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# Convergence rates of numerical methods for solving Laplace's equation

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Thesis

Convergence Rates of Numerical Methods  
for Solving Laplace's Equation

by

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## I Introduction

Let us consider the partial differential equation

$$(1.1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

which is defined at the region R with boundary S.

This equation is also called as Laplace's Equation which is velocity potential and stream function of an ideal incompressible in two dimensions.

In order to get the numerical solution of the equation (1.1), we will discuss the convergence rate of the several such iterative methods. We divide the methods into two groups:

A) group one method

Richardson Method

Liebmann Method

Extrapolated Liebmann Method

Second-order Richardson Method

B) group two method

Jacobi Method

Gauss-Seidel Method

Successive Overrelaxation Method

where group one and group two method are defined at the

rectangular region  $R^h$  and at the square region  $R^s$  respectively.

We now from the proof of the existence of convergence, discuss the rate of convergence of the above methods. One of the important contribution to the rate of convergence is made by the convergence factor  $\alpha$  in the group one and  $\omega$  in the group two. The value  $\alpha$  is depend on the spectrum eigenvalue  $\bar{\lambda}$ , and the value  $\omega$  is depend on the spectral norm of  $\bar{A}$ .

From the comparison of the rate of convergence in those various methods, we will predict the best methods for the numerical solution of the equation (1.1).

## II. Difference Equation

For the discussion of finite difference methods, we let the  $x$   $y$  plane be divided into squares of side  $h$  by drawing the two families of parallel lines:

$$(2.1) \quad \begin{array}{ll} x = fh & f = 0, 1, 2, \dots \\ y = kh & k = 0, 1, 2, \dots \end{array}$$

Here  $h$  is a fixed number, often known as mesh size, and intersection of these two families of lines are called mesh points, and two such points  $(x, y)$  and  $(x_1, y_1)$  are said to be adjacent if

$$(2.2) \quad \left( \frac{x - x_1}{h} \right)^2 + \left( \frac{y - y_1}{h} \right)^2 = 1$$

Let us consider a class of regions, each member of which has a connected interior, and a boundary which may be traced on the net by an unbroken segmental path joining certain adjacent points. We assume this region  $R$  lies entirely in the first quadrant, including axes, and denote  $R_h$  to be interior points of  $R$  and  $S_h$  be boundary points of  $R$ .

In order to get the difference equation which satisfy the equation (1.1), we replace the second partial

derivatives by their corresponding difference equations <sup>4</sup>

$$(2.3) \quad \frac{\partial^2 U}{\partial x^2} \sim \frac{U(x+h, y) - 2U(x, y) + U(x-h, y)}{h^2}$$

$$\frac{\partial^2 U}{\partial y^2} \sim \frac{U(x, y+h) - 2U(x, y) + U(x, y-h)}{h^2}$$

We obtain following difference equation,

$$(2.4) \quad LU = \frac{1}{h^2} U(x+h, y) + U(x-h, y) + U(x, y+h) + U(x, y-h) - 4U(x, y) = 0$$

The solution of

$$(2.5) \quad LU = U(x+h, y) + U(x-h, y) + U(x, y+h) + U(x, y-h) - 4U(x, y) = 0$$

satisfies equation (2.4), when  $0 < h < \infty$ .

The difference formula (2.5)

is the difference expression corresponding to equation (1.1) which involves values of  $U$  corresponding to the five points shown in Figure 1.

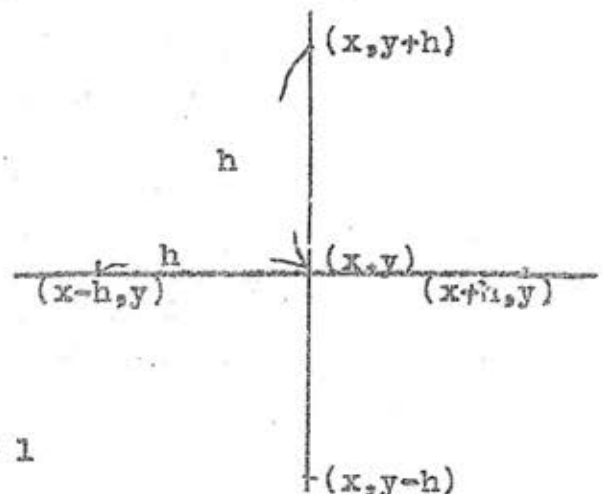


Figure 1

Clearly, other more accurate difference expressions could be found if one considers additional points, but at a cost in simplicity.

### Accuracy

We now consider the truncation error between the difference equation (2.4) and the differential equation (1.1). Assuming that  $U(x,y)$  has continuous partial derivatives of the fourth order throughout the closed region  $R$ , we have by Taylor's theorem

$$(2.6) \quad U(x+h,y) = U(x,y) + h \frac{\partial U}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 U}{\partial x^2} + \frac{h^3}{3!} \frac{\partial^3 U}{\partial x^3} + \frac{h^4}{4!} \frac{\partial^4 U}{\partial x^4}$$

$$U(x-h,y) = U(x,y) - h \frac{\partial U}{\partial x} + \frac{h^2}{2!} \frac{\partial^2 U}{\partial x^2} - \frac{h^3}{3!} \frac{\partial^3 U}{\partial x^3} + \frac{h^4}{4!} \frac{\partial^4 U}{\partial x^4}$$

$$U(x,y+h) = U(x,y) + h \frac{\partial U}{\partial y} + \frac{h^2}{2!} \frac{\partial^2 U}{\partial y^2} + \frac{h^3}{3!} \frac{\partial^3 U}{\partial y^3} + \frac{h^4}{4!} \frac{\partial^4 U}{\partial y^4}$$

$$U(x,y-h) = U(x,y) - h \frac{\partial U}{\partial y} + \frac{h^2}{2!} \frac{\partial^2 U}{\partial y^2} - \frac{h^3}{3!} \frac{\partial^3 U}{\partial y^3} + \frac{h^4}{4!} \frac{\partial^4 U}{\partial y^4}$$

