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Linear Arrangement of Halin Graphs

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Abstract

We study the Optimal Linear Arrangement (OLA) problem of Halin graphs, one of the simplest classes of non-outerplanar graphs. We present several properties of OLA of general Halin graphs. We prove a lower bound on the cost of OLA of any Halin graph, and define classes of Halin graphs for which the cost of OLA matches this lower bound. We show for these classes of Halin graphs, OLA can be computed in $n \log n$, where n is the number of vertices.

1 Introduction

Given graph $G = (V, E)$, a *linear arrangement* or simply a *layout* of vertices is defined as a bijective function $\varphi : V \rightarrow \{1, \dots, |V|\}$. In the Optimal Linear Arrangement (OLA) problem, a special case of more general vertex layout problems, the goal is to find the layout φ minimizing $\sum_{\{v,u\} \in E} |\varphi(v) - \varphi(u)|$. The OLA problem is known to be NP-hard for general graphs [5], for *bipartite graphs* [4], and for more specific classes of graphs such as *interval graphs* and *permutation graphs* [2].

Defining interesting classes of graphs for which the OLA problem is polynomially solvable has been a notoriously hard task. The results have been few and spread over several decades [5, 4, 2, 1, 10]. Three decades ago, it was suggested that a good candidate for polynomial-solvable OLA are interval graphs, a class of graph for which no NP-hardness results were known at that time (page 13 of [8]). Efforts in that direction were in vain, as some two decades later, the OLA problem of interval graphs was shown to be NP-hard [2]. Another candidate for polynomial-solvable OLA are the so-called *recursively constructed graphs* [7], given that most NP-hard problems on general graphs are easily solvable for this class.

Halin graphs are planar graphs which the degree of every vertex is at least 3 and can be constructed using an underlying tree T and a cycle C which connects leaf nodes of the tree T . Figure 1 presents a Halin graph. Throughout this report, the edges of cycle C are presented in bold and the edges of the tree T are depicted in dashed lines. Halin graphs can be considered as one of the simplest class of graphs that are not outerplanar ¹.

To the best of our knowledge, the OLA problem of Halin graphs is only studied for the simple case where the underlying tree is a caterpillar [3]. After introducing our notations and preliminary definitions in Section 2, we present several properties of OLA of Halin graphs in Section 3, including a lower bound on the cost of OLA for Halin graphs. In Section 4, we define and study a class of Halin graphs for which the cost of their OLA meets this lower bound. We also present an algorithm which, given a Halin graph in this class, returns an OLA in $O(n \log n)$ where n is the number of vertices.

2 Preliminaries

We only consider simple, undirected graphs. For a finite graph $G = (V, E)$ where V and E are respectively the sets of vertices and edges, we show $|V|$ by n and $|E|$ as m . For a given vertex $v \in V$, $d_G(v)$ presents the degree of v in G . For a subgraph $G' \subseteq G$, $V(G')$ and $E(G')$ respectively present the set of vertices and edges of G' .

¹A Halin graph is a 2-outerplanar graph.

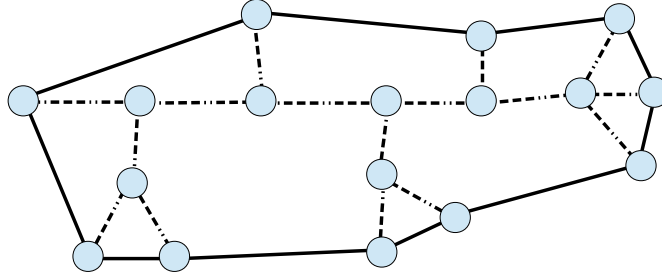


Figure 1: H_1 , an example of a Halin graph. The cycle which connect the leaf nodes is shown in bold and the underlying tree is presented in dashed lines.

We denote by $\Phi(G)$ the set of all possible layouts for the graph G . A layout φ can be considered as an ordering (w_1, w_2, \dots, w_n) of vertices of V . Accordingly for $v = w_i \in V$, $\varphi(v) = i$. Without loss of generality we assume the left most and right most vertices are recursively labeled as 1 and n and we call them the *extreme vertices* based on φ .

Notation 2.1. Let V_1, \dots, V_k be a partitioning of V . We say a layout φ is of type (V_1, V_2, \dots, V_k) if:

$$\forall 1 \leq i < j \leq k, \forall v \in V_i, \forall u \in V_j \Rightarrow \varphi(v) < \varphi(u)$$

Notation 2.2. Given layout φ for $G = (V, E)$ and an edge $e = \{u, v\} \in E$ we define the expand of e as:

$$\lambda(e, \varphi) = |\varphi(u) - \varphi(v)|$$

Several cost functions have been defined on a given graph G and layout φ . For a comprehensive list refer to [9]. In this report we focus on *Optimal Linear Arrangement* problem (OLA) defined as follows.

Definition 2.3 (Optimal Linear Arrangement). Given an undirected graph $G = (V, E)$ and a layout φ the *linear arrangement cost* (LA) of φ is:

$$LA(\varphi, G) = \sum_{\{v,u\} \in E(G)} \lambda(\{u, v\}, \varphi)$$

A layout φ^* is optimal if:

$$LA(\varphi^*, G) = \min_{\varphi \in \Phi(G)} LA(\varphi, G)$$

□

Lemma 2.4 present a lower bound on the cost of optimal linear arrangement which will be useful in presenting some properties and proofs in the rest of the paper.

Lemma 2.4. Given graph $G = (V, E)$ and two induced subgraph $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ s.t. $E_1 \cap E_2 = \emptyset$ and $E_1 \uplus E_2 = E$, assume φ^* , φ_1^* and φ_2^* are respectively the optimal linear arrangement for G , G_1 and G_2 . Then $LA(\varphi^*, G) \geq LA(\varphi_1^*, G_1) + LA(\varphi_2^*, G_2)$.

A Halin graph $H = (V, E)$ is constructed based on an underlying tree $T = (V, E')$ embedded in plane in a planar manner where all the leaf nodes are connected using a cycle $C = (V', E'')$. A Halin graph H is shown as $H = T \uplus C$. As depicted in Figure 2 it's easy to see that for a given tree T , there may exist finitely many non-isomorphic Halin graphs.

Based on the structure of Halin graphs, one may suspect that they inherit many of properties of their underlying tree. Accordingly in the following subsection we present some properties and well-known facts regarding the OLA of trees.

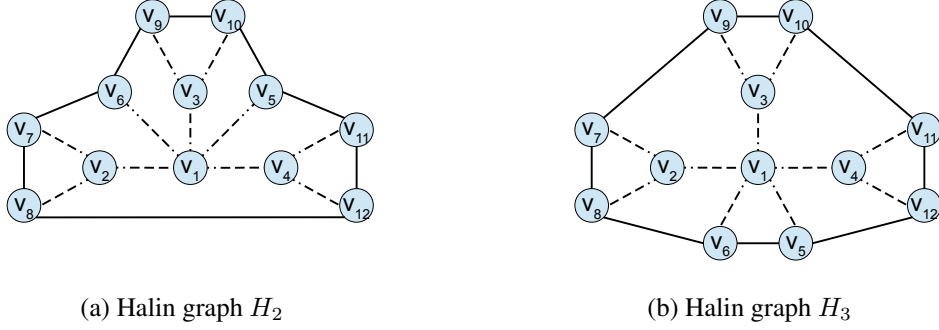


Figure 2: Two non-isomorphic Halin graphs constructed from the same tree using different embedding.

2.1 Linear Arrangement of Trees

As one of non-trivial results, the OLA of trees was first shown to be polynomially solvable in [6] and more efficient algorithms were later presented in [11, 1]. In this section we present some well-known properties of OLA of trees which simplify the understanding of linear arrangement of Halin graphs in the rest of the report. For more details, one can refer to [1].

Property 2.5. Given an OLA φ^* for tree $T = (V, E)$, vertices that are assigned label 1 and n are leaf nodes (the two extreme vertices are leafs in T). \square

Definition 2.6 (Spinal path). Given layout φ for a tree $T = (V, E)$, and vertices $v, u \in V$ where $\varphi(v) = 1$ and $\varphi(u) = n$, we define the path $P = (w_1 = v, w_2, \dots, w_l = u)$ connecting v and u , the **spine** of tree T corresponding to φ . \square

Property 2.7. Having OLA φ^* for tree $T = (V, E)$ and spinal path $P = (w_1 = v, w_2, \dots, w_l = u)$, it is the case that $\forall 1 \leq i < l : \varphi^*(w_i) < \varphi^*(w_{i+1})$. In other word, the function φ^* is monotonic along the path P . \square

Definition 2.8 (Spinal rooted subtree and anchored branches). Given layout φ for a tree $T = (V, E)$ and spinal path $P = (w_1 = v, w_2, \dots, w_l = u)$, Removing all edges of P , leaves us with a set of subtrees T_1, T_2, \dots, T_l respectively rooted at w_1, w_2, \dots, w_l . Removing spinal vertex w_i from T_i where $d_T(w_i) = k > 2$, results in a set of branches $B_{i,1}, B_{i,2}, \dots, B_{i,k-2}$. Each branch $B_{i,j}$ is anchored at a vertex v_j which is connected to w_i . \square

Property 2.9. Consider a tree $T = (V, E)$ and its OLA φ^* and the corresponding spinal path $P = (w_1 = v, w_2, \dots, w_l = u)$. Removing all the edges of P results in a set of l subtrees T_1, \dots, T_l , respectively rooted at w_1, \dots, w_l . Then based on φ^* , for a fixed i , the vertices of of T_i are labeled by contentious integers. Formally speaking:

$$\forall 1 < i < l, \forall u \in T_{i-1}, v \in T_i, w \in T_{i+1} \Rightarrow \varphi^*(u) < \varphi^*(v) < \varphi^*(w)$$

Moreover φ^* restricted to $V(T_i)$ (denoted by φ_i^*) is optimal for T_i . \square

3 Some Properties of OLA of Halin Graphs

Halin graphs are the example of edge-minimal 3-connected graphs. Hence, in a Halin graph $H = T \uplus C$, for any two vertices v and u , there are exactly three edge-disjoint paths connecting v and u where one comprises only edges of $E(T)$.

Definition 3.1 (Spinal path in Halin graphs). Given layout the φ for a Halin graph H and two vertices $v, u \in E(H)$ where $\varphi(v) = 1$ and $\varphi(u) = n$, the *spinal path* based on φ , is defined as the path $P = (w_1 = v, w_2, \dots, w_l = u)$ where for every $1 \leq i < l$, $\{w_i, w_{i+1}\}$ is an edge in T . \square

Definition 3.2 (*Spinal rooted subtree and anchored branches in Halin graphs*). Given layout φ for a Halin graph $H = T \uplus C$ and spinal path $P = (w_1 = v, w_2, \dots, w_l = u)$, removing all edges of P and $E(C)$ results in a set of subtrees T_1, T_2, \dots, T_l , respectively rooted at w_1, w_2, \dots, w_l . Removing spinal vertex w_i from T_i where $d_T(w_i) = k > 2$, give us a set of branches $B_{i,1}, B_{i,2}, \dots, B_{i,k-2}$. Also each branch $B_{i,j}$ is anchored at a vertex v_j , connected to w_i . \square

Lemma 3.3. Consider the Halin graph $H = T \uplus C$ and the spinal path $P = (w_1, w_2, \dots, w_l)$ based on a given OLA φ^* . Removing all the edges of P and C results in a set of l subtrees T_1, \dots, T_l respectively rooted at w_1, \dots, w_l . For a fixed i , the vertices of T_i are labeled by contentious integers by OLA φ^* . Formally speaking:

$$\forall 1 < i < l, \forall u \in T_{i-1}, v \in T_i, w \in T_{i+1} \Rightarrow \varphi^*(u) < \varphi^*(v) < \varphi^*(w)$$

See proof A.6 in Appendix A.

