that $\mathcal{C}(f)$ is finite. Under a transversality condition, the number of points in $\mathcal{C}(f)$ can be estimated from below. In fact, for all but very special metrics, the transversality condition implies that $\mathcal{C}(f)$ is infinite for diffeomorphisms with $L(f) \neq \chi(M)$.

4.1 A current on the cut locus

We first construct the largest (topological) tubular neighborhood of the diagonal Δ in $M \times M$. A tubular neighborhood is given by points of the form $(\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$, where $v \in T_{\bar{x}}M, \bar{x} \in M$, and $|v|$ is small. For $x, y \in M$, we say that y is inside $\mathcal{C}_x$ if there is a unique minimal geodesic from x to y. Let $N_{\bar{x}} = \{\exp_{\bar{x}} v : \exp_{\bar{x}} v \text{ is inside } \mathcal{C}_{\exp_{\bar{x}}(-v)}\}$. Define $T \subset M \times M$ by $T = \{(\exp_{\bar{x}} v, \exp_{\bar{x}}(-v)) : \exp_{\bar{x}} v \in N_{\bar{x}}, \bar{x} \in M\}$. We call T the *cut locus tubular neighborhood.*

Lemma 4.1 *(i) T is a topological tubular neighborhood of the diagonal.*
(ii) $(x, y) \in T$ iff y is inside $\mathcal{C}_x$.
(iii) The vertical fiber $T \cap (\{x\} \times M)$ of T at x equals $\{x\} \times (M \setminus \mathcal{C}_x)$.

Proof: We prove (ii) first. The forward implication is from the definition of T. Conversely, if y is inside $\mathcal{C}_x$, then there is a unique minimal geodesic from x to y. Then $(x, y) = (\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$, where $\bar{x}$ is the midpoint of the geodesic. Thus $x, y \in N_{\bar{x}}$, so $(x, y) \in T$.

For (i), the standard argument that the interior of the cut locus is a topological sphere immediately extends to show that $\exp_x^{-1}(N_x) \subset T_xM$ is the interior of a topological sphere. This argument in turn extends to show that the radius of this sphere is a continuous function on the unit tangent sphere, which implies that $\exp^{-1}(T)$ is a topological disk bundle.

To finish the proof, we must show that exp on $M \times M$ is injective on $\{(v, -v)\} \subset \coprod_x (\exp_x^{-1}(N_x) \times \exp_x^{-1}(N_x))$. If not, there exists $\bar{x}, \bar{y}, v, w$ with $\alpha = \exp_{\bar{x}} v = \exp_{\bar{y}} w$ and $\beta = \exp_{\bar{x}}(-v) = \exp_{\bar{y}}(-w)$. By the definition of $N_{\bar{x}}, N_{\bar{y}}$, this gives two minimal geodesics from α to β, a contradiction.

For (iii), $(\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$ is in the vertical fiber over $(\exp_{\bar{x}} v, \exp_{\bar{x}} v)$, and at ∂T, $\exp_{\bar{x}}(-v) \in \mathcal{C}_{\exp_{\bar{x}} v}$. □

We now define tf. Fix a diffeomorphism $\mu : [0,1) \to [0,\infty)$ with $\mu(0) = 0, \mu(1) = 1$ and such that the derivative of μ^{-1} grows at most polynomially. For the moment, let T denote any smooth tubular neighborhood of Δ given by geodesics of the form $(x,y) = (\gamma(t), \gamma(-t))$. (The range of t is a smooth function on the unit tangent bundle.) For such (x,y), set

$$d_{x,y} = \min\{t : (\gamma(t), \gamma(-t)) \in \partial T\}.$$

For $x \in M$ and $t \in [0,\infty)$, define $t_x : M \to M$ by

$$t_x(y) = \begin{cases} \exp_x[\mu^{-1}(\mu(d_{x,y}^{-1}|v|)t)d_{x,y}\frac{v}{|v|}], & (x,y) \in T,\ y = \exp_x v,\ y \neq x, \\ y, & (x,y) \notin T, \\ x, & y = x. \end{cases}$$

Thus for x, y close, t_x pushes y towards ∂T as $t \to \infty$ along their minimal geodesic, but fixes y if it is far from x, as measured by T.

For $f : M \to M$, define $tf : M \to M$ by

$$(tf)(x) = t_x(f(x)).$$

The maps tf are smooth for $t > 0$. Note that $(1f)(x) = \exp_x v = f(x)$ if $(x, f(x)) \in T$ and $(1f)(x) = f(x)$ otherwise, so $1f = f$. Similarly, we have $(0f)(x) = x$ if $(x, f(x)) \in T$, and $(0f)(x) = f(x)$ otherwise. Thus $0f$ is discontinuous on $\{x : (x, f(x)) \in \partial T\}$. We remark that (Id, tf) can be used in place of F_t to prove the Lefschetz fixed submanifold formula as $t \to \infty$.

We now examine the $t \to 0$ limit of pullbacks of Mathai-Quillen forms. Fix ϵ, and let $\mathrm{MQ}_\Delta = \mathrm{MQ}_{\Delta_\epsilon}$ be the Mathai-Quillen form on the ϵ-neighborhood of the diagonal. Then

$$\begin{aligned} L(f) &= \int_\Delta (\mathrm{Id}, f)^* \mathrm{MQ}_\Delta \\ &= \lim_{t \to 0} \int_\Delta (\mathrm{Id}, tf)^* \mathrm{MQ}_\Delta \\ &= \lim_{t \to 0} \int_{\{(x,x) : (x, f(x)) \in T\}} (\mathrm{Id}, tf)^* \mathrm{MQ}_\Delta + \lim_{t \to 0} \int_{\{(x,x) : (x, f(x)) \notin T\}} (\mathrm{Id}, tf)^* \mathrm{MQ}_\Delta. \end{aligned}$$

As usual $(\mathrm{Id}, tf)^*(\mathrm{MQ}_\Delta)_{(x,x)} = (\mathrm{MQ}_\Delta)_{(x,f(x))} \circ (\mathrm{Id}, tf)_* = 0$ if $(x, f(x)) \notin T$, so

$$L(f) = \lim_{t \to 0} \int_{\{(x,x) : (x, f(x)) \in T\}} (\mathrm{Id}, tf)^* \mathrm{MQ}_\Delta.$$