$0 < \xi < x \quad 0 < \eta < y$

By substituting these expressions into difference equation (2.4), we obtain

$$(2.7) \quad LU = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{h^2}{4!} \left[ 2 \left( \frac{\partial^4 U}{\partial x^4} + \frac{\partial^4 U}{\partial y^4} \right) \right]$$

Subtracting differential equation (1.1) from equation (2.7), we find the difference to be the order  $h^2$ .

This implies that the difference equation (2.4) approximate differential equation (1.1) with an accuracy of  $h^2$ . Since region  $R_h$  is regular the truncation error (maximum error) does not exceed in the absolute value

$$(2.8) \quad E_T = \frac{1}{12} M_4 r^2 h^2$$

where  $r$  is the radius of any circles which contains the region and

$$(2.9) \quad M_4 = \text{Max} \left[ \frac{\text{Max}_{(x,y) \text{ R\&S}} \frac{\partial^4 U}{\partial x^4} ; \frac{\text{Max}_{(x,y) \text{ R\&S}} \frac{\partial^4 U}{\partial y^4} \right]$$

$$0 < \xi < x \quad 0 < \eta < y$$

In the next we will reduce the difference equation to the convenient form for each method. We discuss the difference equation in the two different regions, one to be in the rectangular region, and the other to be in the square region.

#### Difference formula for rectangular region

We now consider points which mentioned

$$(2.1) \quad \begin{cases} x = fh \\ x = kh \end{cases}$$

of a rectangular lattice which lies within the region  $R'$

$$R'' = \begin{cases} f = 1, 2, \dots, p \\ k = 1, 2, \dots, q \end{cases}$$

We denote  $U(fh, kh)$  by  $U_{f,k}$ . On the boundary of this region is assigned by the fixed values of  $U_{f,k}$  by  $b_{f,k}$ . Then difference equation (2.5) can be shown as

$$(2.10) \quad LU = U_{f+1,k} + U_{f-1,k} + U_{f,k+1} + U_{f,k-1} - 4U_{f,k} = 0$$

$$f = 1, 2, \dots, p - 1$$

$$k = 1, 2, \dots, q - 1$$

$$= U_{f,k} = b_{f,k}$$

$$f = 0 \quad \text{or} \quad p$$

$$k = 0 \quad \text{or} \quad q$$

This difference formula will be used in the iterative methods of the Group one, Richardson, Liebmann, Extrapolated Liebmann and Second-order Richardson Methods.

#### Difference formula for the square region

Let us order the  $N$  interior mesh points by an arbitrary, but fixed ordering  $(x_1, y_1)$   $(x_2, y_2)$   $\dots$   $(x_N, y_N)$  in the square region  $R'$ . If we write the difference

equation (2.5) for each of these  $N$  interior mesh points, we obtain linear systems of  $N$  linear equations

$$(2.11) \quad \sum_{j=1}^N a_{i,j} u_j + d_i = 0$$

$$i = 1, 2, \dots, N$$

$$j = 1, 2, \dots, N$$

where  $a_{i,j}$  to be the real number coefficient and  $d_i$  to be the linear combination of boundary values are known and  $u_1, u_2, \dots, u_N$  are unknown. These linear systems are used for the Group two iteration methods, Jacobi, Gauss-Seidal Method and, Successive Overrelaxation Method.

In all methods  $U_{f,k}^0$  and  $u_j^0$  is the first approximation to  $U_{f,k}$  and  $u_j$  respectively. Each of the succeeding approximations,  $U^1, U^2, \dots$ , is calculated on the basis of its predecessors by some process which guarantees the convergence of  $U_{f,k}^n$  and  $u_j^n$  to  $U_{f,k}$  and  $u_j$  respectively. We now define the iterative formula for each methods.

### III Iterative Method

#### A) Group one method

We have seen that little is known about the accuracy of solutions of difference equations. Because of this uncertainty it would appear desirable to use an extremely fine mesh size in order to obtain a desired accuracy. On the other hand, since the number of points of  $R_n''$  increases rapidly as the mesh size is decreased, the number of linear algebraic equations which must be solved also increases rapidly. However, we are concerned only with the rate of convergence to the solution of this set of the algebraic equations in each method and not with closeness of that approximation to the solution of the differential equation. In the following we will determine the iterative equation from the difference equation (2.10) where each method are concerned.