Corollary 3.4. Having OLA φ^* for Halin graph H and spinal path $P = (w_1, w_2, \dots, w_l)$, it is the case that $\forall 1 \leq i < l : \varphi^*(w_i) < \varphi^*(w_{i+1})$. In other word, the function φ^* is monotonic along the path P .

Lemma 3.5. Consider an OLA φ^* for a Halin graph $H = T \uplus C$ and the set of subtrees T_1, \dots, T_l resulted after removing the edges of C and the spinal path $P = (w_1, w_2, \dots, w_l)$. Let $\{B_{i,1}, \dots, B_{i,k-2}\}$ be the set of branches of T_i connected to a spinal vertex w_i with degree $k > 2$. For two branches $B_{i,j}$ and $B_{i,j'}$:

- If φ^* is of type $(\dots, w_i, \dots, V(B_{i,j}) \cup V(B_{i,j'}), \dots)$ then it is of type $(\dots, w_i, \dots, V(B_{i,j}), V(B_{i,j'}), \dots)$ or $(\dots, w_i, \dots, V(B_{i,j'}), V(B_{i,j}), \dots)$
- If φ^* is of type $(\dots, V(B_{i,j}) \cup V(B_{i,j'}), \dots, w_i, \dots)$ then it is of type $(\dots, V(B_{i,j}), V(B_{i,j'}), \dots, w_i, \dots)$ or $(\dots, V(B_{i,j'}), V(B_{i,j}), \dots, w_i, \dots)$

In other word the two branches $B_{i,j}$ and $B_{i,j'}$ which are on the same side of w_i (either their vertices are all labeled after w_i or all before it), do not overlap.

For proof refer to Appendix A, proof A.11.

Theorem 3.6. Given an OLA φ^* for a Halin graph $H = T \uplus C$ and the vertices v and u where $\varphi^*(v) = 1$ and $\varphi^*(u) = n$, it is always the case that:

- v and u are both leaves in T or
- if v (or u) is not a leaf vertex in T , then degree of v is three and it is connected to exactly two leaves in T . Accordingly replacing the label of v (or u) with one of it's leaf nodes, we get another OLA φ^{\otimes} where the extreme nodes are leaves in T .

This lemma is proven in proof A.15 in Appendix A.

Corollary 3.7. Consider an OLA φ^* for a Halin graph $H = (V, E)$ constructed from tree $T = (V, E')$ and cycle C , then:

$$LA(\varphi^*, H) \geq 2 \times (n - 1) + LA(\varphi_T^*, T)$$

where φ_T^* is the OLA for T .

4 Halin Graphs With Polynomially Solvable LA Algorithm

As mentioned before, the OLA problem is polynomially solvable for trees. The OLA of a Halin graph $H = T \uplus C$, depends both on the underlying tree and the planar embedding of T . Motivated by the work in [11], in this section we study some classes of Halin graphs where OLA problem can be solved in polynomial time. More specifically we show that for these classes of Halin graphs, the equality in corollary 3.7 holds.

Definition 4.1 (*Recursively Balanced Trees*). Consider the tree T and the vertex v_r , designated as the root of the tree, and the set of vertices $v_{r,0}, \dots, v_{r,k}$ connected to the v_r as it's direct children. Removing the set of edges $\{v_r, v_{r,0}\}, \dots, \{v_r, v_{r,k}\}$ results in the set of subtrees $T_{r,0}, \dots, T_{r,k}$, respectively rooted at $v_{r,0}, \dots, v_{r,k}$. T is recursively balanced if:

- $\mathcal{T}_{r,0} = \mathcal{T}_{r,1} = \dots = \mathcal{T}_{r,k}$ ² and
- $T_{r,i}$, rooted at $v_{r,i}$, is recursively balanced for $i = 0, 1, \dots, k$

□

The root vertex v_r of a Recursively Balanced Tree (RBT), is the only vertex satisfying the properties of the *central vertex* in the following theorem.

Theorem 4.2. *Given a tree $T = (V, E)$, there exists a vertex v_r where the set of subtrees T_0, \dots, T_k resulted by removing v_r from T , satisfies:*

$$\mathcal{T}_i \leq \lfloor \frac{n}{2} \rfloor \quad \text{for } i = 0, \dots, k$$

For proof see [11].

Considered a tree T rooted at v_r and the corresponding subtrees T_0, \dots, T_k after removing v_r , where $\mathcal{T}_0 \geq \mathcal{T}_2 \geq \dots \geq \mathcal{T}_k$. Assume that an OLA φ^* for T is of type $(\dots, T_i, \dots, v_r, \dots)$ or $(\dots, v_r, \dots, T_i, \dots)$ for some subtree T_i . T_i is called (respectively right or left) *anchored subtree*, rooted at v_i connecting T_i to v_r . A tree T which is not anchored is called a *free tree*. In theorem 4.3, which is the motivating theorem and the heart of the OLA algorithm for trees in [11], we show the root of a tree by v_r . Vertex v_r is the central vertex in the case of free trees, or the anchor vertex if the tree is an anchored subtree. Also the parameter α is 0 for free trees and 1 otherwise.

Theorem 4.3. *Given a free or (right) anchored tree $T = (V, E)$ ³, let ρ be the largest integer that satisfies the following:*

$$\mathcal{T}_i > \lfloor \frac{\mathcal{T}_1 + 2}{2} \rfloor + \lfloor \frac{\mathcal{T}_* + 2}{2} \rfloor \quad \text{for } i = 1, \dots, 2\rho - \alpha$$

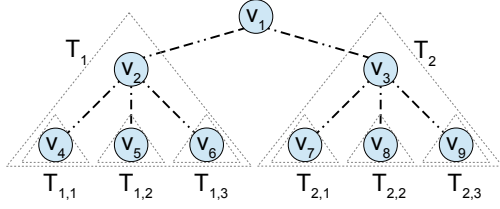
where:

$$\mathcal{T}_* = n - \sum_0^{2\rho - \alpha} \mathcal{T}_i$$

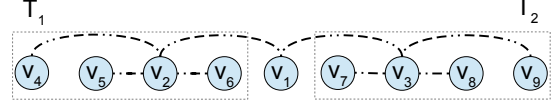
- If $\rho = 0$, the OLA of T is of type (T_0, v_r, \dots)
- If $\rho > 0$, then T has an OLA of type either (T_0, v_r, \dots) or $(T_1, T_3, \dots, T_{2\rho-1}, \dots, v_r, \dots, T_{2\rho-2\alpha}, \dots, T_4, T_2)$

² $\mathcal{T}_i = |V(T_i)|$. See notation A.1 in Appendix A.

³The theorem symmetrically holds in case of left anchored subtrees.



(a) An example of a recursively balance tree T rooted at v_1 . The two subtrees T_1 and T_2 of v_1 are highlighted by larger enclosing triangles.



(b) An OLA φ^* for tree T .

Figure 3: An example of a recursively balance tree and it's corresponding OLA.

Notation 4.4. Consider the layout φ of type $(T_1, \dots, T_i, \dots, T_j, \dots, T_k)$ for the tree T rooted at the central vertex v_r . Swapping the arrangement of vertices of two subtrees T_i and T_j , while keeping the relative order of the vertices of each subtree unchanged (or reversed), is presented using operator $\sigma(\varphi, T_i, T_j)$ which is of type $(T_1, \dots, T_j, \dots, T_i, \dots, T_k)$.

Lemma 4.5. Given a recursively balanced tree T rooted at the central vertex v_r , and the corresponding subtrees $T_{r,1}, \dots, T_{r,k}$, there exists an OLA φ^* of type $(T_{r,1}, \dots, T_{r,\bar{k}}, v_r, T_{r,\bar{k}+1}, \dots, T_k)$, where $\bar{k} = \lceil \frac{k+1}{2} \rceil$.

Proof. We know that the subtrees $T_{r,1}, \dots, T_{r,k}$ have the same size and v_r satisfy the central vertex theorem 4.2. Accordingly, considering the theorem 4.3, it's easy to see that there exists an OLA φ_0 where half of subtrees are labeled before v_r and the other half are labeled after v_r while the vertices of no two subtrees overlap.

Based on the structure of φ_0 , there exists a sequence of layouts $(\varphi_0, \varphi_1, \dots, \varphi_l = \varphi^*)$ where for $k = 1, \dots, l$, $\varphi_k = \sigma(\varphi_{k-1}, T_{r,i}, T_{r,j})$ for some subtrees $T_{r,i}, T_{r,j}$. Since all the subtrees have the same size, then $LA(\varphi_0, T) = LA(\varphi_1, T) = \dots = LA(\varphi^*, T)$.

Generally, given an OLA φ^* of type $(T_{r,1}, \dots, T_{r,\bar{k}}, v_r, T_{r,\bar{k}+1}, \dots, T_k)$ for the RBT T and two subtrees $T_{r,i}$ and $T_{r,j}$, $\sigma(\varphi^*, T_{r,i}, T_{r,j})$ is also an OLA for T . \square

Lemma 4.6. Let $T_{r,1}, \dots, T_{r,k}$ be the set of subtrees of the RBT T resulted by removing the root vertex v_r . Given two leaf vertices $v \in V(T_{r,i}), u \in V(T_{r,j})$ for $i \neq j$, the simple path $P = (v, \dots, v_r, \dots, u)$ connecting v and u (via v_r) is the spinal path for some OLA φ^* . In other word there is an OLA φ^* , where $\varphi^*(v) = 1$ and $\varphi^*(u) = n$.

Proof. An immediate result of lemma 4.5 is that there exists an OLA of type $(T_{r,i}, \dots, v_r, \dots, T_j)$ for tree T . Also note that the two subtrees $T_{r,i}$ and $T_{r,j}$ are recursively balanced trees. Applying the same approach recursively and excluding all the the details, one can deduce that there exists an OLA for T which is of type $(v, \dots, v_r, \dots, u)$. \square

Example 4.7. Figure 3 depicts an example of a recursively balance tree (in 3a) and it's corresponding OLA φ^* (in 3b). As you see the operation $\sigma(\varphi^*, T_1, T_2)$ will result in another layout with the same value. Generally, given an OLA φ^* for a recursively balanced tree $T = (V, E)$ and $v \in V$ and any two rooted subtrees $T_{i,j}$ and $T_{i,j'}$ connected to v , it is the case that $\sigma(\varphi^*, T_{i,j}, T_{i,j'})$ is also an OLA.

Following the results of lemmas 4.5 and 4.6, In following we present an approach to find an OLA for recursively balanced tree in linear time.

Theorem 4.8. Having a recursively balanced tree $T = (V, E)$ rooted at v_r , an OLA φ^* for T can be found in linear time.

Proof. Assume $T_{r,1}, \dots, T_{r,k}$ are the subtrees connected to v_r respectively via $v_{r,1}, v_{r,2}, \dots, v_{r,k} \in V$. From lemma 4.5 an OLA φ^* of type $(T_{r,1}, \dots, T_{r,\bar{k}}, v_r, T_{r,\bar{k}+1}, \dots, T_k)$ exists for T . Each subtree has exactly $\frac{|V|-1}{k}$ vertices, hence $\varphi^*(v_r) = \bar{k} \times \frac{|V|-1}{k} + 1$, where $\bar{k} = \lceil \frac{k+1}{2} \rceil$.

Also based on the definition 4.1 every subtree $T_{r,i}$, for $1 \leq i \leq k$, is recursively balance with the central vertex $v_{r,i}$. Therefore, using the same approach one can go on with constructing OLA φ^* by finding the label of $v_{r,i}$, for $1 \leq i \leq k$. Applying this method recursively OLA φ^* can be found while every vertex of the tree is visited $O(1)$ times. \square

The following two theorems conclude this section by presenting some classes of Halin graphs which there exists a polynomial OLA algorithm for them. More specifically, given a Halin graph $H = T \uplus C$ from these classes, an OLA for H can be derived given any optimal layout for the underlying tree T ⁴.