Note that $d(x,y)/d_{x,y} < 2|v|/|v| = 2$ for $(x,y) = (\exp_{\bar{x}} v, \exp_{\bar{x}}(-v))$. Fix $\delta < 1$, set

$$A_\delta = \{x : (x, f(x)) \in T, \frac{d(x, f(x))}{d_{x,f(x)}} \leq 2\delta\},$$

and set $B_\delta = \{x : (x, f(x)) \in T\} \setminus A_\delta$. It is easy to check that $(\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta \to (\mathrm{Id}, \mathrm{Id})^*\mathrm{MQ}_\Delta = \mathrm{Pf}(\Omega)$ as $t \to 0$ uniformly on the compact set A_δ. Thus

$$L(f) = \int_{A_\delta} \mathrm{Pf}(\Omega) + \lim_{t\to 0} \int_{B_\delta} i^*(\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta,$$

where $i : M \to \Delta$ is the inclusion; we will omit this map from here on. Since $\mathrm{Pf}(\Omega)$ is smooth on M and A_δ exhausts the open set $\{x : (x, f(x)) \in T\}$ as $\delta \to 1$, we get

$$L(f) = \int_{\{x:(x,f(x))\in T\}} \mathrm{Pf}(\Omega) + \lim_{\delta\to 1}\lim_{t\to 0} \int_{B_\delta} (\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta. \tag{4.2}$$

This construction extends to the topological tubular neighborhood T of Lemma 4.1. Fix $\epsilon > 0$ and pick a smooth disk bundle $D^\epsilon \subset \nu_\Delta$ such that $T^\epsilon = \exp D^\epsilon$ is inside T and is within ϵ of filling T – i.e. for all $(v, -v) \in \partial D^\epsilon$, we have $d_{M\times M}((\exp v, \exp(-v)), (\exp(tv), \exp(-tv))) < \epsilon$, where t is the smallest positive number such that $(\exp tv, \exp(-tv)) \in \partial T$. To define the Mathai-Quillen form on T^ϵ, we have to choose a diffeomorphism $\alpha^\epsilon : \nu_\Delta \to D^\epsilon$ and pull back the Mathai-Quillen form MQ_ν from ν_Δ. As $\epsilon \to 0$, D^ϵ fills out a continuous disk bundle in ν_Δ, and we demand that for all $R > 0$, there exists $\epsilon_0 = \epsilon_0(R)$ such that $\alpha^\epsilon(B_R(\nu_\Delta))$ is constant for all $\epsilon < \epsilon_0$, where $B_R(\nu_\Delta)$ is the R-ball around the zero section in ν_Δ. For this choice of α^ϵ, it is immediate that

$$\mathrm{MQ}^0_\Delta(v_1, \ldots, v_n) \equiv \lim_{\epsilon\to 0}[((\alpha^\epsilon)^{-1})^*\mathrm{MQ}_\nu(v_1, \ldots, v_n)]$$

exists and is smooth, and that $\mathrm{MQ}^\epsilon_\Delta \equiv ((\alpha^\epsilon)^{-1})^*\mathrm{MQ}_\nu \to \mathrm{MQ}^0_\Delta$ pointwise. In fact, since MQ_ν decays exponentially at infinity in ν_Δ, it is easy to check that this convergence is uniform. This yields

$$L(f) = \lim_{\epsilon\to 0} \int_\Delta (\mathrm{Id}, tf)^*\mathrm{MQ}^\epsilon_\Delta = \int_\Delta (\mathrm{Id}, tf)^*\mathrm{MQ}^0_\Delta.$$

We can now repeat the argument leading to (4.2), noting that for the topological neighborhood T, $d_{x,y}$ is just continuous in x, y. By Lemma 4.1, we obtain

$$L(f) = \int_{M\setminus\mathcal{C}(f)} \mathrm{Pf}(\Omega) + \lim_{\delta\to 1}\lim_{t\to 0} \int_{B_\delta} (\mathrm{Id}, tf)^*\mathrm{MQ}^0_\Delta, \tag{4.3}$$

where $\mathcal{C}(f)$ is given by (4.1). Note that $\mathcal{C}(f)$ is closed, so the first integral exists. By the Chern-Gauss-Bonnet theorem, we get

$$L(f) = \chi(M) - \int_{\mathcal{C}(f)} \mathrm{Pf}(\Omega) + \lim_{\delta\to 1}\lim_{t\to 0} \int_M \chi_{B_\delta} \cdot (\mathrm{Id}, tf)^*\mathrm{MQ}^0_\Delta. \tag{4.4}$$

We now define zero currents L^{tf}, E, on M via their action on $g \in C^\infty(M) \simeq C^\infty(\Delta)$:

$$L^{tf}(g) = \int_M g \cdot (\mathrm{Id}, tf)^* \mathrm{MQ}^0_\Delta,$$
$$E(g) = \int_M g \cdot \mathrm{Pf}(\Omega).$$

We also set

$$\mathcal{C}^f(g) = -\int_{\mathcal{C}(f)} g \cdot \mathrm{Pf}(\Omega) + \lim_{\delta \to 1} \lim_{t \to 0} \int_M g \cdot \chi_{B_\delta} \cdot (\mathrm{Id}, tf)^* \mathrm{MQ}^0_\Delta,$$

whenever the right hand side exists. We define the limit of currents by pointwise convergence: $\lim_{t\to 0} L^{tf} = L^0$ if $\lim_{t\to 0} L^{tf}(g) = L^0(g)$ for all smooth g.

Lemma 4.2 *As a current,* $(\lim_{t\to 0} L^{tf}) - \mathcal{C}^f$ *exists and equals* E. *In particular,* $\lim_{t\to 0} L^{tf}(g)$ *exists whenever* supp $g \cap \mathcal{C}(f) = \emptyset$.

Proof: We have

$$L^{tf}(g) = \int_M g \cdot (\mathrm{Id}, tf)^* \mathrm{MQ}^0_\Delta = \int_{A_\delta} g \cdot (\mathrm{Id}, tf)^* \mathrm{MQ}^0_\Delta + \int_{B_\delta} g \cdot (\mathrm{Id}, tf)^* \mathrm{MQ}^0_\Delta,$$

and so

$$\lim_{t\to 0} L^{tf}(g) - \lim_{\delta \to 1} \lim_{t \to 0} \int_{B_\delta} g \cdot (\mathrm{Id}, tf)^* \mathrm{MQ}^0_\Delta = \int_{M \setminus \mathcal{C}(f)} g \cdot \mathrm{Pf}(\Omega)$$
$$= E(g) - \int_{\mathcal{C}(f)} g \cdot \mathrm{Pf}(\Omega),$$

as in (4.3). This gives the first statement. For the second statement, if supp $g \cap \mathcal{C}(f) = \emptyset$, then $g \cdot \chi_{D_\delta} = 0$ for $\delta \approx 1$, and so $\mathcal{C}^f(g) = 0$ for such g. □

In view of this lemma, we think of L^0 as a singular current, with $\mathcal{C}^f$ the singular part of L^0 and E the finite part. Note that $L^{tf}(1) = L(f)$ for all t. This gives:

Theorem 4.1 *For every Riemannian metric on a closed manifold M and every smooth function $f : M \to M$, there exists a canonical singular part $\mathcal{C}^f$ to $L^0 = \lim_{t\to 0} L^{tf}$, with supp $\mathcal{C}^f \subset \mathcal{C}(f)$. Moreover, we have*

$$L(f) = \chi(M) + \mathcal{C}^f(1).$$

Remarks:

(1) The previous discussion can be watered down to apply to the degree of f, defined by $\deg(f) = \int_M f^*\omega / \int_M \omega$ for any top degree form ω. By its homotpy invariance, the degree of f is one if the graph of f never intersects $\mathcal{C}_x$, so we expect

that a singular 0-current, with singular part supported on $\mathcal{C}(f)$, computes $\deg(f)-1$. Taking ω to be the volume form dvol of a Riemannian metric on M, we get

$$\begin{aligned}\mathrm{vol}(M)\cdot\deg(f) &= \lim_{t\to 0}\int_M (tf)^*\mathrm{dvol}\\ &= \int_{M\setminus\mathcal{C}(f)}\mathrm{dvol} + \lim_{\delta\to 1}\lim_{t\to 0}\int_M \chi_{B_\delta}(tf)^*\mathrm{dvol}.\end{aligned}$$

Setting

$$\mathcal{D}^f(g) = -\int_{\mathcal{C}(f)} g\cdot\mathrm{dvol} + \lim_{\delta\to 1}\lim_{t\to 0}\int_M g\cdot\chi_{B_\delta}\cdot(tf)^*\mathrm{dvol},$$

whenever the right hand side exists, we see that the support of $\mathcal{D}^f$ is contained in $\mathcal{C}(f)$, and that

$$\deg(f)-1 = \frac{\mathcal{D}^f(1)}{\mathrm{vol}(M)}.$$

(2) In (1) and in the previous section, we have compared f to (the homotopy class of) the identity map. We can also compare f to a constant map $c(x)=x_0$. In this case the Lefschetz number (resp. degree) of f is 1 (resp. 0) if the graph of f misses $\mathcal{C}_{x_0}$. Again there are singular 0-currents, with singular part supported on $\mathcal{C}_{x_0}$, which measure $L(f)-1$ and $\deg(f)$.

(3) Finally, we can compare f to a fixed map f_0. We obtain

$$L(f)-L(f_0) = \mathcal{E}^{f,f_0}(1),$$

where the singular 0-current $\mathcal{E}^{f,f_0}$ has singular part supported on $\{x : f_0(x)\in\mathcal{C}_{f(x)}\}$. There is a similar result for degrees. As a simple well-known example, note that if $M=S^n$ and f, f_0 have different Lefschetz numbers (equiv. different degrees), then there exists $x\in S^n$ such that $f(x), f_0(x)$ are antipodal.

$\mathcal{C}^f(1)$ can be identified in the simplest case where f is Lefschetz (i.e. the graph Γ of f is transverse to Δ–a generic condition) and $\mathcal{C}(f)$ consists of isolated points $\{x_1,\ldots,x_n\}$ (i.e. $\Gamma\cap\partial T = \{(x_i, f(x_i))\}$, which we will see is a non-generic condition). Since T is homeomorphic to TM, and diffeomorphic away from ∂T, we can consider f as a smooth vector field V_f on M with singularities at the x_i. At each fixed point x of f, the local Lefschetz number $L_x(f)$ equals the Hopf index $\mathrm{ind}_x(V_f)$ of V_f [6, p. 135]. We modify V_f by multiplying the vectors in a neighborhood of each x_i by a smooth function which is one on the boundary of the neighborhood and which vanishes to all orders at x_i. The modified vector field V'_f extends to a smooth vector field, also denoted V'_f, on all of M with zeros at the fixed points of f and at the x_i. We have

$$\chi(M) = \sum_{\{x:V'_f(x)=0\}}\mathrm{ind}_x(V'_f) = \sum_{\{x:f(x)=x\}} L_x(f) + \sum_i \mathrm{ind}_{x_i}(V'_f) = L(f) + \sum_i \mathrm{ind}_{x_i}(V'_f).$$

PROPOSITION 4.1 *Let $f : M \to M$ be a Lefschetz map with $\mathcal{C}(f)$ consisting of isolated points. Then*

$$\mathcal{C}^f(1) = -\sum_{x_i \in \mathcal{C}(f)} \mathrm{ind}_{x_i}(V_f'),$$

and in particular

$$L(f) = \chi(M) - \sum_{x_i \in \mathcal{C}(f)} \mathrm{ind}_{x_i}(V_f').$$

Remark: This proposition is related to the proof of the Hopf index formula in [8]. As in the beginning of §3, for a vector field s we have $\chi(M) = \int_M (ts)^*\mathrm{MQ}_{TM}$ for all t. At $t = 0$ we recover the Chern-Gauss-Bonnet formula, so $\chi(M) = \lim_{t\to\infty} \int_M (ts)^*\mathrm{MQ}_{TM}$. Let B_ϵ be the ϵ-neighborhood of the zero set of s. Then by the uniform decay of $(ts)^*\mathrm{MQ}_{TM}$ off B_ϵ, $\chi(M) = \lim_{\epsilon\to 0}\lim_{t\to\infty} \int_{B_\epsilon} (ts)^*\mathrm{MQ}_{TM}$. Define a family of Euler currents by

$$E_s^t(g) = \int_M g \cdot (ts)^*\mathrm{MQ}_{TM}.$$

Then $E_s^t(1) = \chi(M)$ and

$$\lim_{t\to\infty} E_s^t - \lim_{\epsilon\to 0}\lim_{t\to\infty} \int_{B_\epsilon} (ts)^*\mathrm{MQ}_{TM} = 0.$$

Thus the singular 0-current $\lim_{t\to\infty} E_s^t$ is supported on the zero set of s, and the Euler characteristic as a 0-current localizes to the zero set. If the zero set consists of nondegenerate points, this singular part is given by $\pm\delta$-functions at the zeros, and the Hopf index formula is recovered.

4.2 Isolated cut points

Assume that (i) $\mathcal{C}(f)$ consists of a finite set of points, and (ii) the graph Γ of f is transverse to $M \times \{f(x)\}$ for all $x \in \mathcal{C}(f)$. In this case, we say that f *is transverse to the cut locus.* Under this assumption, we will show that $|\mathcal{C}(f)|$ can be bounded from below.

Condition (ii) is equivalent to df_x being invertible, as $(v, df_x v) \in T(M \times \{f(x)\})$ implies $df_x v = 0$. In particular, a diffeomorphism of M satisfies (ii).