#### Richardson method

The most elementary of the familiar iterative procedure is termed as the Richardson method. In this method the correction process applied to each  $U^n$  consists of the addition of a positive multiple of  $LU^n$  (for each point). Thus [1]

$$(3.1) \quad \begin{aligned} U_{f,k}^{n+1} &= U_{f,k}^n + \alpha LU_{f,k}^n \\ &= U_{f,k}^n + \alpha \left[ U_{f+1,k}^n + U_{f-1,k}^n + U_{f,k+1}^n + U_{f,k-1}^n - 4U_{f,k}^n \right] \end{aligned} \quad (\text{in } R_n'')$$

$$(3.2) \quad U_{f,k}^{n+1} = b_{f,k} \quad (\text{in } S_n'')$$

The most familiar form of the Richardson Method is that obtained by setting  $\alpha = \frac{1}{2}$  which is optimum value of  $\alpha$ . Then the equation (3.1) become

$$(3.3) \quad U_{f,k}^{n+1} = \frac{1}{2} \left[ U_{f+1,k}^n + U_{f-1,k}^n + U_{f,k+1}^n + U_{f,k-1}^n \right] \quad (\text{in } R_h^n) \\ = b_{f,k} \quad (\text{in } S_h^n)$$

This form is not only numerically more convenient by the reason of disappearance of  $U_{f,k}^n$ , but also in one sense the most effective form. If we consider  $U_{f,k}^n$  to be the true solution of (2.10) then we have

$$(3.4) \quad U_{f,k} = \frac{1}{4} \left[ U_{f+1,k} + U_{f-1,k} + U_{f,k+1} + U_{f,k-1} \right]$$

In order to get error formula, we will get the difference between approximated value  $U_{f,k}^{n+1}$  of equation (3.3) and true value  $U_{f,k}$  of equation (3.4) will be

$$(3.5) \quad e_{f,k}^{n+1} = U_{f,k}^{n+1} - U_{f,k} \\ = e_{f,k}^n + \frac{1}{4} L e_{f,k}^n \\ = \frac{1}{4} \left[ e_{f+1,k}^n + e_{f-1,k}^n + e_{f,k+1}^n + e_{f,k-1}^n \right]$$

Since the boundary value is fixed, the error at boundary equals to zero. The equation (2.10),  $LU_{f,k} = 0$ , then we also can write (3.5) as

$$(3.6) \quad e^{n+1} = (1 + \frac{1}{2}L)e^n \quad (\text{in } R'_h) \\ = 0 \quad (\text{in } S'_h)$$

Now we may reduce error formula (3.6) with operational form

$$(3.7) \quad e^{n+1} = Ke^n$$

where K is linear operator.

#### Liebmann Method

This method is also one of the successive displacement method, because of the sense of which the new value of  $U_{f,k}^{n+1}$  is used as soon as it is computed. Thus, as applied to the Laplace equation and boundary conditions described above, the Liebmann iteration process may be written,

$$(3.8) \quad U_{f,k}^{n+1} = U_{f,k}^n + \alpha \left[ U_{f+1,k}^n + U_{f-1,k}^{n+1} + U_{f,k+1}^n + U_{f,k-1}^{n+1} - 4U_{f,k}^n \right] \\ = b_{f,k} \quad \begin{array}{l} (\text{in } R'_h) \\ (\text{in } S''_h) \end{array}$$

The case of the using optimum value  $\alpha = \frac{1}{2}$  reduces the equation (3.8) to

$$(3.9) \quad U_{f,k}^{n+1} = \frac{1}{2} \left[ U_{f+1,k}^n + U_{f-1,k}^{n+1} + U_{f,k+1}^n + U_{f,k-1}^{n+1} \right] \quad (\text{in } R''_h) \\ = b_{f,k} \quad (\text{in } S''_h)$$

In this Liebmann iterative formula may be regarded as very mechanical application of the relaxation method.

The error formula of iterative method (3.9) is obtained by the  $U_{f,k}^{n+1}$  of approximated value and  $U$  of true value as following formula

$$(3.10) \quad e_{f,k}^{n+1} = U_{f,k}^{n+1} - U_{f,k} \\ = \frac{1}{4} \left[ e_{f+1,k}^n + e_{f-1,k}^{n+1} + e_{f,k+1}^n + e_{f,k-1}^{n+1} \right] \quad (\text{in } R_h'')$$

and since the boundary value is fixed, error formula at boundary condition will be zero. This error formula (3.10) can be again written as

$$(3.11) \quad e_{f,k}^{n+1} = Ke^n$$

$K$  is denoted as a linear transformation operator. This  $K$  is not simply related to the Laplace's operator  $L$ . Since  $K$  is the eigenvalue of the (3.11), then its eigenfunction  $e_{f,k}$  must satisfy the following equation, obtained by substituting (3.11) in (3.10)

$$(3.12) \quad Ke_{f,k}^n = \frac{1}{4} \left[ e_{f+1,k}^n + Ke_{f-1,k}^n + e_{f,k+1}^n + Ke_{f,k-1}^n \right] \quad (\text{in } R_h'') \\ e_{0,k} = e_{p,0} = e_{f,0} = e_{f,q} = 0$$

This equation is the reduced error formula.