Theorem 4.9. *Consider a Halin graph $H = T \uplus C$, where the underlying tree T is recursively balanced, rooted at v_r . Let φ^\otimes be an OLA for T . There exist a linear arrangement φ^* s.t.*

- $LA(\varphi^*, H) = LA(\varphi^\otimes, T) + 2 \times (|V| - 1)$. Hence, based on corollary 3.7, φ^* is an OLA for H
- φ^* can be constructed from φ^\otimes in $O(|V| \log |V|)$

Proof. We know that for every layout φ , it is the case that $LA(\varphi, C) \geq 2 \times (n - 1)$, where $n = |V|$. Hence, given the OLA φ^\otimes for T , if $LA(\varphi^\otimes, H) = LA(\varphi^\otimes, T) + 2 \times (n - 1)$, then φ^\otimes is also an OLA for H as well. Otherwise, starting from φ^\otimes , we present an iterative approach where using a sequence of swapping operations, an OLA is found for H . In this sequence of swapping, after each swap operation the value of arrangement stays unchanged for T , and decreases for H .

This procedure is presented in algorithm 1. Assume the underlying tree T , rooted at central vertex v_r , has height h ⁵

Correctness of the algorithm. Starting with $\varphi^* = \varphi^\otimes$, after execution of line 8, we will end up with a potentially modified layout φ^* of type $(T_1, \dots, v_r, \dots, T_k)$ where T_1 is directly connected to T_k via an edge from $E(C)$ and for $1 \leq i < k$, T_i is directly connected to T_{i+1} through $E(C)$. So if we collapse every subtree T_i for $1 \leq i < k$ into one vertex, the resulted φ^* is an OLA for the corresponding Halin graph.

So far, Based on the resulted layout φ^* , $T_L = T_1$ and $T_R = T_k$, respectively defined in lines 3 and 4, are the two left and right boundary subtrees and all other subtrees are middle subtrees.

Lines 9 to 17 of algorithm, guarantee that in a recursive approach, for every subtree T of height $1 \leq h < h$, based on the final φ^* :

- If T is a left side subtree (i.e. if φ^* is of type (T, \dots)), consider $v \in V(T)$, where $\varphi^*(v) = 1$. v is connected to T_R via $e \in E(C)$
- If T is a right side subtree (i.e. if φ^* is of type (\dots, T)), consider $v \in V(T)$, where $\varphi^*(v) = |V|$. v is connected to T_L via $e \in E(C)$
- Otherwise φ^* is of type $(\dots, T_1, T, T_2, \dots)$. Let $v_L \in V(T)$ be the vertex of V s.t. $\forall v \in V(T), \varphi^*(v_L) \leq \varphi^*(v)$. Similarly let $v_R \in V(T)$ be the vertex s.t. $\forall v \in V(T), \varphi^*(v) \leq \varphi^*(v_R)$. Then v_L and v_R are respectively directly connected to T_1 and T_2 through $E(C)$

Therefore it can be inferred that based on the final layout φ^* , $LA(\varphi^*, H) = LA(\varphi^*, T) + LA(\varphi^*, C) = LA(\varphi^*, T) + 2 \times (n - 1)$. But we know that the swap operation σ does not change the value of linear arrangement for the underlying tree T . Hence $LA(\varphi^*, H) = LA(\varphi^\otimes, T) + 2 \times (n - 1)$ which induces the optimality of φ^* .

⁴Remember that OLA problem is polynomially solvable for trees.

⁵We consider the height of a tree with consist of only vertex is 1.

Algorithm 1 Finding OLA φ^* for Halin graph $H = (T, C)$ given OLA φ^\otimes for RBT T

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1:  $\varphi^* \leftarrow \varphi^\otimes$ 
2: let  $\{T_1, \dots, T_k\}$  be subtree of height  $h - 1$  as  $\varphi^*$  is of type  $(T_1, \dots, v_r, \dots, T_k)$ 
3: let  $T_L$  be  $T_1$ 
4: let  $T_R$  be one of the two subtrees connected to  $T_L$  via  $E(C)$ 
5:  $\sigma(\varphi^*, T_R, T_k)$ 
6: for  $i = 1$  to  $k - 2$ 
7:   Let  $T_{i,R} \in \{T_{i+1}, \dots, T_{k-1}\}$  be the subtree connected to  $T_i$  via and edge in  $E(C)$ 
8:    $\sigma(\varphi^*, T_{i+1}, T_{i,R})$ 

9: for  $h = h$  to 2:
10:  for every subtree  $T$  of height  $h$  rooted at  $v_r$ :
11:    Let  $(T_{r,1}, \dots, v_r, \dots, T_{r,k})$  be  $\varphi^*$  restricted to  $T$ 
12:    if  $\varphi^*$  is of type  $(T, \dots)$ :
13:      ReArrLeftSubTree $((T_{r,1}, \dots, T_{r,k}), \varphi^*)$ 
14:    else if  $\varphi^*$  is of type  $(\dots, T)$ :
15:      ReArrRightSubTree $((T_{r,1}, \dots, T_{r,k}), \varphi^*)$ 
16:    else:
17:      ReArrMidSubTree $((T_{r,1}, \dots, T_{r,k}), \varphi^*)$ 

18: ReArrMidSubTree $((T_{r,1}, \dots, T_{r,k}), \varphi^*)$ :
19:  Let  $T_{r,L}$  be the subtree connected to some vertex  $v_L$ , where based on  $\varphi^*$ ,  $v_L$  is labeled before  $T_{r,1}$ 
6Hence,  $v_L \notin \{V(T_{r,1}) \uplus \dots \uplus T_{r,k}\}$  and  $\forall v \in \{V(T_{r,1}) \uplus \dots \uplus T_{r,k}\}, \varphi^*(v_L) < \varphi^*(v)$ 
20:  Let  $T_{r,R}$  be the subtree connected to some vertex  $v_R$ , where based on  $\varphi^*$ ,  $v_R$  is labeled after  $T_{r,k}$  7Hence,
 $v_R \notin \{V(T_{r,1}) \uplus \dots \uplus T_{r,k}\}$  and  $\forall v \in \{V(T_{r,1}) \uplus \dots \uplus T_{r,k}\}, \varphi^*(v) < \varphi^*(v_R)$ 
21:   $\sigma(\varphi^*, T_{r,1}, T_{r,L})$ 
22:   $\sigma(\varphi^*, T_{r,k}, T_{r,R})$ 
23:  if  $k > 3$ :
24:    ReArrMidSubTree $((T_{r,2}, \dots, T_{r,k-1}), \varphi^*)$ 

25: ReArrLeftSubTree $((T_{r,1}, \dots, T_{r,k}), \varphi^*)$ :
26:  Let  $T_{r,L}$  be the subtree connected to  $T_R$  via  $E(C)$ 
27:   $\sigma(\varphi^*, T_{r,1}, T_{r,L})$ 
28:  for  $i = 1$  to  $k - 2$ 
29:    Let  $T_{i,R} \in \{T_{i+1}, \dots, T_k\}$  be the subtree connected to  $T_i$ 
30:     $\sigma(\varphi^*, T_{i+1}, T_{i,R})$ 