For simplicity, write $\mathcal{C}(f) = \{x\}$. The transversality assumption implies that the differential of f is invertible at x, so on some $\epsilon-$neighborhood $B_\epsilon(x)$, $\Gamma|_{B_\epsilon(x)}$ is a graph over the neighborhood $U = \{x\} \times f(B_\epsilon(x))$ of $(x, f(x)) \in \{x\} \times M$. Thus the projection $p_2 : M \times M \to M$ onto the second factor restricts to a diffeomorphism $p_2 : \Gamma|_{B_\epsilon(x)} \to U$, which of course has degree ± 1. For MQ_Δ the Mathai-Quillen form of $\nu_\Delta^{M\times M}$, considered as a form on the cut locus tubular neighborhood, we have

$$L(f) = \lim_{\epsilon\to 0}\lim_{t\to 0} \int_{M\setminus B_\epsilon(x)} (\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta + \lim_{\epsilon\to 0}\lim_{t\to 0} \int_{B_\epsilon(x)} (\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\int_{M\setminus B_\epsilon(x)} \mathrm{Pf}(\Omega) + \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{B_\epsilon(x)} (\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta \qquad (4.5)\\
&= \chi(M) + \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{B_\epsilon(x)} (\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta.
\end{aligned}
$$

Here we do not distinguish between integrals over M and integrals over the diagonal $\Delta \subset M \times M$. To justify (4.5), we need that $(\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta$ converges uniformly to $\mathrm{Pf}(\Omega)$ on $M \setminus B_\epsilon(x)$ as $t \to 0$.

Lemma 4.3 *Let $\imath : \Delta \to M \times M$ be the inclusion. Then*

$$\lim_{t\to 0} \imath^*(\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta = \mathrm{Pf}(\Omega)$$

uniformly on $M \setminus B_\epsilon(x)$.

Proof: On $M \setminus B_\epsilon(x)$, we have $(tf)(y) \to y$ uniformly as $t \to 0$. Thus if $\gamma(s)$ is a short curve with $\gamma(0) = y, \dot\gamma(0) = w$, then $(tf)(\gamma(s)) \to \gamma(s)$ uniformly as $t \to 0$, and

$$
\begin{aligned}
\lim_{t\to 0}(tf)_*(w) &= \lim_{t\to 0}\lim_{s\to 0}\frac{(tf)(\gamma(s)) - (tf)(y)}{s}\\
&= \lim_{s\to 0}\lim_{t\to 0}\frac{(tf)(\gamma(s)) - (tf)(y)}{s}\\
&= \lim_{s\to 0}\frac{\gamma(s) - \gamma(0)}{s} = \dot\gamma(0) = w.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
&[(\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta]_{(y,y)}((v_1, w_1), \ldots, (v_n, w_n)) =\\
&\qquad (\mathrm{MQ}_\Delta)_{(y,f(y))}((v_1, (tf)_*w_1), \ldots, (v_n, (tf)_*w_n))
\end{aligned}
$$

converges uniformly in y to

$$(\mathrm{MQ}_\Delta)_{(y,y)}((v_1, w_1), \ldots, (v_n, w_n))$$

as $t \to 0$. Since $\imath^*\mathrm{MQ}_\Delta = \mathrm{Pf}(\Omega)$, the lemma follows. □

Let $TM^\uparrow$ denote $\{0\} \times TM \subset T(M \times M)|_\Delta$, and let $\mathrm{MQ}_{TM^\uparrow}$ denote the Mathai-Quillen form of $TM^\uparrow$, considered as a form supported on the cut locus neighborhood. Thus $\mathrm{MQ}_{TM^\uparrow} = (\exp^{-1})^*\beta^*\mathrm{MQ}$, where MQ is the Mathai-Quillen form on $TM^\uparrow$, exp is the exponential map from $TM^\uparrow$ to $M \times M$, and β is a homeomorphism from the neighborhood of zero in $TM^\uparrow$ with fiber $\exp_x^{-1}(M \setminus \mathcal{C}_x)$ to TM. Here we have used Lemma 4.1 (iii). As in the last section, β is a limit of diffeomorphisms, and because of the decay of MQ we may treat β as a diffeomorphism.

Note that $p_2^*\mathrm{MQ}_{TM\uparrow} = \mathrm{MQ}_\Delta$, since (i) $p_2^*\mathrm{MQ}_{TM\uparrow}$ is closed and (ii) for a fiber $F = \{(\exp_x(-v), \exp_x v) : v \in N_x\}$ of the cut locus tubular neighborhood, we have

$$\begin{aligned}\int_F p_2^*\mathrm{MQ}_{TM\uparrow} &= \int_{p_2 F}\mathrm{MQ}_{TM\uparrow} = \int_{M\setminus\mathcal{C}_x}\mathrm{MQ}_{TM\uparrow}\\ &= \int_{\beta\exp_x^{-1}(M\setminus\mathcal{C}_x)}\mathrm{MQ} = \int_{T_xM\uparrow}\mathrm{MQ} = 1. \end{aligned}\tag{4.6}$$

Thus

$$\begin{aligned}\int_{B_\epsilon(x)}(\mathrm{Id}, tf)^*\mathrm{MQ}_\Delta &= \int_{(\mathrm{Id},tf)B_\epsilon(x)}\mathrm{MQ}_\Delta = \pm\int_{p_2(\mathrm{Id},tf)B_\epsilon(x)}(p_2^{-1})^*\mathrm{MQ}_\Delta \qquad (4.7)\\ &= \pm\int_{(tf)(B_\epsilon(x))}\mathrm{MQ}_{TM\uparrow} = \pm\int_{\exp_x^{-1}(tf)(B_\epsilon(x))}\exp_x^*\mathrm{MQ}_{TM\uparrow}.\end{aligned}$$

The last step is valid since for each t, $\exp_x^{-1}$ is well defined except on $tf(B_\epsilon(x))\cap\mathcal{C}_x$, which has measure zero. By (4.5), (4.7),

$$L(f) = \chi(M) \pm \lim_{\epsilon\to 0}\lim_{t\to 0}\int_{\exp_x^{-1}(tf)(B_\epsilon(x))}\exp_x^*\mathrm{MQ}_{TM\uparrow}. \tag{4.8}$$

We modify this equation to handle the non-uniformity of the integral. Since Γ is transverse to $M\times\{f(x)\}$, for a fixed $\epsilon' < \epsilon$ there exists $\delta = \delta(\epsilon')$ such that any δ perturbation of Γ in the C^1 topology is still a graph over a set $U_{\epsilon'} \subset \{x\}\times M$ containing $(x, f(x))$. Also, for any sequence $\epsilon_n \to 0$, there exists a sequence $t_n \to 0$ such that the graph Γ_{t_n} of t_nf is a $\delta_n = \delta(\epsilon_n)$ perturbation of Γ. Thus there is a set $U_n \subset (t_nf)(B_{\epsilon_n}(x))$ such that Γ_{t_n} is a graph over U_n. Set $W_n = (t_nf)^{-1}(U_n)\cap B_{\epsilon_n}(x)$. Then

$$\begin{aligned}\lim_{\epsilon\to0}\lim_{t\to0}\int_{B_\epsilon(x)}(\mathrm{Id},tf)^*\mathrm{MQ}_\Delta &= \lim_{n\to\infty}\lim_{t\to0}\Big[\int_{B_{\epsilon_n}(x)\setminus W_n}(\mathrm{Id},tf)^*\mathrm{MQ}_\Delta\\ &\qquad + \int_{W_n}(\mathrm{Id},tf)^*\mathrm{MQ}_\Delta\Big]\\ &= \lim_{n\to\infty}\int_{B_{\epsilon_n}(x)\setminus W_n}\mathrm{Pf}(\Omega) \qquad (4.9)\\ &\qquad + \lim_{n\to\infty}\lim_{t\to0}\int_{W_n}(\mathrm{Id},tf)^*\mathrm{MQ}_\Delta\\ &= \lim_{n\to\infty}\lim_{t\to0}\int_{W_n}(\mathrm{Id},tf)^*\mathrm{MQ}_\Delta.\end{aligned}$$