Extrapolated Liebmann Method

This method has again to improve the efficiency by the increasing  $\alpha$  value. This iteration method is same as equation (3.8) in Liebmann Method

$$\begin{aligned}
 (3.8) \quad U_{f,k}^{n+1} &= U_{f,k}^n + \alpha \left[ U_{f+1,k}^n + U_{f-1,k}^{n+1} + U_{f,k+1}^n \right. \\
 &\quad \left. + U_{f,k-1}^{n+1} - 4U_{f,k}^n \right] \quad (\text{in } R_n'') \\
 &= b_{f,k} \quad (\text{in } S_n'')
 \end{aligned}$$

Riley [5] mentions that the terms are collected as follows

$$\begin{aligned}
 (3.13) \quad U_{f,k}^{n+1} &= \alpha \left[ U_{f+1,k}^n + U_{f-1,k}^{n+1} + U_{f,k+1}^n + U_{f,k-1}^{n+1} \right] \\
 &\quad - (4\alpha - 1) U_{f,k}^n \quad (\text{in } R_n'') \\
 &= b_{f,k} \quad (\text{in } S_n'')
 \end{aligned}$$

reduce the mean time required for each arithmetic operations for the evaluation of one  $U_{f,k}^{n+1}$  ( $7J$  to  $6J$ , where  $J$  is mean time for one operation).

The error formula of (3.13) between approximate value  $U_{f,k}^{n+1}$  and true value  $U_{f,k}$  will be

$$(3.14) \quad e_{f,k}^{n+1} = U_{f,k}^{n+1} - U_{f,k}$$

$$= \alpha \left[ e_{f+1,k}^n + e_{f-1,k}^{n+1} + e_{f,k+1}^n + e_{f,k-1}^{n+1} \right] \\ - (4\alpha - 1) e_{f,k}^n \quad (\text{in } R_n'')$$

and zero in boundary point  $S_n''$ . The formula (3.13) also can be written as

$$(3.15) \quad e^{n+1} = K(\alpha) e^n$$

where operator  $K(\alpha)$  depending on parameter  $\alpha$ , and this operator  $K(\alpha)$  is not simply Laplace's operator  $L$ . Hence  $K$  is the eigenvalue of the (3.15), then its eigenfunction  $e_{f,k}$  must satisfy the following equation, obtained by substitution (3.15) in (3.14)

$$(3.16) \quad Ke_{f,k} = \alpha \left[ e_{f+1,k} + Ke_{f-1,k} + e_{f,k+1} + Ke_{f,k-1} \right] \\ - (4\alpha - 1) e_{f,k}$$

This is the reduced error formula.

#### Second-order Richardson Method

Second-order Richardson Method is a method to improve the convergence rate of Richardson Method by using not only  $U_{f,k}^n$ , but also  $U_{f,k}^{n+1}$ . This modification is suggested by the formula equivalence between the Richardson Method and the solution of the time-dependant equation  $\frac{\partial U}{\partial t} = LU$ . In Frankel [1] the modification

method of Second-order Richardson Method is written as

$$(3.17) \quad \begin{aligned} U_{f,k}^{n+1} &= U_{f,k}^n + \alpha LU_{f,k}^n + \beta (U_{f,k}^n - U_{f,k}^{n-1}) && (\text{in } R_h'') \\ &= b_{f,k} && (\text{in } S_h'') \end{aligned}$$

The error formula will be

$$(3.18) \quad e_{f,k}^{n+1} = e_{f,k}^n + \alpha L e_{f,k}^n + \beta (e_{f,k}^n - e_{f,k}^{n-1}) \quad (\text{in } R_h'')$$

and zero in boundary condition we again can write (3.16) with an operator notation

$$(3.19) \quad e^{n+1} = K(\alpha, \beta) e^n$$

where  $K(\alpha, \beta)$  is the operator depending on parameter  $\alpha$  and  $\beta$ .

In equation (3.15), it appears that two lines of starting values are necessary. It is not clear what relation they must bear to one another or how this relation will effect later iteration. From Riley [5] paper, the disadvantages of the Second-order Richardson procedure as compared to the Extrapolated Liebmann procedure are removed for the case of the Laplace's difference equation.

One of the formulas defining the optimum  $\alpha$  and  $\beta$  in Frankel [1]

$$(3.20) \quad \alpha(L_0 + L_m) = -2(1 + \beta)$$

and

$$(3.21) \quad L_0 + L_m = -8$$

where  $L_0$  &  $L_m$  are the eigenvalue of the smallest and largest magnitude of the eigenfunction  $L$  (This will be discussed at section IV). Then equation (3.20) and (3.21) are simplified as

$$(3.22) \quad \beta = 4\alpha - 1$$

By substituting (3.22) to (3.17) we obtain

$$(3.23) \quad U_{f,k}^{n+1} = U_{f,k}^n + LU_{f,k}^n + (4\alpha - 1)(U_{f,k}^n - U_{f,k}^{n-1})$$

$$= \alpha \left[ U_{f+1,k}^n + U_{f-1,k}^n + U_{f,k+1}^n + U_{f,k-1}^n \right]$$

$$+ (1 - 4\alpha)U_{f,k}^{n-1} \quad (\text{in } R''_h)$$

$$= b_{f,k} \quad (\text{in } S''_h)$$

This formula has already simplified considerably. The complicated operational disadvantage in the Second-order Richardson iterative process is disappeared by that  $U_{f,k}^n$  has drops out. As simple Richardson Method, it is not necessary to compute it at all. That is, if one has values at alternate points, said where  $f - k$  is odd, and values in the next iteration at points where  $f - k$  is even, again. The missing values are independent of the values

computed, and vice versa. Thus only half of the computation is necessary, therefore, the improved Second-order Richardson Method requires fewer operations for each steps than the Richardson Method, and requires equal number of operation as the Extrapolated Liebmann Method.