31: ReArrRightSubTree $((T_{r,1}, \dots, T_{r,k}), \varphi^*)$ :
32:  Let  $T_{r,R}$  be the subtree connected to  $T_L$  via  $E(C)$ 
33:   $\sigma(\varphi^*, T_{r,k}, T_{r,R})$ 
34:  for  $i = k$  to 2
35:    Let  $T_{i,R} \in \{T_1, \dots, T_{i-1}\}$  be the subtree connected to  $T_i$ 
36:     $\sigma(\varphi^*, T_{i-1}, T_{i,R})$ 

```

Time complexity analysis. The time complexity of the algorithm depends on the two major **For** loops in lines 6 and 9.

- **Analysis of the first loop in line 6.** Assuming every basic swap operation is an atomic operation with cost $O(1)$, then the cost of every $\sigma(\varphi^*, T_1, T_2)$ is $O(|V(T_1)|) = O(|V(T_2)|)$. Hence, the cost of the loop at line 6 is $O(k \times \frac{n-1}{k}) = O(n)$ ⁸.
- **Analysis of the second loop in line 9.** At every iteration, if there are k subtrees $\{T_{r,1}, \dots, T_{r,k}\}$, In worst-case scenario at most $O(k)$ swap operations σ are carried out. For $1 \leq i \leq k$, $|V(T_{r,i})| = \frac{n}{k}$. Therefore the cost of each iteration is $O(k \times \frac{n}{k}) = O(n)$.

Having the fact that $h = O(\log n)$, we conclude the time complexity of the loop in line 9 as $O(n \log n)$, which dominates the time complexity of the whole algorithm. □

Example 4.10. In figure 3, two layouts are presented for the Halin graph $H_2 = T \uplus C$ (Figure 2a in section 2). Layout φ_1 in Figure 4a, is an OLA for the underlying tree of graph H_2 while it is not an optimal layout for H_2 itself (the OLA for H_2 is shown in Figure 4b). Enumerating all the OLAs for the underlying tree of H_2 , it can be verified that non is an OLA for H_2 .

$P = (v_7, v_2, v_2, v_4, v_{12})$ is the spinal path corresponding to the layout φ_1 . After removing the edges of the spinal path P and cycle C , each vertex v_i of the path P corresponds to a subtree T_i . Notice that subtree T_1 , rooted at v_1 , is not an RBT. Hence based on the order of arrangement of the three branches connected to v_1 , we may get different values for the linear arrangement, and an ordering of the branches with the optimal arrangement values for T , is not necessary optimal after adding the edge of cycle C and path P back.

As opposed to the OLA of the underlying tree of H_2 , given an OLA φ for an arbitrary RBT T' , and an arbitrary subtree T_i rooted at some spinal vertex v_i , all the branches of T_i connected to v_i have the same number of vertices and are also recursively balanced. Hence for every Halin graph $H' = T' \uplus C'$ based on T' , the layout φ can be modified by changing the order of the branches of the subtrees where the value of linear arrangement for T' stays unchanged (Let's call the modified layout φ'), while the value of linear arrangement for the edges of cycle C' (i.e. $\sum_{\{v,u\} \in E(C')} \lambda(\{v,u\}, \varphi')$) is equal to $2 \times (n-1)$ ⁹. Consequently the value of OLA for H' is equal to $LA(\varphi, T) + 2 \times (n-1)$. □

Corollary 4.11. *Let T be the underlying tree for some Halin graph $H = T \uplus C$ and let φ^\circledast be an OLA for T where $P = (w_1, \dots, w_l)$ is respectively the spinal path and based on φ^\circledast , and $\{T_1, \dots, T_l\}$ is the set of subtrees remaining after removing all edges of C and P .*

If for some OLA φ^\circledast of T , T_i rooted at w_i is a recursively balance tree for $i = 1, \dots, l$, then there exists an OLA φ^ for H where $LA(\varphi^*, H) = LA(\varphi^\circledast, T) + 2 \times (n-1)$.*

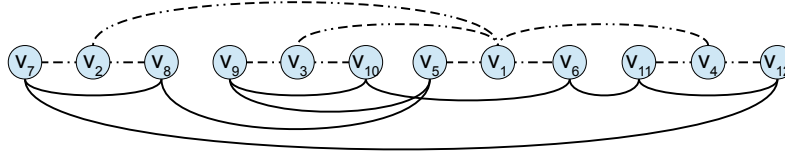
Another class of Halin graphs which their underlying trees satisfy the sufficient property of corollary 4.11, are the Halin graphs based on *caterpillar trees*. This class of Halin graphs is studied in [3] and presented result on value of their OLA, testifies the corollary 4.11.

5 Conclusion and Future Work

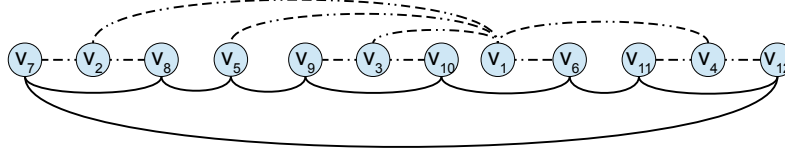
As one of the simplest classes of non-outerplanar graphs, in this work we studied some properties of OLA of Halin graphs and we presented a lower bound for the value of OLA for Halin graphs. We also introduced some

⁸Every subtree T_i has exactly $\frac{n-1}{k}$ vertices

⁹So there is no redundant crossing exists based on φ' .



(a) Layout φ_1 is an OLA for underlying tree of Halin graph H_2 .



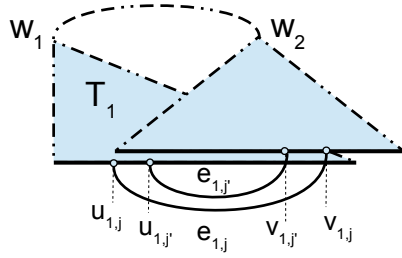
(b) Layout φ_2 is an OLA for Halin graph H_2 .

Figure 4: Two layout φ_1 and φ_2 for Halin graph $H_2 = T \uplus C$ (Figure 2a in section 2), where $LA(\varphi_1, H_2) > LA(\varphi_2, H_2)$ while $LA(\varphi_1, T) < LA(\varphi_2, T)$. More specifically φ_1 is an OLA for T and φ_2 is the OLA for H_2 .

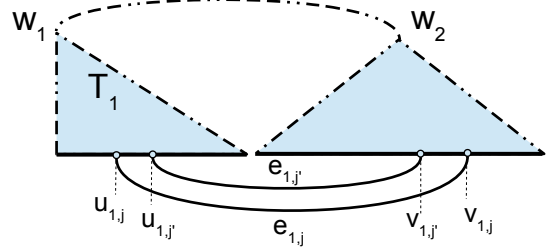
classes of Halin graphs which the OLA can be found in $O(n \log n)$. The problem of OLA of general Halin graphs is still open and we believe a solution for the OLA of general Halin graphs gives good insights into the properties of OLA of the more general class of k -outerplanar graphs.

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(a) OLA layout φ^* which labels some vertex $u \in V(T_1)$ with integers larger than the label of some vertex $v \in V(H) - V(T_1)$. In other word the arrangement of T_1 has overlapping area with arrangement of $H - T_1$.



(b) In the layout φ^\diamond all the vertices of T_1 are labeled before the vertices of $H - T_1$.

Figure 5: The OLA φ^* and the alternative layout φ^\diamond .

A Appendix: Proofs of Supporting Lemmas for Section 3

Notation A.1. Consider the layout φ for a Halin graph $H = T \uplus C$ and it's corresponding spinal path $P = (w_1, w_2, \dots, w_l)$. $\mathcal{T}_i = |\{u \text{ s.t. } u \in V(T_i)\}|$ presents the number of vertices of an spinal subtree T_i . Similarly for branch $B_{i,j}$, $\beta_{i,j} = |\{u \text{ s.t. } u \in V(B_{i,j})\}|$ stands for the number of vertices of branch $B_{i,j}$.

Notation A.2. Given vertex $v \in V$, subset $\mathcal{V} \subseteq V$ and a layout φ , we define:

$$\delta_\varphi(\underline{v}, \bar{v}, \mathcal{V}) = |\{u \text{ s.t. } u \in \mathcal{V} \text{ and } \varphi(\underline{v}) < \varphi(u) < \varphi(\bar{v})\}|$$

In other word, $\delta_\varphi(\underline{v}, \bar{v}, \mathcal{V})$ is the number of vertices in \mathcal{V} , which based on φ are labeled with integers greater than the label of \underline{v} and smaller than the label of \bar{v} . Respectively $\delta_\varphi(-, \bar{v}, \mathcal{V})$ and $\delta_\varphi(\underline{v}, -, \mathcal{V})$ can be interpreted as the number of vertices of \mathcal{V} labeled before \bar{v} and after \underline{v} .

In what follows we present an auxiliary lemma and its proof that will be helpful in simplifying and understanding of the proof of lemma 3.3.

Lemma A.3. Consider the OLA φ^* for the Halin graph $H = T \uplus C$ and the corresponding spinal path $P = (w_1, w_2, \dots, w_l)$. Removing all the edges of P and C results in a set of l subtrees T_1, \dots, T_l respectively rooted at w_1, \dots, w_l . In the layout φ^* , the vertices of T_1 are labeled by contentious integers and before all the vertices of $H - T_1$. Formally speaking:

$$\forall v \in T_1, u \notin T_1 \Rightarrow \varphi^*(v) < \varphi^*(u)$$

Proof. We prove this lemma by showing that the opposing assumption contradicts the optimality of φ^* . In other other word, if $\exists v \in V(T_1), u \notin V(T_1), \varphi^*(u) < \varphi^*(v)$, we suggest an alternative layout φ^\diamond where $LA(\varphi^*, H) > LA(\varphi^\diamond, H)$. Two layouts φ^* and φ^\diamond are respectively shown in Figure 5a and 5b. In layout φ^\diamond , defined as it follows, all the vertices of T_1 are labeled with integers smaller than all the labels of vertices in $V(H) - V(T_1)$ by shifting them to the left while keeping their relative orders unchanged.

$$\forall v \in V, \varphi^\diamond = \begin{cases} \varphi^*(v) - \delta_{\varphi^*}(-, v, V(H) - V(T_1)) & \text{if } v \in V(T_1) \\ \varphi^*(v) + \delta_{\varphi^*}(v, -, V(T_1)) & \text{if } v \notin V(T_1) \end{cases}$$

Going from layout φ^* to φ^\diamond , the equation 1 can be inferred.

$$(1) \quad LA(\varphi^*, H) - LA(\varphi^\diamond, H) = \\ \Delta + \\ \lambda(\{w_1, w_2\}, \varphi^*) - \lambda(\{w_1, w_2\}, \varphi^\diamond) + \\ \lambda(e_{1,j}, \varphi^*) - \lambda(e_{1,j}, \varphi^\diamond) + \\ \lambda(e_{1,j'}, \varphi^*) - \lambda(e_{1,j'}, \varphi^\diamond) +$$

Where Δ is the increase in the value of linear arrangement due to overlapping vertices.¹⁰

Value of Δ : We define Δ_1 and Δ_2 respectively as the number of vertices of $V(T_1)$ and $V(H) - V(T_1)$ in the overlapping area. More specifically:

$$\Delta_1 = |\{v \in V(T_1) \text{ s.t. } \exists u, u' \in V(H) - V(T_1), \varphi^*(u) < \varphi^*(v) < \varphi^*(u')\}| \\ \Delta_2 = |\{v \in V(H) - V(T_1) \text{ s.t. } \exists u, u' \in V(T_1), \varphi^*(u) < \varphi^*(v) < \varphi^*(u')\}|$$