The next technical lemma replaces tf by a family of maps deforming $f(y)$ towards x rather than towards y, for x, y close.

LEMMA 4.4 *For $\mu > 0$, there exists a neighborhood $U = U_\mu$ of x such that for all $y_0 \in U$, there exists a unique minimal geodesic $\gamma_{f(y_0),x}$ from $f(y_0)$ to x which is μ close in the C^1 topology to the unique minimal geodesic $\gamma_{f(y_0),y_0}$ from $f(y_0)$ to y_0.*

PROOF: The lemma is obvious unless $f(y_0) \in \mathcal{C}_x$. In general, fix y_0 close to x and let y denote a point on the minimal geodesic from y_0 to x. Since $f(y_0) \notin \mathcal{C}_{y_0}$, we have $y_0 \notin \mathcal{C}_{f(y_0)}$, and in particular y_0 is not in the conjugate locus of $f(y_0)$. Thus the exponential map $\exp_{f(y_0)} : T_{f(y_0)}M \to M$ surjects onto some neighborhood of y_0. For y close to y_0, there is a unique minimal geodesic $\gamma_{f(y_0),y}$ from $f(y_0)$ to y, and the family of such geodesics is C^1 close. Now take a curve γ_ϵ which is a smoothed approximation to $\gamma_{f(y_0),y}$ followed by the minimal geodesic from y to x such that the length of γ_ϵ satisfies $\ell(\gamma_\epsilon) \leq d(f(y_0), y) + d(y, x) + \epsilon$. Parametrizing all curves by arclength, we see that for y_0 close enough to x, the new family of curves is still C^1 close. By the Ascoli theorem, a subsequence of this family converges in C^0 as $y \to x$ and as $\epsilon \to 0$ to a curve $\gamma_{f(y_0),x}$ from $f(y_0)$ to x of length $d(f(y_0), x)$–i.e. $\gamma_{f(y_0),x}$ is a minimal geodesic from $f(y_0)$ to x. Since $\gamma_{f(y_0),x}$ is smooth and since the tangent vectors $\exp^{-1}_{f(y_0)} y$ lie on the unit sphere in $T_{f(y_0)}M$, it follows easily that a subsequence of $\exp^{-1}_{f(y_0)} y$ converges to a vector v with $\exp_{f(y_0)}(sv) = \gamma_{f(y_0),x}$. By the smooth dependence of geodesics on initial conditions, the minimal geodesic $\gamma_{f(y_0),x}$ is C^1 close to the minimal geodesic from $f(y_0)$ to y, and hence C^1 close to $\gamma_{f(y_0),y_0}$.

This shows that along any radial geodesic r centered at x, there exists a distance $\delta = \delta(r)$ such that if y is a point on r with $d(x, y) < \delta$, then there is a minimal geodesic from $f(y)$ to x which is μ close to the minimal geodesic from $f(y)$ to y in the C^1 topolgy. A similar argument shows that we may take δ to be a continuous function of the radial direction. □

We now fix n large enough so that the lemma applies to all $y \in W_n$. Define a family of maps $g_t : W_n \to M$, $t \in (0, 1]$, which are approximations to tf as follows. For $x \in \mathcal{C}(f)$, set $g_t(x) = (tf)(x) = x$. For $y \notin \mathcal{C}(f)$ and $v_y = \exp^{-1}_{f(y)} y$, define α_t by $(tf)(y) = \exp_{f(y)}(\alpha_t v_y)$. Now set $g_t(y) = \exp_{f(y)}(\alpha_t v_x)$, where $\exp_{f(y)}(sv_x)$ is the minimal geodesic from $f(y)$ to x just constructed. By the smooth dependence of geodesics on parameters, we see that for n large enough, tf is arbitrarily C^1 close to g_t for all $y \in W_n$ and for all $t \in (0, 1]$. This implies

$$\begin{aligned} L(f) - \chi(M) &= \lim_{n\to\infty} \lim_{t\to 0} \int_{W_n} (\mathrm{Id}, tf)^* \mathrm{MQ}_\Delta \\ &= \lim_{n\to\infty} \lim_{t\to 0} \int_{W_n} (\mathrm{Id}, g_t)^* \mathrm{MQ}_\Delta \\ &\qquad + \lim_{n\to\infty} \lim_{t\to 0} \int_{W_n} [(\mathrm{Id}, tf)^* - (\mathrm{Id}, g_t)^*] \mathrm{MQ}_\Delta \\ &= \lim_{n\to\infty} \lim_{t\to 0} \int_{W_n} (\mathrm{Id}, g_t)^* \mathrm{MQ}_\Delta \\ &\qquad + \lim_{n\to\infty} \int_{W_n} \lim_{t\to 0} [(\mathrm{Id}, tf)^* - (\mathrm{Id}, g_t)^*] \mathrm{MQ}_\Delta \\ &= \lim_{n\to\infty} \lim_{t\to 0} \int_{W_n} (\mathrm{Id}, g_t)^* \mathrm{MQ}_\Delta. \end{aligned} \tag{4.10}$$

As in (4.7) we have

$$\begin{aligned}\int_{W_n}(\mathrm{Id},g_t)^*\mathrm{MQ}_\Delta &= \pm\int_{p_2(\mathrm{Id},g_t)W_n}(p_2^{-1})^*\mathrm{MQ}_\Delta\\ &= \pm\int_{\beta\exp_\cdot^{-1}g_t(W_n)}\mathrm{MQ} \qquad (4.11)\\ &\approx \pm\int_{\beta\exp_x^{-1}g_t(W_n)}\mathrm{MQ},\end{aligned}$$

where $\exp_\cdot$ denotes the exponential map from $TM^\uparrow|_{W_n}$ to $M\times M$. Since $\exp_q^{-1}$ is C^1 close to $\exp_x^{-1}$ for q close to x, the error in the last line goes to zero as $n\to\infty, t\to 0$. (We use parallel translation to compare maps with different ranges.) Since $\beta\exp_x^{-1}g_tW_n\subset T_xM^\uparrow$,

$$\left|\lim_{n\to\infty}\lim_{t\to 0}\int_{\beta\exp_x^{-1}g_tW_n}\mathrm{MQ}\right|\le\lim_{n\to\infty}\lim_{t\to 0}\int_{T_xM^\uparrow}\left|\mathrm{MQ}\right|=1, \qquad (4.12)$$

as MQ is a positive multiple of the volume form in $T_xM^\uparrow$. Thus by (4.10)–(4.12), we get

$$|L(f)-\chi(M)|\le 1. \qquad (4.13)$$

Summing over the finite number of points in $\mathcal{C}(f)$ gives the main theorem.