We now consider the error formula for the equation (3.23) as other methods, and we get

$$(3.24) \quad e_{f,k}^{n+1} = \alpha \left[ e_{f+1,k}^n + e_{f-1,k}^n + e_{f,k+1}^n - e_{f,k-1}^n \right] + (1 - 4\alpha)e_{f,k}^{n-1} \quad (\text{in } R_h'')$$

and zero at the boundary points. This equation is presented as same convergent rate [5] as the Extrapolated Liebmann Method. Hence (3.24) equation implies (3.16) and

$$(3.25) \quad e^{n+1} = K(\alpha)e^n$$

where  $K(\alpha)$  is the operation depending on  $\alpha$ .

#### B) Group two method

Given the value of the function  $U(x,y)$  at the boundary points of the region  $R'$ , let us assign arbitrary initial values to the function at each of the interior points of the region  $R'$ . We calculate at each interior point a sequence of "improved" values converging to

exact solution of the difference equation (2.5) at that point. In each methods an improved value is computed at every interior point in some established order, and then the process is repeated many times with a systematic manner keeping the same order. We now describe the iterative method to determine the solution of (2.11). The methods of Jacobi, Gauss-Seidal, and Successive Overrelaxation are applicable if  $|A| \neq 0$ ; for when this condition holds, a solution of (2.11) exists and is unique. Therefore, in this section we shall assume that  $|A| \neq 0$ .

Jacobi Method (simultaneous displacement)

In this method one starts with the initial approximation  $U^0$  and computes the sequence of improved values  $U^n$  using the following formula which reduced from (2.11)

$$(3.26) \quad U_i^{n+1} = \sum_{j=1}^N b_{i,j} U_j^n + c_i$$

$i = 1, 2, \dots, N$

where

$$(3.27) \quad b_{i,j} = \begin{cases} -\frac{a_{i,j}}{a_{i,i}} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

$$c_i = -\frac{d_i}{a_{i,i}}$$

$c_i$  is linear combination of the known boundary values. In this method the improved values are not used until the entire process is through; i.e. the values of all  $U_i^n$  are modified "simultaneously". This is similar to the Richardson's elementary iterative process which is mentioned above section III (A) .

If we consider the  $U = (U_1, U_2, \dots, U_N)$  to be the true solution of (2.11), then we have

$$(3.27a) \quad U_i = \sum_{j=1}^N b_{i,j} U_j + c_i$$

$i = 1, 2, \dots, N$

If we assume  $U = U_i = (U_1, \dots, U_N)$ , the error between the true solution of (3.27a) and approximated value by the iterative method (3.26) will be

$$(3.28) \quad e_i^{n+1} = U_i^{n+1} - U_i$$

$$= \sum_{j=1}^N b_{i,j} e_j^n$$

$n \geq 0$

$i = 1, 2, \dots, N$

Since the boundary values are known constant, the error is zero. The equivalent form of (3.28) is

$$(3.29) \quad e^{n+1} = K [e^n]$$

where

$$e^n = (e_1^n, \dots, e_N^n)$$

and  $K$  is a linear transformation whose matrix is  $(b_{i,j})$ .

Gauss-Seidel Method (successive displacement)

In this method one modifies the values of the  $U_i^n$  cyclically in an arbitrary, but fixed, order, which we denote by  $\sigma$ , and the values  $U_i^{n+1}$  instead of  $U_i^n$ , are used when available. This process is same as the Liebmann iterative process. If we assume that the rows and corresponding columns of  $(b_{i,j})$  have been arranged to conform with this ordering, the sequence is given by

$$(3.30) \quad U_i^{n+1} = \sum_{j=1}^{i-1} b_{i,j} U_j^{n+1} + \sum_{j=i+1}^N b_{i,j} U_j^n + c_i$$

$$N \geq 0$$

$$i = 1, 2, \dots, N.$$

Equation (3.30) may be written in the form of

$$(3.31) \quad U^{n+1} = L_{\sigma,1} [U^n] + f$$

where

$$U^n = (U_1^n, U_2^n, \dots, U_N^n)$$

$f = (f_1, f_2, \dots, f_N)$ .  $f$  is fixed, and  $L$ , denote a linear operator (or transformation) with order  $\sigma$ .

If we assume again  $U = U_1$  to be the true solution

of (2.11), then we obtain in the same manner that we have obtained in the equation of (3.28), and the error function at  $(n+1)^{\text{st}}$  step is determined as

$$(3.32) \quad e_i^{n+1} = u_i^{n+1} - u_i \\ = \sum_{j=1}^{i-1} b_{i,j} e_j^{n+1} + \sum_{j=i+1}^N b_{i,j} e_j^n$$

we can rewrite this as

$$(3.33) \quad e^{n+1} = L_{\sigma,1} [e^n]$$

where  $e^n = (e_1^n, e_2^n, \dots, e_N^n)$  and  $L_{\sigma,1}$  is the linear transformation for the order  $\sigma$ .

#### Successive Overrelaxation Method

This method consists in the form of successive corrections as the Gauss-Seidel Method in the definitely fixed order  $\sigma$ . Assuming that the rows and corresponding columns of  $(b_{i,j})$  have been arranged to conform with the ordering  $\sigma$ , the iterative sequence is given by

$$(3.34) \quad u_i^{n+1} = \omega \left[ \sum_{j=1}^{i-1} b_{i,j} u_j^{n+1} + \sum_{j=i+1}^N b_{i,j} u_j^n + e_i \right]$$

$$n \geq 0$$

$$i = 1, 2, \dots, N.$$

The factor  $\omega$  appearing in this expression is called a

"relaxation factor". The equation of (3.34) may be written in the form

$$(3.35) \quad u^{n+1} = L_{\sigma, \omega}[u^n] + f$$

$$n \geq 0$$

where  $u^n = (u_1^n, u_2^n, \dots, u_N^n)$ ,  $f = (f_1, f_2, \dots, f_N)$ ,  $f$  is fixed, and  $L_{\sigma, \omega}$  denotes the linear operator.