Fact A.4. *As presented in Figure 5, the set of vertices $V(T_1)$ and $V(H) - V(T_1)$ are connected via exactly three outgoing edges $\{w_1, w_2\}$, $e_{1,j}$ and $e_{1,j'}$. Based on the three-connectivity of Halin graphs, any subset $\mathcal{V} \subset V(T_1)$ not incident to the outgoing edges, is connected to the rest of $V(T_1)$ by at least three edge disjoint paths. Also Any subset $\mathcal{V} \subset V(T_1)$ incident to some of outgoing edges $\{w_1, w_2\}$, $e_{1,j}$ and $e_{1,j'}$, is connected to $V(T_1) - \mathcal{V}$ via at least two edge-disjoint paths. the same property hold for any $\mathcal{V} \subset V(H) - V(T_1)$.*

According to fact A.4, any vertex $v \in V(T_1)$ in the overlapping area participates one unit in increasing the expand of at least two edges of $E(H/T_1)$. Similarly any vertex $v \in V(H) - V(T_1)$ in overlapping area, increases the expand of at least two edges from $E(T_1)$. Hence:

$$(2) \quad \Delta \geq 2 \times (\Delta_1 + \Delta_2)$$

Change in the expands of $\{w_1, w_2\}$, $e_{1,j}$ and $e_{1,j'}$: Based on the procedure that φ^\diamond is constructed from φ^* it's easy to validate the following equations.

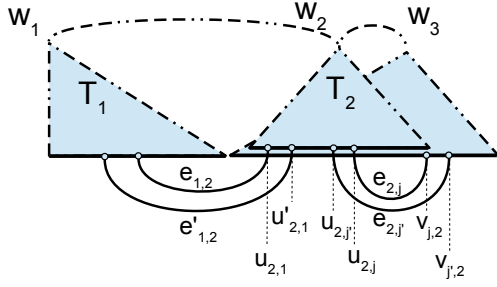
$$(3) \quad (\lambda(\{w_1, w_2\}, \varphi^*) - \lambda(\{w_1, w_2\}, \varphi^\diamond)) = -\delta(w_2, -, V(T_1)) \geq -\Delta_1 \\ (4) \quad (\lambda(e_{1,j}, \varphi^*) - \lambda(e_{1,j}, \varphi^\diamond)) = -(\delta(-, u_{1,j}, V(H) - V(T_1)) + \delta(v_{1,j}, -, V(T_1))) > -(\Delta_2 + \Delta_1) \\ (5) \quad (\lambda(e_{1,j'}, \varphi^*) - \lambda(e_{1,j'}, \varphi^\diamond)) = -(\delta(-, u_{1,j'}, V(H) - V(T_1)) + \delta(v_{1,j'}, -, V(T_1))) > -(\Delta_2 + \Delta_1)$$

Remark A.5. *Let $v \in \{w_2, v_{1,j}, v_{1,j'}\}$ be the vertex with largest label among the three. The rearrangement of v increases the expand of the corresponding edges by $\delta(v, -, V(T_1))$. But notice that based on fact A.4 the set of vertices of $V(H) - V(T_1)$ labeled after v are connected to rest of vertices (vertices on left side according to φ^*) using at least three vertices. Hence each vertex of $V(T_1)$ after v (labeled with integers larger than label of v) add one unit to the expands of at least three edges of H/T_1 , while only the expands of two edges where considered in equation 2. Therefore the value $\delta(v, -, V(T_1))$ in the increase of the expand of the edge incident to v must be ignored in the calculation of $LA(\varphi^*, H) - LA(\varphi^\diamond, H)$.*

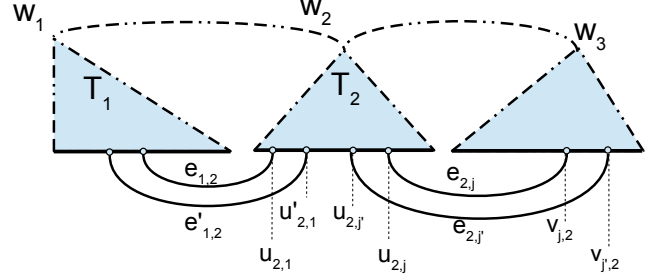
Considering the remark A.8, we finalize the proof by the following contradictory result.

$$(6) \quad LA(\varphi^*, H) - LA(\varphi^\diamond, H) > \\ (7) \quad 2 \times (\Delta_1 + \Delta_2) - \Delta_1 - (\Delta_2 + \Delta_1) - \Delta_2 \\ (8) \quad > 0$$

¹⁰Let $v \in V(T_1)$ be the vertex with largest label and $v' \in T(H) - V(T_1)$ with smallest label according to φ^* . $u \in V(H)$ is in overlapping area if $\varphi^*(v') < \varphi^*(u) < \varphi^*(v)$.



(a) OLA layout φ^* which labels some vertex $u \in V(T_2)$ with integers larger than the label of some vertex $v \in V(\overline{T}_{1,2})$. In other word the arrangement of T_2 has overlapping area with arrangement of $T_3 \uplus \dots \uplus T_1$.



(b) In the layout φ^\diamond all the vertices of T_2 are labeled after the vertices of T_1 and before all the vertices of $V(\overline{T}_{1,2})$.

Figure 6: The OLA φ^* and the alternative layout φ^\diamond . Subtree T_2 is connected to T_1 via at most two edges $e_{1,2}, e'_{1,2} \in E(C)$, (respectively incident to $u_{1,2}, u'_{1,2} \in V(T_2)$) and exactly one edge $\{w_1, w_2\} \in E(T)$. T_2 is also connected to the rest of graph by the same number of edges $e_{2,j}, e_{2,j'} \in E(C)$ and $\{w_2, w_3\}$. End points of edge $e_{2,j}$ are $u_{2,j} \in V(T_2)$ and $v_{2,j} \in V(\overline{T}_{1,2})$ while $u_{2,j'} \in V(T_2)$ and $v_{2,j'} \in V(\overline{T}_{1,2})$ are the two end points of $e_{2,j'}$.

□

Proof A.6. (Proof of lemma 3.3) In lemma A.3, it is shown that, given an OLA φ^* for the Halin graph H , all the vertices of T_1 are labeled with continuous integers and hence are arranged before all other vertices in the graph. Using a similar approach as in lemma A.3, we show that in an OLA φ^* all the vertices in $V(T_2)$ are labeled before all the vertices of $V(T_3) \cup \dots \cup V(T_l)$. Proof is complete as the same approach can be carried out to show that $\forall 2 < i < l$ all the vertices of $V(T_i)$ are labeled with integers smaller than the labels of vertices of $V(T_{i+1}) \cup \dots \cup V(T_l)$.

In the rest of this proof we present $T_3 \uplus \dots \uplus T_l$ by $\overline{T}_{1,2}$. We show that a layout φ^* , where the arrangement of vertices of T_2 overlap with the arrangement of vertices in $V(\overline{T}_{1,2})$ (as shown in Figure 6a) cannot be optimal. In order to do so, based on this allegedly optimal layout φ^* , we define the modified layout φ^\diamond (presented by Figure 6b) and we show that $LA(\varphi^*) > LA(\varphi^\diamond)$.

The subtree T_2 is connected to T_1 through one edge $e_{1,2} \in E(T)$ and one or two edges from $E(C)$. T_2 is also connected to the rest of graph by exactly the same number of edges. As presented in Figure 6, we only consider the case where T_2 is connected to each of subgraphs T_2 and $T_3 \uplus \dots \uplus T_l$ by one edge of $E(T)$ and two edges of $E(C)$. The other case can be analyze in the same way and is omitted.

Layout φ^\diamond is formally defined as it follows.

$$\forall v \in V, \varphi^\diamond = \begin{cases} \varphi^*(v) - \delta_{\varphi^*}(-, v, V(\overline{T}_{1,2})) & \text{if } v \in V(T_2) \\ \varphi^*(v) + \delta_{\varphi^*}(v, -, V(T_2)) & \text{if } v \in V(\overline{T}_{1,2}) \end{cases}$$

Based on this definition it's easy to see that:

$$(9) \quad LA(\varphi^*, H) - LA(\varphi^\diamond, H) = \Delta +$$

$$(\lambda(\{w_1, w_2\}, \varphi^*) - \lambda(\{w_1, w_2\}, \varphi^\diamond)) +$$

$$(\lambda(\{w_2, w_3\}, \varphi^*) - \lambda(\{w_2, w_3\}, \varphi^\diamond)) +$$

$$(\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi^\diamond)) +$$

$$(\lambda(e'_{1,2}, \varphi^*) - \lambda(e'_{1,2}, \varphi^\diamond)) +$$

$$(\lambda(e_{2,j}, \varphi^*) - \lambda(e_{2,j}, \varphi^\diamond)) +$$

$$(\lambda(e_{2,j'}, \varphi^*) - \lambda(e_{2,j'}, \varphi^\diamond)) +$$

As in lemma A.3 the increase in the value of linear arrangement due to overlap is presented by Δ and we define Δ_2 and Δ_3 as:

$$\Delta_2 = |\{v \in V(T_2) \text{ s.t. } \exists u, u' \in V(\overline{T}_{1,2}), \varphi^*(u) < \varphi^*(v) < \varphi^*(u')\}|$$

$$\Delta_3 = |\{v \in V(\overline{T}_{1,2}) \text{ s.t. } \exists u, u' \in V(T_2), \varphi^*(u) < \varphi^*(v) < \varphi^*(u')\}|$$

Hence Δ_2 and Δ_3 respectively correspond to the number of vertices of T_2 and $T_3 \uplus \dots \uplus T_l$ which are in the overlapping area based on φ^* .

Fact A.7. Similar to the fact A.4 and according to the three-connectivity of Halin graphs, any subset $\mathcal{V} \subset V(\overline{T}_{1,2})$ not incident to the outgoing edges $\{w_2, w_3\}$, $e_{2,j}$ and $e_{2,j'}$, is connected to the rest of $V(\overline{T}_{1,2})$ through at least three edge disjoint paths. Also Any subset $\mathcal{V} \subset V(\overline{T}_{1,2})$ incident to some of the outgoing edges, is connected to $V(\overline{T}_{1,2}) - \mathcal{V}$ via at least two edge-disjoint paths. The same property hold for any $\mathcal{V} \subset V(T_1) \cup V(T_2)$.

Value of Δ : Using the fact A.7 one can see that each vertex in the overlapping area, increases the expand of at least two edges by one unit. Hence the following equation can be deduced.

$$(10) \quad \Delta \geq 2 \times (\Delta_2 + \Delta_3)$$

Change in the expands of outgoing edges: As in lemma A.3, the change in the expand of edges linking T_2 to the rest of graph can be derived as:

$$(11) \quad (\lambda(\{w_1, w_2\}, \varphi^*) - \lambda(\{w_1, w_2\}, \varphi^\diamond)) = \delta(-, w_2, V(\overline{T}_{1,2}))$$

$$(12) \quad (\lambda(\{w_2, w_3\}, \varphi^*) - \lambda(\{w_2, w_3\}, \varphi^\diamond)) = -(\delta(-, w_2, V(\overline{T}_{1,2})) + \delta(w_3, -, V(T_2)))$$

$$(13) \quad (\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi^\diamond)) = \delta(-, u_{1,2}, V(\overline{T}_{1,2}))$$

$$(14) \quad (\lambda(e'_{1,2}, \varphi^*) - \lambda(e'_{1,2}, \varphi^\diamond)) = \delta(-, u'_{1,2}, V(\overline{T}_{1,2}))$$

$$(15) \quad (\lambda(e_{2,j}, \varphi^*) - \lambda(e_{2,j}, \varphi^\diamond)) \geq -(\delta(-, u_{2,j}, V(\overline{T}_{1,2})) + \delta(v_{j,2}, -, V(T_2)))$$

$$(16) \quad (\lambda(e_{2,j'}, \varphi^*) - \lambda(e_{2,j'}, \varphi^\diamond)) \geq -(\delta(-, u_{2,j'}, V(\overline{T}_{1,2})) + \delta(v_{j',2}, -, V(T_2)))$$