Theorem 4.2 *Assume that f is transverse to the cut locus. Then*

$$|L(f)-\chi(M)|\le|\mathcal{C}(f)|.$$

Remarks: 1) The theorem is trivially sharp by setting $f=\mathrm{Id}$. The result is also sharp for $f: z\mapsto z^n$ on S^1, and f is transverse to the cut locus. The two point suspension of f to S^2 is transverse to the cut locus and gives equality in Theorem 4.2, and iterating this procedure gives sharp maps in all dimensions.

2) The inequality in Theorem 4.2 can be refined to an equality for maps on S^n with the standard metric, since $\mathcal{C}_x=\{-x\}$ easily implies $\lim_{n\to\infty}\lim_{t\to 0}\beta\exp_x^{-1}(\mathrm{Id},g_t)W_n=T_xM^\uparrow$. Thus as in (4.6) the left hand side of (4.12) is ± 1. If we denote $+1$ (-1) by sqn_x for $x\in\mathcal{C}(f)$ if p_2 is orientation preserving (reversing) on Γ at $(x,f(x))$, then we have

$$L(f)-\chi(M)=\sum_{x\in\mathcal{C}(f)}\mathrm{sgn}_x.$$

We now show that the existence of a function f transverse to the cut locus and with $L(f)\neq\chi(M)$ imposes strong restrictions on the metric. Recall that $\exp_x^{-1}g_t(y)$ lies on the radial line joining $\exp_x^{-1}y$ to 0 in T_xM. Looking back at (4.12), we have

$$\left|\lim_{n\to\infty}\lim_{t\to 0}\int_{\beta\exp_x^{-1}(\mathrm{Id},g_t)W_n}\mathrm{MQ}\right|=1$$

iff $\exp_x^{-1}(\mathrm{Id}, f)B_\epsilon(x)$ contains an interior collar of the cut locus in T_xM, as only in this case will $\lim_{n\to\infty}\lim_{t\to 0}\beta\exp_x^{-1}(\mathrm{Id}, g_t)W_n = T_xM^\uparrow$. Letting ϵ shrink, we see that this collar condition occurs only if the cut locus of x in M is contained in an arbitrary neighborhood of $f(x)$–i.e. the cut locus of x in M is precisely $f(x)$. Thus M is homeomorphic to the one point compactification of $\exp_x^{-1}(M\setminus\{f(x)\})$, so $M\approx S^n$.

Corollary 4.1 *(i) Let $f : M\to M$ be a smooth map which is transverse to the cut locus. If $M\not\approx S^n$ and $\mathcal{C}(f)\neq\emptyset$, then*

$$|L(f)-\chi(M)| < |\mathcal{C}(f)|.$$

(ii) Let $f : M\to M$ be a smooth map which is Lefschetz and transverse to the cut locus. Let $\mathrm{Fix}(f)$ *be the fixed point set of f. Then*

$$|\chi(M)|\leq|\mathrm{Fix}(f)|+|\mathcal{C}(f)|,$$

with strict inequality if $M\not\approx S^n$ and $\mathcal{C}(f)\neq\emptyset$.

(ii) follows from $|\chi(M)|\leq|L(f)-\chi(M)|+|L(f)|$ and the Lefschetz fixed point theorem. As an application of (i), let f be transverse to the cut locus and have $L(f)\neq\chi(M)$. Then if $M\not\approx S^n$, either $\mathcal{C}(f)=\emptyset$ or $|\mathcal{C}(f)|\geq 2$. This fails on S^n for the suspension of $z\mapsto z^2$.

As in (4.12), set

$$\alpha_x = (\mathrm{sgn}\ p_2)\lim_{n\to\infty}\lim_{t\to 0}\int_{\beta\exp_x^{-1} g_tW_n}\mathrm{MQ}. \tag{4.14}$$

Then $\alpha_x\in[-1,1]$, and $\sum_i\alpha_{x_i}\in L(f)-\chi(M)\in\mathbf{Z}$ by (4.10), (4.11). In general, we expect α_x to vanish. For let A_ϵ be the radial projection of $\exp_x^{-1}f(B_\epsilon(x))$ onto $\mathcal{C}'_x=\exp_x^{-1}\mathcal{C}_x$ in T_xM. Since the Mathai-Quillen form is a radially symmetric fractional, positive multiple of the volume form in each fiber, we have

$$|\alpha_x|\leq\frac{\lim_{\epsilon\to 0}|A_\epsilon|}{|\mathcal{C}'_x|},$$

where $|A_\epsilon|, |\mathcal{C}'_x|$, denote the $(n-1)$-dimensional measure of $A_\epsilon, \mathcal{C}'_x$. (Alternatively, we can project $\exp_x^{-1}f(B_\epsilon(x))$ onto a small sphere centered at $0\in T_xM$ and compare its measure to the measure of the sphere.) Thus if f is transverse to the cut locus, and $L(f)\neq\chi(M)$, there exists $x\in\mathcal{C}(f)$ such that

$$\frac{\lim_{\epsilon\to 0}|A_\epsilon|}{|\mathcal{C}'_x|}\neq 0. \tag{4.15}$$

By an application of Lemma 4.4, for ϵ small, for all $y\in B_\epsilon(x)$, every minimal geodesic from $f(y)$ to x is C^1 close to a minimal geodesic from $f(x)$ to x. Thus the unique minimal geodesic γ from $f(y)$ to $f(x)$ lifts under $\exp_x^{-1}$ to a curve of length at most a constant times the length of γ, where the constant depends on sectional curvature bounds for M. Therefore there exist $y' \in \exp_x^{-1}(f(y)),\ z \in \exp_x^{-1}(f(x))$ such that $d(y', z) \to 0$ uniformly in y as $\epsilon \to 0$. It follows that (4.15) can occur only if $\lim_{\delta\to 0} |B_\delta(\exp_x^{-1} f(x))| \neq 0$, where B_δ denotes the delta ball in $\mathcal{C}'_x$. This implies that $\exp_x^{-1} f(x)$ has positive $(n-1)$ dimensional Hausdorff measure.