Hence  $\sigma$  denotes the ordering of the equations and  $\omega$  denotes the relaxation factor. We again assuming  $u = u_1$  to be the true solution of the equation of (2.11), then the error formula will be same as the error formula of the Gauss-Seidel iterative process with the relaxation factor  $\omega$ . Therefore the error formula is

$$(3.36) \quad e_i^{n+1} = u_i^{n+1} - u_i$$

$$= \omega \left[ \sum_{j=1}^{i-1} b_{i,j} e_j^{n+1} - \sum_{j=i+1}^N b_{i,j} e_j^n \right]$$

$$n \geq 0$$

$$i = 1, 2, \dots, N$$

The equivalent form of (3.36) is

$$(3.37) \quad e^{n+1} = L_{\sigma, \omega}[e^n]$$

where  $e^n = (e_1^n, e_2^n, \dots, e_N^n)$  and  $L_{\sigma, \omega}$  is the linear transformation for the order  $\sigma$  and the relaxation factor  $\omega$ .

Now we will discuss the convergence about these methods and their conditions for convergence in next section.

#### IV Convergence

##### A) Group one method

We will discuss the convergence in the rectangular region. In order to examine the convergence, we now consider the error which is defined as equation (3.5). The error eigenfunction of operator L for a  $p \times q$  rectangle are [1],

$$(4.1) \quad e_{f,k}^{(r,s)} = \sin(\pi r f / p) \sin(\pi s k / q)$$

$$r = 1, 2, \dots, p-1$$

$$s = 1, 2, \dots, q-1$$

The corresponding eigenvalue is  $L$  we denote by  $L(r,s)$

$$(4.2) \quad L(r,s) e_{f,k}^{(r,s)} = e_{f+1,k}^{(r,s)} + e_{f-1,k}^{(r,s)} + e_{f,k+1}^{(r,s)} + e_{f,k-1}^{(r,s)} - 4 e_{f,k}^{(r,s)}$$

Substitue (4.1) into (4.2) we get

$$(4.3) \quad L(r,s) e_{f,k}^{(r,s)} = \left[ \sin \frac{\pi r (f-1)}{p} + \sin \frac{\pi r (f+1)}{p} - 2 \sin \left( \frac{\pi r f}{p} \right) \right]$$

$$\sin \left( \frac{\pi s k}{q} \right) + \left[ \sin \frac{\pi s (k-1)}{q} + \sin \frac{\pi s (k+1)}{q} - 2 \sin \left( \frac{\pi s k}{q} \right) \right]$$

$$\sin \left( \frac{\pi r f}{p} \right)$$

$$= \left[ 2 \sin \left( \frac{\pi r f}{p} \right) \cos \left( \frac{\pi r}{p} \right) - 2 \sin \left( \frac{\pi r f}{p} \right) \right] \sin \left( \frac{\pi s k}{q} \right)$$

$$+ \left[ 2 \sin \left( \frac{\pi s k}{q} \right) \cos \left( \frac{\pi s}{q} \right) - 2 \sin \left( \frac{\pi s k}{q} \right) \right] \sin \left( \frac{\pi r f}{p} \right)$$

$$= \left[ 2 \cos\left(\frac{\pi r}{p}\right) + 2 \cos\left(\frac{\pi s}{q}\right) - 4 \right] \sin\left(\frac{\pi r f}{p}\right) \sin\left(\frac{\pi s k}{q}\right)$$

Since  $e_{f,k}^{(r,s)}$  is the equation (4.1).

$$(4.4) \quad L(r,s) = 2 \cos\left(\frac{\pi r}{p}\right) + 2 \cos\left(\frac{\pi s}{q}\right) - 4$$

and the equivalent form of (4.4) is

$$(4.5) \quad L(r,s) = -4 \left[ \sin^2\left(\frac{\pi r}{2p}\right) + \sin^2\left(\frac{\pi s}{2q}\right) \right]$$

All of these eigenvalues are negative.

The smallest in magnitude is denoted by  $L_0$ , and the largest in magnitude is denoted by  $L_m$  which belong to  $r = s = 1$  and to  $r = p - 1, s = q - 1$ , respectively.