Remark A.8. Let $v \in \{w_3, v_{j,2}, v_{j',2}\}$ be the vertex with largest label among the three. According to the fact A.7 and using the same reasoning as in remark A.8, each vertex in $V(T_2)$ labeled after v takes part in the increase of expand of at least three edges of $V(\overline{T}_{1,2})$, while only two were considered in the calculation of Δ in equation 10. Hence the value $\delta(v, -, V(T_2))$, considered for the change in expand of the edge incident to v , should be added back to the calculation of $LA(\varphi^*, H) - LA(\varphi^\diamond, H)$. Without loss of generality in the rest of the proof we assume $v = v_{j',2}$.

Accordingly equation 9 can be simplified as it follows.

$$\begin{aligned}
(17) \quad LA(\varphi^*, H) - LA(\varphi^\diamond, H) &\geq \\
&2 \times (\Delta_2 + \Delta_3) \\
&\quad - \delta(w_3, -, V(T_2)) \\
&\quad - \delta(u_{1,2}, u_{2,j}, V(\overline{T}_{1,2})) \\
&\quad - \delta(u'_{1,2}, u_{2,j'}, V(\overline{T}_{1,2})) \\
(18) \quad &\quad - \delta(v_{j,2}, -, V(T_{1,2}))
\end{aligned}$$

Remark A.9. Depending on the order of labels of $u_{1,2}$ and $u_{2,j}$, $\delta(-, u_{1,2}, V(\overline{T}_{1,2})) - \delta(-, u_{2,j}, V(\overline{T}_{1,2}))$ is either equal to $\delta(u_{2,j}, u_{1,2}, V(\overline{T}_{1,2}))$ or $-\delta(u_{1,2}, u_{2,j}, V(\overline{T}_{1,2}))$. The same way we can reason about $\delta(-, u'_{1,2}, V(\overline{T}_{1,2})) - \delta(-, u_{2,j'}, V(\overline{T}_{1,2}))$ as following:

$$(19) \quad \delta(-, u_{1,2}, V(\overline{T}_{1,2})) - \delta(-, u_{2,j}, V(\overline{T}_{1,2})) \geq -\delta(u_{1,2}, u_{2,j}, V(\overline{T}_{1,2})) \geq -\Delta_3$$

$$(20) \quad \delta(-, u'_{1,2}, V(\overline{T}_{1,2})) - \delta(-, u_{2,j'}, V(\overline{T}_{1,2})) \geq -\delta(u'_{1,2}, u_{2,j'}, V(\overline{T}_{1,2})) \geq -\Delta_3$$

Also it is easy to see that:

$$(21) \quad -\delta(w_3, -, V(T_2)) \geq \Delta_2$$

$$(22) \quad -\delta(v_{2,j}, -, V(T_{1,2})) \geq \Delta_2$$

Remark A.10. The equality in equation 15 holds only if $u_{2,j}$ is labeled before $v_{j,2}$ (namely $\varphi^*(u_{2,j}) < \varphi^*(v_{2,j})$)¹¹. But the equalities in equations 19 and 22 can hold simultaneously only if $v_{2,j}$ is labeled before all the vertices $V(T_2)$ in the overlapping area and $u_{2,j}$ is labeled after all the vertices of $\overline{T}_{1,2}$ which are in the overlapping area. In other word, both the equalities in equations 19 and 22 hold only if $\varphi^*(u_{2,j}) > \varphi^*(v_{2,j})$. Accordingly the equalities in equations 15, 19 and 22 never simultaneously hold.

Consequently the equation 17 can be simplified as following with the contradictory result that completes the proof.

$$\begin{aligned}
(23) \quad LA(\varphi^*, H) - LA(\varphi^\diamond, H) &> \\
&2 \times (\Delta_2 + \Delta_3) \\
&\quad - \Delta_2 \\
&\quad - \Delta_3 \\
&\quad - \Delta_3 \\
(24) \quad &\quad - \Delta_2 = 0
\end{aligned}$$

Proof A.11. (Proof of lemma 3.5) Each spinal branch $B_{i,j}$ is connected to the rest of the graph using three edges. $e_{j,w} = \{v_j, w_i\} \in E(T)$ connecting it to the spinal vertex w_i and two edge $e_{j,j'} \in E(C)$ and $e_{j,j''} \in E(C)$ connecting $B_{i,j}$ to two other branches $B_{i',j'}$ and $B_{i'',j''}$ ¹².

Without loss of generality we assume $j = 1$ and $j' = 2$ and we consider two branches $B_{i,1}$ and $B_{i,2}$ are anchored at v_1 and v_2 , and we only show the following for the case where φ^* is of type $(\dots, w_i, \dots, V(B_{i,1}) \cup V(B_{i,2}), \dots)$.

$$\begin{aligned}
\forall v \in V(B_{i,1}) \cup V(B_{i,2}); \varphi^*(w_i) < \varphi^*(v) &\Rightarrow \\
(\forall v \in V(B_{i,1}), u \in V(B_{i,2}) : \varphi^*(v) < \varphi^*(u)) &\vee \\
(\forall v \in V(B_{i,1}), u \in V(B_{i,2}) : \varphi^*(v) > \varphi^*(u)) &
\end{aligned}$$

Assume two vertices $\underline{v}, \overline{v} \in V(B_{i,1}) \cup V(B_{i,2})$ such that $\forall u \in V(B_{i,1}) \cup V(B_{i,2}), \varphi^*(\underline{v}) \leq u \leq \varphi^*(\overline{v})$.

¹¹Similarly the equality in equation 20 hold only if $\varphi^*(u'_{2,j}) < \varphi^*(v_{2,j'})$.

¹²Note that based on the structure of Halin graphs, $B_{i',j'}$ and $B_{i'',j''}$ may belong to the same subtree T_i as $B_{i,j}$, but both can not be a part of the same subtree T'_i different from T_i

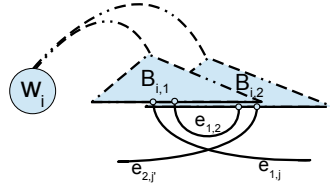
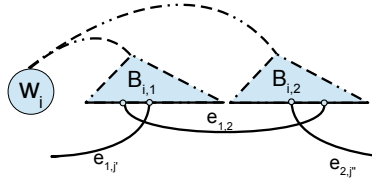
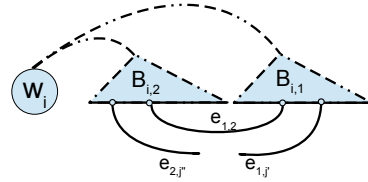


Figure 7: An branch overlapping layout where the two branches $B_{i,1}$ and $B_{i,2}$ are connected.



(a) The alternative non-overlapping layout φ_L^\diamond .



(b) The alternative non-overlapping layout φ_R^\diamond .

Figure 8: In the layout φ_L^\diamond all the vertices of $B_{i,1}$ are labeled on the left side of vertices of $B_{i,2}$ and on the right side in the layout φ_R^\diamond .

Case 1: $\underline{v} \in B_{i,1}$ and $\bar{v} \in B_{i,2}$. There are two possible sub-cases: 1.1) there is no edge connecting $B_{i,1}$ and $B_{i,2}$. 1.2) $B_{i,1}$ is connected to $B_{i,2}$ using exactly one edge $e_{1,2} \in E(C)$ (Figure 7). We present the proof for the later case. The analysis of proof of former case is similar and omitted. We show that the assumption of lemma being false, in other word if two branches of $B_{i,1}$ and $B_{i,1}$ overlap, contradicts the optimality assumption of φ^* . Accordingly for an overlapping OLA φ^* , we present an alternative layout φ_L^\diamond and finish the proof by showing the contradictory result $LA(\varphi^*) > LA(\varphi_L^\diamond)$. In the alternative layout φ_L^\diamond all the vertices of $B_{i,1}$ are labeled before all the vertices of $B_{i,2}$ while the relative order of labels of other vertices are preserved the same. Formally φ_L^\diamond is defined as follows.

$$\forall v \in V, \varphi_L^\diamond = \begin{cases} \varphi^*(v) & \text{if } v \notin V(B_{i,1}) \cup V(B_{i,2}) \\ \varphi^*(v) - \delta_{\varphi^*}(-, v, V(B_{i,2})) & \text{if } v \in V(B_{i,1}) \\ \varphi^*(v) + \delta_{\varphi^*}(v, -, V(B_{i,1})) & \text{if } v \in V(B_{i,2}) \end{cases}$$

Based on the definition of φ_L^\diamond and from Figure 8a the following holds:

$$(25) \quad LA(\varphi^*) - LA(\varphi_L^\diamond) = \Delta + (\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi_L^\diamond)) + (\lambda(e_{1,j'}, \varphi^*) - \lambda(e_{1,j'}, \varphi_L^\diamond)) + (\lambda(e_{2,j}, \varphi^*) - \lambda(e_{2,j}, \varphi_L^\diamond)) + (\lambda(e_{1,w}, \varphi^*) - \lambda(e_{1,w}, \varphi_L^\diamond)) + (\lambda(e_{2,w}, \varphi^*) - \lambda(e_{2,w}, \varphi_L^\diamond))$$

As before, Δ represents the increase in the value of linear arrangement due to overlap.

Let Δ_1 and Δ_2 respectively be the number of vertices of $B_{i,1}$ and $B_{i,2}$ in the overlapping area. Formally speaking:

$$\begin{aligned}\Delta_1 &= |\{v \in B_{i,1} | \exists u, u' \in B_{i,2}, \varphi^*(u) < \varphi^*(v) < \varphi^*(u')\}| \\ \Delta_2 &= |\{v \in B_{i,2} | \exists u, u' \in B_{i,1}, \varphi^*(u) < \varphi^*(v) < \varphi^*(u')\}| \end{aligned}$$

Fact A.12. Every spinal branch $B_{i,j}$ of a Halin graph $H = T \uplus C$, anchored at $v_{i,j}$, is connected to the rest of H via three outgoing edges, $\{w_i, v_{i,j}\} \in E(T)$, $e_{j,j'} \in E(C)$ and $e_{j,j''} \in E(C)$. The two edges $e_{j,j'}$ and $e_{j,j''}$, respectively incident to v_R and v_L , connect $B_{i,j}$ to two other spinal branches $B_{i',j'}$ and $B_{i'',j''}$ ¹³. Every vertex of a non-empty sub-branch $\mathcal{B} \subset B_{i,j} - \{v_{i,j}, v_{j,R}, v_{j,L}\}$ is connected to $B_{i,j}/\mathcal{B}$ via at least three edge disjoint paths. Consequently \mathcal{B} is connected to $B_{i,j}/\mathcal{B}$ by at least three edges.

In general every vertex of a non-empty set $\mathcal{B} \subset B_{i,j}$ (which may contains some of the vertices of $\{v_{i,j}, v_{j,R}, v_{j,L}\}$) is connected to $B_{i,j}/\mathcal{B}$ by at least two edges-disjoint paths.

Value of Δ : As a consequence of fact A.12, each vertex of a branch in the overlapping area contribute one unit to the increase in the expand of at least two edges from the other branch. Hence it is the case that:

$$(26) \quad \Delta \geq 2 \times (\Delta_1 + \Delta_2)$$

Change in the expands of $e_{1,2}$, $e_{1,j'}$, $e_{2,j''}$ and $e_{2,w}$: The increase in the expand of each of these edges is equivalent to how much the two end points drift apart in construction of φ_L^\diamond . Accordingly the following equations are easy to verify:

$$(27) \quad (\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi_L^\diamond)) \geq -(\Delta_1 + \Delta_2)$$

$$(28) \quad (\lambda(e_{1,j'}, \varphi^*) - \lambda(e_{1,j'}, \varphi_L^\diamond)) \geq -\Delta_2$$

$$(29) \quad (\lambda(e_{2,j''}, \varphi^*) - \lambda(e_{2,j''}, \varphi_L^\diamond)) \geq -\Delta_1$$

$$(30) \quad (\lambda(e_{2,w}, \varphi^*) - \lambda(e_{2,w}, \varphi_L^\diamond)) \geq -\Delta_1$$