PROPOSITION 4.2 *Assume that f is transverse to the cut locus and $L(f) \neq \chi(M)$. Then there exists $x \in \mathcal{C}(f)$ such that $\exp_x^{-1} f(x)$ has positive $(n-1)$ dimensional Hausdorff measure in $\mathcal{C}'_x$.*

We call a metric on M *somewhere (nowhere) sphere-like* if there exist (do not exist) $x, y \in M$ such that $\exp_x^{-1} y$ has positive $(n-1)$ dimensional Hausdorff measure in $\mathcal{C}'_x$. One would expect a typical metric to be nowhere sphere-like, but it is easy to construct a somewhere sphere-like metric on any M, by considering M as the connect sum $M\#S^n$.

THEOREM 4.3 *(i) A metric of non-positive curvature is nowhere sphere-like.*

(ii) Let $f : M \to M$ be a diffeomorphism with $L(f) \neq \chi(M)$. If M is nowhere sphere-like, then $|\mathcal{C}(f)| = \infty$.

PROOF: (i) The map $\exp_x$ is a covering map for metrics of non-positive curvature, so the inverse image of any $y \in M$ is discrete in T_xM.

(ii) By the proposition, f is not transverse to the cut locus. Since condition (ii) for this transversality is satisfied, (i) must fail. Thus $|\mathcal{C}(f)| = \infty$. □

Examples: (i) The flat torus and a constant negative curvature surface are nowhere sphere-like, so no self-map of these spaces satisfies the hypothesis of Proposition 4.2. For example, for $x = (\theta, \psi) \in T^2 = [0, 2\pi] \times [0, 2\pi]$, the cut locus of x in T^2 is $(\{\theta \pm \pi\} \times [0, 2\pi]) \cup ([0, 2\pi] \times \{\psi \pm \pi\})$. For $(n, m) \in \mathbf{Z}^2$, $f(\theta, \psi) = (n\theta, m\psi)$ is a local diffeomorphism with $L(f) = 2-n-m$. Theorem 4.3 applies to local diffeomorphisms, so for $n + m \neq 2$, we conclude that $|\mathcal{C}(f)| = \infty$. In fact, it is easy to check that $\mathcal{C}(f)$ is uncountable for $(n, m) \neq (1, 1)$.

(ii) Let Σ^g be a genus $g > 1$ surface symmetric about a plane passing through the g holes. For $f : \Sigma^g \to \Sigma^g$ the diffeomorphism given by reflection through this plane, $L(f) = 0$. For example, Σ^g can be the hyperelliptic curve $y^2 = \prod_{i=1}^{2g+1}(x - a_i)$ with a_i real and distinct, with f the involution $(x, y) \mapsto (x, -y)$. For any metric on Σ^g we have either (i) f is not transverse to the cut locus and so $|\mathcal{C}(f)| = \infty$, or (ii) the metric is somewhere sphere-like and $|\mathcal{C}(f)| > 2g - 2$. Thus for any metric, $|\mathcal{C}(f)| > 2g - 2$, and for most metrics $\mathcal{C}(f)$ is infinite.

(iii) Let $f : S^2 \to S^2$ be a holomorphic map of degree n. Then f is a branched covering and so Γ is transverse to $M \times \{f(x)\}$ except at the branch points B. As above, for any metric on S^2 with $f^{-1}(B) \cap \mathcal{C}(f) = \emptyset$, we have $|\mathcal{C}(f)| \geq |L(f) - \chi(S^2)| = n-1$.

In §3, the topological limit as $t \to \infty$ was shown to have a geometric refinement. We do not have a topological interpretation for the geometric $t \to 0$ limit; presumably, the singular part of the current of §4.1 represents a cohomology class in some theory.

A Hodge theoretic techniques

As mentioned in §2.3, the upper bound for the Lefschetz number of a flat manifold can be extended to arbitrary metrics. Using sectional curvature bounds to control the Jacobi fields and the curvature tensor, one can extract an upper bound from the integral formula Theorem 2.4 in terms of the sectional curvature. In contrast, there is an easier Hodge theory argument which constructs a better upper bound in terms of Ricci curvature.

Let $\mathcal{N} = \mathcal{N}(n, C, D, V)$ be the class of Riemannian n-manifolds (M, g) with Ricci curvature $\mathrm{Ric} \geq C$, $\mathrm{diam}(M) \leq D$ and $\mathrm{vol}(M) \geq V$.

Proposition A.1 *There exist constants $C = C(k, n)$ and $D = D(\mathcal{N})$ such that for all $(M, g) \in \mathcal{N}$,*

$$|L(f)| \leq 1 + D \sum_{k=1}^{n} C(k, n) \binom{n}{k} \beta_k \cdot \sup_{x \in M} |df_x|_\infty^k,$$

where β_k is the k^{th} Betti number of M.

Before the proof, we compare two norms for differential forms. For $\alpha \in \Lambda^k T_x^* M$, we have the L^2 (Hodge) norm $|\alpha|_2^2 = *(\alpha \wedge *\alpha)$ and the L^∞ norm

$$|\alpha|_\infty = \sup_{v \in (T_x M)^{\otimes k} \setminus \{0\}} \frac{|\alpha(v)|}{|v|},$$

where $v = v_1 \otimes \ldots \otimes v_k$ has norm $|v| = \prod |v_i|$. Here we consider α as a linear functional on $(T_x M)^{\otimes k}$. Of course, there exists $C = C(g)$ such that $C^{-1}|\alpha|_\infty \leq |\alpha|_2 \leq C|\alpha|_\infty$, but we want this constant to depend only on k, n.

Lemma A.1 *There exists a constant $C = C(k, n)$ such that*

$$\binom{n}{k}^{-1/2} |\alpha|_2 \leq |\alpha|_\infty \leq C(k, n)|\alpha|_2.$$

Proof: Let $\{\theta^i\}$ be an orthonormal basis of $T_x^* M$ with dual basis $\{X_i\}$ of $T_x M$. For $\alpha = \alpha_I \theta^I$, we have

$$|\alpha|_\infty \geq \frac{|(\alpha_I \theta^I)(X_{i_1} \otimes \ldots \otimes X_{i_k})|}{|X_{i_1} \otimes \ldots \otimes X_{i_k}|} = |\alpha_{I_0}|,$$

where $I_0 = (i_1, \ldots, i_k)$. Thus

$$|\alpha|_\infty \geq \sup_I |\alpha_I| \geq \binom{n}{k}^{-1/2} \left(\sum_I |\alpha_I|^2\right)^{1/2} = \binom{n}{k}^{-1/2} |\alpha|_2.$$

For the other estimate

$$|\alpha|_\infty^2 \leq \sup_{v=v_1\otimes\ldots\otimes v_k\neq 0} \frac{\sum_I |\alpha_I|^2 |\theta^I(v_1\otimes\ldots\otimes v_k)|^2}{|v_1\otimes\ldots\otimes v_k|^2}.$$