Thus,

$$(4.6) \quad L_0 = -4 \left[ \sin^2\left(\frac{\pi}{2p}\right) + \sin^2\left(\frac{\pi}{2q}\right) \right]$$

$$\approx -\frac{4}{\pi^2} (p^{-2} + q^{-2})$$

$$(4.7) \quad L_m = -4 \left[ \sin^2\left(\frac{\pi(p-1)}{2p}\right) + \sin^2\left(\frac{\pi(q-1)}{2q}\right) \right]$$

$$= -4 \left[ 2 - \sin^2\left(\frac{\pi}{2p}\right) - \sin^2\left(\frac{\pi}{2q}\right) \right]$$

$$= -8 + 4 \left[ \sin^2\left(\frac{\pi}{2p}\right) + \sin^2\left(\frac{\pi}{2q}\right) \right]$$

$$\tilde{\alpha} = -8 - \frac{2}{\pi} (p^{-2} + q^{-2})$$

The eigenfunctions of  $L$  are also eigenfunctions of the iteration operator  $K \equiv (1 + \alpha L)$ , corresponding to the eigenvalues

$$(4.8) \quad K(r,s) = 1 + L(r,s)$$

Each error-eigenfunction component is multiplied by its  $K(r,s)$  at each iteration. Thus if

$$(4.9) \quad \begin{aligned} \phi_{f,k}^n &= K(r,s) \phi_{f,k}^{n-1} \\ &= K^n(r,s) \phi_{f,k}^{(r,s)} \end{aligned}$$

With increasing  $n$  those components,  $\phi_{f,k}^{(r,s)}$ , for which

$$(4.10) \quad |K(r,s)| < 1$$

This iteration converge if and only if equation (4.10) satisfied for all  $r,s$ , and the more rapidly the smaller

$$|K(r,s)|.$$

Since equation (4.10) condition hold

$$(4.11) \quad \lim_{n \rightarrow \infty} \phi_{f,k}^n = 0$$

Hence this equation of (4.11) proves the necessary and sufficient condition for the convergence.

From the above consequence we can be able to discuss that Richardson Iterative Method converges to the difference equation (2.10). Since its error formula (3.7) of the Richardson iterative formul is

$$(4.12) \quad \lim_{n \rightarrow \infty} e_{f,k}^n = \lim_{n \rightarrow \infty} K e_{f,k}^{n-1} = \lim_{n \rightarrow \infty} K^n e_{f,k} = 0$$

if and only if condition  $|K| < 1$ . Hence under this condition Richardson Method converges to the solution of the equation (2.10).

The convergence about Liebmann Method can be discussed under the same condition. The error formula is defined as the equation of (3.12).

$$(4.13) \quad K e_{f,k}^n = \frac{1}{4} \left[ e_{f+1,k}^n + K e_{f-1,k}^n + e_{f,k+1}^n + K e_{f,k-1}^n \right]$$

where

$$e^{n+1} = K e^n$$

In the Richardson Method the error formula has simply (n) power of eigenvalue K, however, this Liebmann Method's error formula has not only (n) power of K, but also combines with (n+1) and (n) powers of eigenvalue K. Since the condition is given that  $|K| < 1$ , the error

formula (4.12) approaches the limit zero then equation (4.13) approaches the limit much faster than the equation (4.12). Therefore we can conclude that if Richardson Method converges then Liebmann Method also converges.

The Extrapolated Liebmann Method's error formula

$$(3.16) \quad e_{f,k}^{n+1} = \alpha \left[ e_{f+1,k}^n + e_{f-1,k}^{n+1} + e_{f,k-1}^n + e_{f,k-1}^{n+1} \right] \\ - (4\alpha - 1) e_{f,k}^n$$

can be reduced as

$$(3.15) \quad e_{f,k}^{n+1} = K(\alpha) e^n$$

Since  $|K| < 1$  is the condition, we can assume that

$|K(\alpha)| < 1$ , and so the convergence is determined.

In the Second-order Richardson Method convergence is maintained with the convergence condition of Richardson Method and also the convergence of the Improved Second-order Richardson Method is maintained with the convergence of the Extrapolated Liebmann Method.

Now we will discuss convergence in the slightly different way. The rectangular region  $R^2$  has

$$R^j = \begin{cases} i = 1, 2, \dots, p \\ k = 1, 2, \dots, q = p \end{cases}$$

then, we can discuss the following way to prove the existence of convergence. According to Young [11] the Extrapolated Liebmann Method (3.13) is written as

$$\begin{aligned} (4.14) \quad U_{f,k}^{n+1} &= \frac{\omega}{4} \left[ U_{f+1,k}^n + U_{f,k}^{n+1} + U_{f,k+1}^n + U_{f,k-1}^{n+1} \right] \\ &\quad - (\omega - 1) U_{f,k}^n \quad (\text{in } R^i_h) \\ &= b_{f,k} \quad (\text{in } S^i_h) \end{aligned}$$

with relaxation factor  $\omega$  which is the same sense as the Successive Over-relaxation Method's relaxation factor  $\omega$ . We shall denote this equation (4.14) as  $I(\omega)$ . When  $\omega = 1$  equation (4.14) becomes  $I(1)$  as

$$\begin{aligned} (4.15) \quad U_{f,k}^{n+1} &= \frac{1}{4} \left[ U_{f+1,k}^n + U_{f-1,k}^{n+1} + U_{f,k+1}^n + U_{f,k-1}^{n+1} \right] \quad (\text{in } R^i_h) \\ &= b_{f,k} \quad (\text{in } S^i_h) \end{aligned}$$

hence this is the Liebmann Iterative formula with

$\alpha = \frac{1}{4}$ . (optimum value  $\alpha$ ). Also we can derive the Richardson Iterative formula from the equation (4.14) by removing that continuous substitution. This formula will be

$$(4.16) \quad u_{f,k}^{n+1} = \frac{1}{4} \left[ u_{f+1,k}^n + u_{f-1,k}^n + u_{f,k+1}^n + u_{f,k-1}^n \right] \quad (\text{in } R_{\frac{1}{2}}^n)$$