Remark A.13. Let $u \in V(B_{i,2})$ be the vertex incident to one of the edges $e_{1,2}$, $e_{2,j''}$ and $e_{2,w}$ with the largest label based on φ^* . Using fact A.12, the set of vertices of $\vec{V}_u \subset V(B_{i,2})$ labeled after u is three-connected to set of vertices labeled before u . Accordingly each vertex of $B_{i,1}$ labeled with an integer larger than $\varphi^*(u)$ contributes one unit to the increase in expand of at least three edges of $B_{i,2}$. Therefore this value cancels out the expand of the edge incident to u .

Remark A.14. It's easy to see that at most only one of the equalities 27 to 30 can hold.

The following contradictory result, from putting the equations 26 to 29 and remarks A.13 and A.14 together, concludes our proof in this case.

$$LA(\varphi^*) - LA(\varphi_L^\diamond) > 2 \times (\Delta_1 + \Delta_2) + -(\Delta_1 + \Delta_2) - \Delta_1 - \Delta_2 = 0$$

Case 2: $\underline{v} \in B_{i,2}$ and $\bar{v} \in B_{i,1}$. This case is symmetric to the previous case and in the alternative layout φ_L^\diamond all the vertices of $B_{i,1}$ are labeled after those of $B_{i,2}$. Hence the proof is similar and is omitted.

¹³ $e_{j,j'}$ and $e_{j,j''}$ are the right and left outgoing edges of $B_{i,j}$

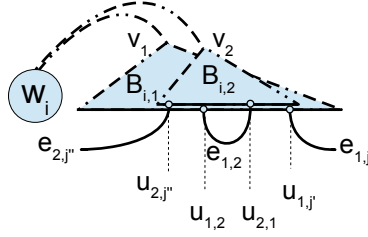


Figure 9: General presentation of an OLA φ^* where $\underline{v}, \bar{v} \in B_{i,1}$. $u_{1,2} \in V(B_{i,1})$ and $u_{2,1} \in V(B_{i,2})$ are the two end points of edge $e_{1,2}$ while $u_{1,j'} \in V(B_{i,1})$ and $u_{2,j''} \in V(B_{i,2})$ are respectively one of the end points of two edge $e_{1,j'}$ and $e_{2,j''}$.

Case 3: $\underline{v}, \bar{v} \in B_{i,1}$. Layout φ^* for this case is presented in Figure 9 where all the vertices of $B_{i,2}$ are enclosed by $B_{i,1}$. In contrast to layout φ^* , we present layout φ_R^\diamond where all vertices of $B_{i,1}$ are labeled on the right side of those of $B_{i,2}$ as formally defined in following.

$$\forall v \in V, \varphi_R^\diamond = \begin{cases} \varphi^*(v) & \text{if } v \notin V(B_{i,1}) \cup V(B_{i,2}) \\ \varphi^*(v) + \delta_{\varphi^*}(v, \bar{v}, V(B_{i,2})) & \text{if } v \in V(B_{i,1}) \\ \varphi^*(v) - \delta_{\varphi^*}(w_i, v, V(B_{i,1})) & \text{if } v \in V(B_{i,2}) \end{cases}$$

We finish the proof by showing that either $LA(\varphi^*) > \varphi_R^\diamond$ or $LA(\varphi^*) > \varphi_L^\diamond$. According to the definitions of φ_R^\diamond and φ_L^\diamond one can infer the equation 31.

$$(31) \quad LA(\varphi^*) - LA(\varphi^\diamond) = \Delta + (\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi^\diamond)) + (\lambda(e_{1,j'}, \varphi^*) - \lambda(e_{1,j'}, \varphi^\diamond)) + (\lambda(e_{2,j}, \varphi^*) - \lambda(e_{2,j}, \varphi^\diamond)) + (\lambda(e_{1,w}, \varphi^*) - \lambda(e_{1,w}, \varphi^\diamond)) + (\lambda(e_{2,w}, \varphi^*) - \lambda(e_{2,w}, \varphi^\diamond))$$

Where the wildcard "-" can be replaced by L or R, and as before Δ represents the increase in the value of linear arrangement due to overlap.

Value of Δ : Considering the same definition for Δ_1 and Δ_2 , then $\Delta \geq 2 \times (\Delta_1 + \Delta_2)$. Since $\Delta_2 = \beta_{i,2}$ then:

$$(32) \quad \Delta \geq 2 \times (\Delta_1 + \beta_{i,2})$$

Change in the expands of $e_{1,2}$, $e_{1,j'}$, $e_{2,j''}$, $e_{1,w}$ and $e_{2,w}$: In the calculation of the change in expand an edge, in should be considered if the two end points are drifting apart or getting closer. Hence, noting the

opposing definitions of φ_L^\diamond and φ_R^\diamond , following equations hold.

$$\begin{aligned}
(33) \quad & (\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi_L^\diamond)) \geq -(\beta_2 + \delta_{\varphi^*}(u_{2,1}, -, V(B_{i,1}))) \\
(34) \quad & (\lambda(e_{1,j'}, \varphi^*) - \lambda(e_{1,j'}, \varphi_L^\diamond)) = -\alpha(e_{1,j'}) \times \delta_{\varphi^*}(-, u_{1,j'}, V(B_{i,2})) \\
(35) \quad & (\lambda(e_{2,j''}, \varphi^*) - \lambda(e_{2,j''}, \varphi_L^\diamond)) = -\alpha(e_{2,j''}) \times \delta_{\varphi^*}(u_{2,j''}, -, V(B_{i,1})) \\
(36) \quad & (\lambda(e_{1,w}, \varphi^*) - \lambda(e_{1,w}, \varphi_L^\diamond)) = \delta_{\varphi^*}(-, v_1, V(B_{i,2})) \\
(37) \quad & (\lambda(e_{2,w}, \varphi^*) - \lambda(e_{2,w}, \varphi_L^\diamond)) = -\delta_{\varphi^*}(v_2, -, V(B_{i,1})) \\
\\
(38) \quad & (\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi_R^\diamond)) \geq -(\beta_2 + \delta_{\varphi^*}(-, u_{2,1}, V(B_{i,1}))) \\
(39) \quad & (\lambda(e_{1,j'}, \varphi^*) - \lambda(e_{1,j'}, \varphi_R^\diamond)) = \alpha(e_{1,j'}) \times \delta_{\varphi^*}(u_{1,j'}, -, V(B_{i,2})) \\
(40) \quad & (\lambda(e_{2,j''}, \varphi^*) - \lambda(e_{2,j''}, \varphi_R^\diamond)) = \alpha(e_{2,j''}) \times \delta_{\varphi^*}(-, u_{2,j''}, V(B_{i,1})) \\
(41) \quad & (\lambda(e_{1,w}, \varphi^*) - \lambda(e_{1,w}, \varphi_R^\diamond)) = -\delta_{\varphi^*}(v_1, -, V(B_{i,2})) \geq -\beta_{i,2} \\
(42) \quad & (\lambda(e_{2,w}, \varphi^*) - \lambda(e_{2,w}, \varphi_R^\diamond)) = \delta_{\varphi^*}(-, v_2, V(B_{i,1}))
\end{aligned}$$

The coefficient $\alpha(e)$ is 1 if the edge e is stretching, -1 if it's expand is decreasing and 0 otherwise. For instance if the expand of edge $e_{1,j'}$ increases based on φ_L^\diamond , it obviously will decrease based on φ_R^\diamond . We break the rest of the proof to different sub-cases according to the signs of $\alpha(e_{1,j'})$ and $\alpha(e_{2,j''})$.

Case 3.1: $\alpha(e_{1,j'}) = 1$ and $\alpha(e_{2,j''}) = 1$. Therefore both edges $e_{1,j'}$ and $e_{2,j''}$ shrink based on φ_R^\diamond . Putting equations 32, 38, 40 and 41 together, we conclude:

$$\begin{aligned}
LA(\varphi^*) - LA(\varphi_R^\diamond) & \geq \\
& 2 \times (\Delta_1 + \beta_{i,2}) \\
& - \beta_2 - \delta_{\varphi^*}(-, u_{2,1}, V(B_{i,1})) \\
& + \delta_{\varphi^*}(-, u_{2,j''}, V(B_{i,1})) \\
& - \beta_{i,2} \geq 2 \times \Delta_1 + (\delta_{\varphi^*}(-, u_{2,j''}, V(B_{i,1})) - \delta_{\varphi^*}(-, u_{2,1}, V(B_{i,1}))) \geq \Delta_1
\end{aligned}$$

Notice that $(\delta_{\varphi^*}(-, u_{2,j''}, V(B_{i,1})) - \delta_{\varphi^*}(-, u_{2,1}, V(B_{i,1}))) = \delta_{\varphi^*}(u_{2,1}, u_{2,1}, V(B_{i,1})) \geq -\Delta_1$. Also if $\Delta_1 = 0$ ¹⁴, then the equality 38 cannot hold¹⁵. Accordingly it is always the case that $LA(\varphi^*) - LA(\varphi_R^\diamond) > 0$.

Case 3.2: $\alpha(e_{1,j'}) = -1$ and $\alpha(e_{2,j''}) = -1$. Thus the expands of both edges $e_{1,j'}$ and $e_{2,j''}$ decrease going from φ^* to φ_L^\diamond . Substituting the results of equations 32 to 37 in 31 gives us:

$$\begin{aligned}
LA(\varphi^*) - LA(\varphi_L^\diamond) & \geq \\
& 2 \times (\Delta_1 + \beta_{i,2}) \\
& - \beta_2 - \delta_{\varphi^*}(u_{2,1}, -, V(B_{i,1})) \\
& + \delta_{\varphi^*}(u_{1,j'}, -, V(B_{i,2})) \\
& + \delta_{\varphi^*}(u_{2,j''}, -, V(B_{i,1})) \\
& + \delta_{\varphi^*}(-, v_1, V(B_{i,2})) \\
& - \delta_{\varphi^*}(v_2, -, V(B_{i,1}))
\end{aligned}$$

¹⁴Layout φ^* labels all the vertices of $B_{i,2}$ with continuous integers.

¹⁵Due to the fact that $u_{1,2}$ is labeled after all the vertices of $B_{i,2}$, hence:

$$\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi_R^\diamond) \geq -(\beta_2 + \delta_{\varphi^*}(-, u_{2,1}, V(B_{i,1})) - 1$$

Considering the worst case scenario when $\Delta_1 = 0$, $\delta_{\varphi^*}(-, v_1, V(B_{i,2})) = 0$ and $\delta_{\varphi^*}(u_{1,j'}, -, V(B_{i,2})) = 0$ ¹⁶, results in:

$$\begin{aligned} LA(\varphi^*) - LA(\varphi_L^\diamond) &\geq \\ &- (\delta_{\varphi^*}(u_{1,2}, -, V(B_{i,1})) + \delta_{\varphi^*}(-, u_{2,1}, V(B_{i,2}))) + 2 \times \beta_{i,2} \geq \\ &- \delta_{\varphi^*}(u_{1,2}, -, V(B_{i,1})) + \beta_{i,2} \end{aligned}$$

Therefore $LA(\varphi^*) \leq LA(\varphi_L^\diamond)$ only if $\delta_{\varphi^*}(u_{2,1}, -, V(B_{i,1})) \leq \beta_{i,2}$. In this case, since $u_{1,2}$ has degree three with two outgoing edges from $E(C)$, and based on φ_L^\diamond , there is a path via edges of $E(C)$ going to the right most vertex and coming back. Due to this redundancy we can rearrange the vertices of $B_{i,1}$ in φ_L^\diamond , without increasing the value of linear arrangement, so that $u_{1,2}$ has the largest label among vertices $B_{i,1}$ (is the right most vertex of $B_{i,1}$). In this new layout φ_{L2}^\diamond the length of edge $e_{1,2}$ will decrease by $\delta_{\varphi^*}(u_{1,2}, -, V(B_{i,1}))$. Finally we have $LA(\varphi^*) - LA(\varphi_{L2}^\diamond) \geq \beta_{i,2}$, which contradicts the optimality of layout φ^* .

Case 3.3: $\alpha(e_{1,j'}) = -1$ and $\alpha(e_{2,j''}) = 1$. In this case we suggest $LA(\varphi_R^\diamond)$ as an alternative for $LA(\varphi^*)$. Using the same approach and having the arithmetic details omitted, it can be verified that the following holds.

$$\begin{aligned} LA(\varphi^*) - LA(\varphi_R^\diamond) &\geq \\ &- (\delta_{\varphi^*}(-, u_{1,2}, V(B_{i,1})) + \delta_{\varphi^*}(u_{2,1}, -, V(B_{i,2}))) + 2 \times \beta_{i,2} \geq \\ &- \delta_{\varphi^*}(-, u_{1,2}, V(B_{i,1})) + \beta_{i,2} \end{aligned}$$