For fixed $I_0 = (i_1, \ldots, i_k)$ and $v_1 = a_1^{j_1} X_{j_1}, \ldots, v_k = a_k^{j_k} X_{j_k}$, we have

$$|\theta^{I_0}(v_1\otimes\ldots\otimes v_k)| \leq \sum_{\substack{j_1,\ldots,j_k \\ \{j_1,\ldots,j_k\}=I_0}} |a_1^{j_1}\cdot\ldots\cdot a_k^{j_k}|.$$

Thus

$$\begin{aligned}
|\alpha|_\infty^2 &\leq \sup_{v\neq 0} \frac{\sum_{I_0} |\alpha_{I_0}|^2 \sum_{\{j_1,\ldots,j_k\}=I_0} |a_1^{j_1}\cdot\ldots\cdot a_k^{j_k}|^2\cdot k!}{|v_1\otimes\ldots\otimes v_k|^2} \\
&= \sup_{v\neq 0} \frac{\sum_{I_0} |\alpha_{I_0}|^2 \sum_{\{j_1,\ldots,j_k\}=I_0} |a_1^{j_1}\cdot\ldots\cdot a_k^{j_k}|^2\cdot k!}{\prod_{q=1}^k (\sum_{l_q} (a_q^{l_q})^2)} \\
&= \sup_{v\neq 0} \sum_{I_0} |\alpha_{I_0}|^2 \left[\frac{k!}{\binom{n}{k}} \frac{\sum_{\{j_1,\ldots,j_k\}=I_0} |a_1^{j_1}\cdot\ldots\cdot a_k^{j_k}|^2}{\prod_{q=1}^k (\sum_{l_q} (a_q^{l_q})^2)}\right].
\end{aligned}$$

For fixed I_0, the term inside the square bracket is a scale invariant function on $\mathbf{R}^{nk} = \{(a_i^j) : i = 1, \ldots, k,\ j = 1, \ldots, n\}$ and so is bounded above by $C'(k,n)$ independent of I_0. Thus

$$|\alpha|_\infty^2 \leq \frac{k!}{\binom{n}{k}} C'(k,n) \sum_I |\alpha_I|^2 = (C(k,n))^2 |\alpha|_2^2.$$

□

Proof of the Proposition: Let $\{\omega_k^i\}$ be an L^2-orthonormal basis of harmonic k-forms. The trace of $f^* : H^k(M;\mathbf{R}) \to H^k(M;\mathbf{R})$ is $\sum_i \langle f^*\omega_k^i, \omega_k^i\rangle$, so

$$\begin{aligned}
|L(f)| &\leq \sum_{k,i} \left|\langle f^*\omega_k^i, \omega_k^i\rangle\right| \leq \sum_{k,i} \|f^*\omega_k^i\| \\
&= \sum_{k,i} \left[\int_M |(f^*\omega_k^i)_x|_2^2 \mathrm{dvol}(x)\right]^{1/2}, \qquad \text{(A.1)}
\end{aligned}$$

by Cauchy-Schwarz. Here $\|\alpha\|^2 = \int_M \alpha\wedge *\alpha$ is the global L^2 norm. When $k = 0$, we have $\|f^*\omega_0^1\| = \|\omega_0^1\| = 1$.

By (A.1) and the lemma, we have

$$|L(f)| \leq 1 + \sum_{k=1}^{n} \sum_{i} \binom{n}{k} \operatorname{vol}^{1/2}(M) \sup_{x \in M} |(f^* \omega_k^i)_x|_\infty. \tag{A.2}$$

Now

$$|(f^*\omega)_x|_\infty = \sup_{v \neq 0} \frac{|(f^*\omega)_x(v_1 \otimes \ldots \otimes v_k)|}{|v_1 \otimes \ldots \otimes v_k|} = \sup_{v \neq 0} \frac{|\omega_{f(x)}(f_* v_1 \otimes \ldots \otimes f_* v_k)|}{|v_1 \otimes \ldots \otimes v_k|},$$

where $f_* = df$. Since the last term vanishes if $f_* v_i = 0$ for some i, we assume $f_* v_i \neq 0$. Then

$$\begin{aligned} |(f^*\omega)_x|_\infty &= \sup_{v \neq 0} \frac{|\omega_{f(x)}(f_* v_1 \otimes \ldots \otimes f_* v_k)|}{|f_* v_i \otimes \ldots \otimes f_* v_k|} \cdot \frac{|f_* v_1 \otimes \ldots \otimes f_* v_k|}{|v_1 \otimes \ldots \otimes v_k|} \\ &\leq |\omega_{f(x)}|_\infty \cdot \sup_{v \neq 0} \frac{\prod_i |f_* v_i|}{\prod_i |v_i|} \\ &\leq |\omega_{f(x)}|_\infty \cdot \sup_{v \neq 0} \frac{\prod_i |df_x|_\infty |v_i|}{\prod_i |v_i|} \\ &\leq |\omega_{f(x)}|_\infty |df_x|_\infty^k. \end{aligned}$$

By (A.2) and the lemma, we get

$$\begin{aligned} |L(f)| &\leq 1 + \sum_{k=1}^{n} \binom{n}{k} \operatorname{vol}^{1/2}(M) \sum_i \sup_{x \in M} |df_x|_\infty^k \cdot |(\omega_k^i)_{f(x)}|_\infty. \\ &\leq 1 + \sum_{k=1}^{n} \binom{n}{k} \operatorname{vol}^{1/2}(M) \sum_i \sup_{x \in M} |df_x|_\infty^k \cdot C(k,n) |(\omega_k^i)_{f(x)}|_2. \end{aligned}$$

By [3], [7], there is an explicit constant $D_1(\mathcal{N})$ such that for all $x \in M$,

$$|(\omega_k^i)_x|_2 \leq D_1(\mathcal{N}) \|\omega_k^i\| = D_1(\mathcal{N}).$$

Thus

$$|L(f)| \leq 1 + \sum_{k=1}^{n} \binom{n}{k} \beta_k \cdot \operatorname{vol}^{1/2}(M) \cdot D_1(\mathcal{N}) C(k,n) \sup_{x \in M} |df_x|_\infty^k.$$

Finally, $\operatorname{vol}(M)$ is bounded above on $\mathcal{N}$ by standard comparison theorems. □

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York, 1982.

[2] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry*, North Holland Publishing Company, Amsterdam, 1975.

[3] C. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. Ec. Norm. Super., Paris **13** (1980), 419–435.

[4] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian Geometry*, Springer-Verlag, Berlin, 1987.

[5] P. B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem*, Publish or Perish, Wilmington, 1984.

[6] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Englewood Cliffs, NJ, 1974.

[7] P. Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. Ec. Norm. Super., Paris **13** (1980), 451–469.

[8] V. Mathai and D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25** (1986), 85–110.