Using the same reasoning, $LA(\varphi_R^\diamond)$ can be partially modified to have $u_{1,2}$ as the left most vertex of $B_{i,1}$ and reduce the length $e_{1,2}$ by $\delta_{\varphi^*}(-, u_{1,2}, V(B_{i,1}))$ and accordingly to have:

$$LA(\varphi^*) - LA(\varphi_R^\diamond) \geq \beta_{i,2} > 0$$

Case 3.4: $\alpha(e_{1,j'}) = -1$ and $\alpha(e_{2,j''}) = 1$. This case is symmetric to the case 3.3 and similarly it can be shown that $LA(\varphi^*) - LA(\varphi_L^\diamond) \geq \beta_{i,2} > 0$.

Proof A.15. (Proof of theorem 3.6) Consider the case where for a given OLA φ^* and extreme vertices v and u , $d_T(v) \neq 1 \vee d_T(u) \neq 1$ or $d_T(v) \neq 1 \wedge d_T(u) \neq 1$.

With no loss of generality we only present the case where $d_T(v) = k \geq 3$ and symmetrically it can be shown for $d_T(u) \geq 3$ as well. Hence there are $k - 1$ spinal branches $B_{1,1}, \dots, B_{1,k-1}$ connected to v . Based on lemma 3.3 and 3.5, $B_{1,1}, \dots, B_{1,k-1}$ are separately labeled on the right side of v . Each branch $B_{1,i}$ is anchored at vertex v_i (is connected to v via edge $\{v_i, v\}$). The set $\{B_{1,1}, \dots, B_{1,k-1}\}$ is connected to the rest of the graph by exactly to edges $e_{1,j} \in E(C)$ and $e_{1,j'} \in E(C)$. Figure 10 generally presents layout φ^* . Note that the two edges $e_{1,j}$ and $e_{1,j'}$ can not be initiated from the same branch. Assume $e_{1,j}$ and $e_{1,j'}$ are respectively connected to $B_{1,j}$ and $B_{1,j'}$ and vertices of $B_{1,j}$ are labeled with integers smaller than the labels of vertices of $B_{1,j}$. Also notice that every two branches $B_{1,i}$ and $B_{1,i'}$ are connected by at most one edge. As apposed to layout φ^* we present the following layout φ^\diamond .

$$\forall u \in V, \varphi^\diamond = \begin{cases} \varphi^*(u) + \sum_{j < i < k} \beta_{1,i} & \text{if } u \in V(B_{1,j}) \\ \mathcal{T}_1 - \varphi^*(u) + 1 & \text{if } u \in V(B_{1,j+1}) \cup \dots \cup V(B_{1,k-1}) \\ \mathcal{T}_1 - \varphi^*(u) - \beta_{1,j} + 1 & \text{if } u \in V(B_{1,1}) \cup \dots \cup V(B_{1,j-1}) \cup \{v\} \\ \varphi^*(u) & \text{otherwise} \end{cases}$$

Where $\beta_{i,j}$ and \mathcal{T}_i are respectively the number of vertices of branch $B_{i,j}$ and subtree T_i ¹⁷.

Informally speaking, layout φ^\diamond is constructed by mirroring the labels of vertices of all the branches about v , except for the vertices of branch $B_{1,j}$. Figure 11 depicts the layout φ^\diamond . According to the construction of φ^\diamond

¹⁶Namely the vertices of branch $B_{i,2}$ are labeled with a set of contentious integers and v_1 and $u_{1,j}$ are labeled before vertices of $B_{i,2}$ so that expands of $e_{1,w}$ and $e_{1,j'}$ stay unchanged.

¹⁷Hence \mathcal{T}_1 is equivalent to the largest possible label for the vertices of T_i

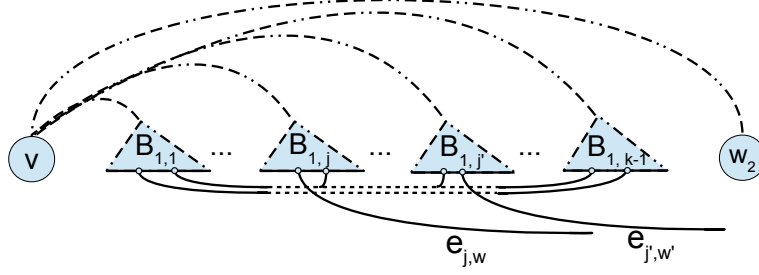


Figure 10: General presentation of an OLA φ^* for $H = T \uplus C$, where $\varphi^*(v)$ while v is not a leaf in tree T .

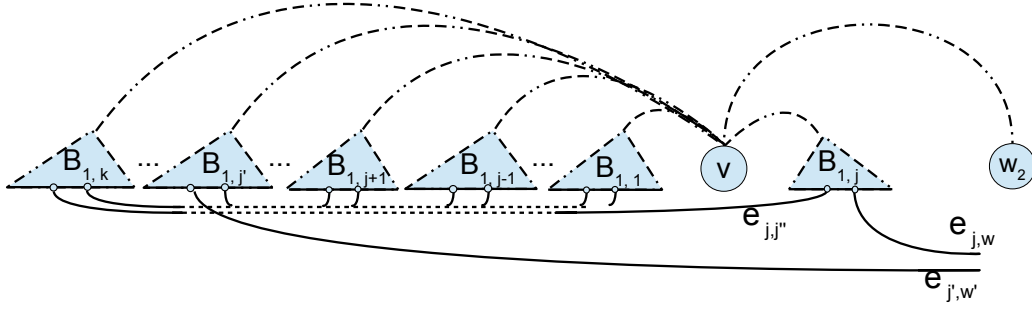


Figure 11: General presentation of an OLA φ^\diamond based on φ^* , where the label of the vertices of branches are mirrored about vertex v except those of branch $B_{1,j}$.

equation 43 holds. Note that the relative inter-orders of vertices of $B_{1,i}$ for $1 \leq i < k$ (accordingly the size of expands of internal edges) stay unchanged in φ^\diamond .

$$(43) \quad LA(\varphi^*) - LA(\varphi^\diamond) \geq$$

$$\begin{aligned} & (\lambda(e_{j,w}, \varphi^*) - \lambda(e_{j,w}, \varphi^\diamond)) + \\ & (\lambda(e_{j',w'}, \varphi^*) - \lambda(e_{j',w'}, \varphi^\diamond)) + \\ & (\lambda(e_{j,j''}, \varphi^*) - \lambda(e_{j,j''}, \varphi^\diamond)) + \\ & (\lambda(\{v, w_2\}, \varphi^*) - \lambda(\{v, w_2\}, \varphi^\diamond)) + \\ & (\lambda(\{v, v_j\}, \varphi^*) - \lambda(\{v, v_j\}, \varphi^\diamond)) + \\ & \sum_{j < i < k} (\lambda(\{v, v_i\}, \varphi^*) - \lambda(\{v, v_i\}, \varphi^\diamond)) \end{aligned}$$

Each term of equation 43 refers to the change in the expands of those edges that their expand may change in the process of constructing φ^\diamond from φ^* .

It's not hard to verify that the following equations hold.

$$(44) \quad (\lambda(e_{j,w}, \varphi^*) - \lambda(e_{j,w}, \varphi^\diamond)) = \sum_{j < i < k} \beta_{1,i}$$

$$(45) \quad (\lambda(e_{j',w'}, \varphi^*) - \lambda(e_{j',w'}, \varphi^\diamond)) \geq - \sum_{1 \leq i < k} \beta_{1,i}$$

$$(46) \quad (\lambda(e_{j,j''}, \varphi^*) - \lambda(e_{j,j''}, \varphi^\diamond)) \geq - \sum_{1 \leq i \leq j} \beta_{1,i}$$

$$(47) \quad (\lambda(\{v, w_2\}, \varphi^*) - \lambda(\{v, w_2\}, \varphi^\diamond)) = \sum_{1 \leq i < k, i \neq j} \beta_{1,i}$$

$$(48) \quad (\lambda(\{v, v_j\}, \varphi^*) - \lambda(\{v, v_j\}, \varphi^\diamond)) = \sum_{1 \leq i < j} \beta_{1,i}$$

$$(49) \quad \sum_{j < i < k} (\lambda(\{v, v_i\}, \varphi^*) - \lambda(\{v, v_i\}, \varphi^\diamond)) = (k - j - 1) \times \beta_{1,j}$$

Consequently equation 43 can be simplified as $LA(\varphi^*) - LA(\varphi^\diamond) \geq (k - j - 3) \times \beta_{1,j} + \sum_{j < i < k} \beta_{1,i}$. The quantity $k - j - 1$ is the number of branches labeled after $B_{1,j}$ and obviously $(k - j - 1) \geq 1$. Hence for $(k - j - 1) > 1$ or $(k - j - 1) = 1 \wedge \beta_{j+1} > \beta_j$, we have $LA(\varphi^*) - LA(\varphi^\diamond) > 0$, which contradicts the optimality of φ^* . Now we analyze the case where $(k - j - 1) = 1 \wedge \beta_{j+1} \leq \beta_j$.

Case 1: $B_{1,j}$ and $B_{1,j+1}$ are not connected. Referring to the structure of Halin graphs, this case holds only if $j > 1$. Informally speaking, there are some branches $B_{1,1}, \dots, B_{1,j-1}$ which based on φ^* their vertices are labeled after v and before $B_{1,j}$. With respect to this case we construct a new layout φ^\otimes where the labels of vertices of $B_{1,1}, \dots, B_{1,j-1}$ are mirrored about v . Formally φ^\otimes is constructed as:

$$\forall u \in V, \varphi^\otimes(u) = \begin{cases} \mathcal{B} + 2 - \varphi^*(u) & \text{if } u \in V(B_{1,1}) \cup \dots \cup V(B_{1,j-1}) \cup \{v\} \\ \varphi^*(u) & \text{otherwise} \end{cases}$$

Where $\mathcal{B} = \sum_{1 \leq i < j} \beta_{1,i}$ is the number of vertices in set $\{V(B_{1,1}) \cup \dots \cup V(B_{1,j-1})\}$.

Following the same approach as before we can show that $LA(\varphi^*) - LA(\varphi^\otimes) \geq \sum_{1 \leq i < j} \beta_{1,i} > 0$. The details of arithmetic calculations are left to the reader.

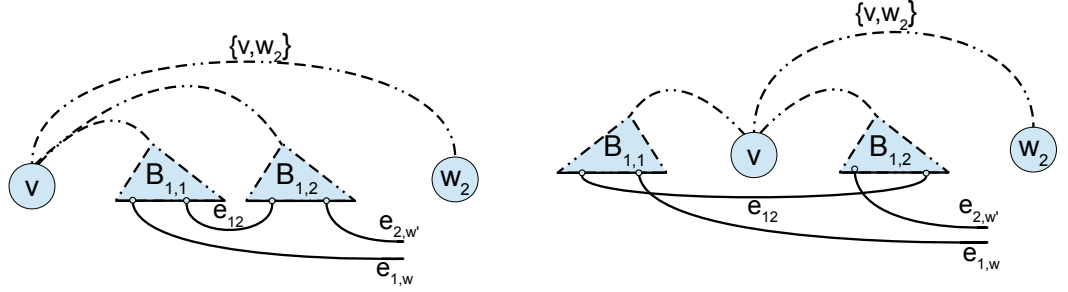
Case 2: $B_{1,j}$ and $B_{1,j+1}$ are connected. Hence $B_{1,j}$ and $B_{1,j+1}$ are the only branches connected to v . Figure 12a shows the layout φ^* corresponding to this case. As schematically shown in figure 12b, we present the the alternative layout φ^\otimes , formally defined as it follows.

$$\forall u \in V, \varphi^\otimes(u) = \begin{cases} \beta_{1,1} + 2 - \varphi^*(u) & \text{if } u \in V(B_{1,1}) \cup \{v\} \\ \varphi^*(u) & \text{otherwise} \end{cases}$$

Equation 50 compares the value of linear arrangements φ^* and φ^\otimes .

$$(50) \quad LA(\varphi^*) - LA(\varphi^\otimes) = \\ (\lambda(\{v, w_2\}, \varphi^*) - \lambda(\{v, w_2\}, \varphi^\otimes)) + \\ (\lambda(\{v, v_2\}, \varphi^*) - \lambda(\{v, v_2\}, \varphi^\otimes)) + \\ (\lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi^\otimes)) + \\ (\lambda(e_{1,w}, \varphi^*) - \lambda(e_{1,w}, \varphi^\otimes))$$

Remember that v_1 and v_2 are the two vertices where $B_{1,1}$ and $B_{1,2}$ are anchored at.



(a) OLA φ^* where $\varphi^*(v) = 1$ and v is connected to exactly two branches $B_{1,1}$ and $B_{1,2}$.

(b) The alternative non-overlapping layout φ^\circledast based on φ^* , where the labels of vertices in $B_{1,1}$ are mirrored about v .

Figure 12: OLA φ^* and the corresponding alternative layout φ^\circledast .

Based on the construction of φ^\circledast from φ^* we have:

$$(51) \quad \lambda(\{v, w_2\}, \varphi^*) - \lambda(\{v, w_2\}, \varphi^\circledast) = \beta_{1,1}$$

$$(52) \quad \lambda(\{v, v_2\}, \varphi^*) - \lambda(\{v, v_2\}, \varphi^\circledast) = \beta_{1,1}$$

$$(53) \quad \lambda(e_{1,2}, \varphi^*) - \lambda(e_{1,2}, \varphi^\circledast) \geq -\beta_{1,1}$$

$$(54) \quad \lambda(e_{1,w}, \varphi^*) - \lambda(e_{1,w}, \varphi^\circledast) \geq -\beta_{1,1}$$

Finally putting equations 50 to 54 together we conclude that $LA(\varphi^*) - LA(\varphi^\circledast) \geq 0$. But the equalities in equations 53 and 54 hold at the same time (and consequently $LA(\varphi^*) - LA(\varphi^\circledast) = 0$), only if the two edges $e_{1,2}$ and $e_{1,w}$ coincide at the left most vertex of $B_{1,1}$. This situation in a Halin graph can only happen when branch $B_{1,1}$ has exactly one vertex. For that reason we conclude that in an OLA for a Halin graph $H = T \uplus C$, a non-leaf vertex v can be a extreme vertex, only if v has exactly two leaves of T as it's children¹⁸.

¹⁸Remember that $\beta_{1,1} \geq \beta_{1,2}$.