$$= b_{f,k} \quad (\text{in } S_{\frac{1}{2}}^n)$$

We may reduce the equation (4.15), (4.16) into the linear systems of  $N$  linear equation by constructing square region where  $N$  are the interior mesh points. Thus, the systems for (4.16) becomes

$$(4.17) \quad u_{f,k}^{n+1} = \frac{1}{4} \left[ \sum_{j=1}^N a_{1,j} u_j^n + d_1 \right] \quad \begin{array}{l} i = 1, 2, \dots, N \\ j = 1, 2, \dots, N \end{array}$$

where  $d_1$  is the boundary value and  $a_{1,j}$  is the real coefficient of unknown  $u_j$ ,

the system for (4.15) becomes

$$(4.18) \quad u_{f,k}^{n+1} = \frac{1}{4} \left[ \sum_{j=1}^{i-1} a_{1,j} u_j^{n+1} + \sum_{j=i+1}^N a_{1,j} u_j^n + d_1 \right]$$

$$\begin{array}{l} i = 1, 2, \dots, N \\ j = 1, 2, \dots, N \end{array}$$

where  $d_1$ ,  $a_{1,j}$  and  $u_n$  have same property and the system for (4.15) becomes

$$(4.19) \quad u_{f,k}^{n+1} = \frac{\omega}{4} \left[ \sum_{j=1}^{i-1} a_{1,j} u_j^{n+1} + \sum_{j=i+1}^N a_{1,j} u_j^n + d_1 \right]$$

$$- (\omega - 1) u_j^n \quad \begin{array}{l} i = 1, 2, \dots, N \\ j = 1, 2, \dots, N \end{array}$$

where  $d_i$ ,  $a_{i,j}$  and  $u_n$  have same property, and  $\omega$  is relaxation factor which vary each iteration.

Hence there exists, the relationship between equation (4.15) and Gauss-Seidel Method, which has same relaxation factor  $\omega = 1$ . Same time, as we assigned the Richardson Method as above, the Jacobi Method has the relationship with Richardson Method. Therefore, according to the Young [12], corollary 2.1 if under the given condition, Gauss-Seidel Iterative Method converge then Jacobi Iterative Method converge to the difference equation, hence the convergence of Liebmann Method implies the convergence of Richardson Method. The next case is that the  $I(\omega)$  is equivalent with the Successive Over-relaxation Method. Therefore, the proof of the Successive Over-relaxation Method's convergence implies that Extrapolated Liebmann Method. Hence we now will prove the existence of the convergence of the Gauss-Seidel Method and the Successive-over-relaxation Method.

#### B) Group two method

An iterative method which computes a sequence of values  $u_1^{(n)}$  at each interior mesh point of the region R is said to converge when

$$\lim_{n \rightarrow \infty} u_1^{(n)} = u_1 \text{ or equivalently, when}$$

$\lim_{n \rightarrow \infty} e_i^{(n)} = 0$ . We now investigate the convergence

of the iterative methods described above, and in the next we discuss the rate at which such convergence takes place.

Firstly we examine the matrix of coefficients  $A = (a_{i,j})$  obtained from equation (2.11) which is

$$\sum_{j=1}^N a_{i,j} u_j + d_i = 0$$

$$i = 1, 2, \dots, N$$

$$j = 1, 2, \dots, N$$

From the equation of (2.5) we have

$$(4.20) \begin{cases} 4 & \text{for } i = j \\ -1 & \text{for } x_j = x_i \pm h \\ & y_j = y_i \pm h \\ 0 & \text{otherwise} \end{cases}$$

The following three conditions on matrix  $(a_{i,j})$  will be important to this discussion

$$(4.21) \begin{cases} (a) & |a_{i,j}| \geq \sum_{j=1, j \neq i}^N |a_{i,j}|, \quad i = 1, 2, \dots, N; \\ & \text{and for some } i \text{ the strict inequality holds.} \\ (b) & \text{Given any two nonempty, disjoint subsets } S \text{ and } T \text{ of } W, \text{ the set of the first } N \text{ positive integers, such that } S \cup T = W. \text{ There exists } a_{i,j} \neq 0 \text{ such that } i \in S \text{ and } j \in T. \end{cases}$$

- (c) There exist two disjoint subsets S and T of W, as defined in (b), such that if  $a_{i,j} \neq 0$ , then either  $i = j$ , or  $i \in S$  and  $j \in T$ , or  $i \in T$  and  $j \in S$ .

Conditions (4.25 a) and (4.25 b) were formulated by Geiringer<sup>[2]</sup> and condition (4.21 c) was formulated by Young [12]. Geiringer [2] proved in (pp 379 - 381) that if the coefficient matrix  $A = (a_{i,j})$  of (2.11) satisfy condition (4.21 a) and (4.21 b), then

$$1) a_{ii} \neq 0 \quad (i = 1, 2, \dots, N)$$

ii)  $A \neq 0$ , which implies the existence and uniqueness of the solution of (2.11). Also Geiringer [2] show that the under this condition Jacobian Method and Gauss-Seidel Method converge to the solution of (2.11) for any starting point and any ordering of the equations. In Young [12] shows that if the matrix A satisfies the conditions (4.21), and if the eigenvalue of A are real, then there exist consistent orderings  $\sigma$ , such that for all real  $\omega$  in the interval  $0 < \omega < 2$ , Successive over-relaxation converges. As we shall see later, for certain value of  $\omega$  (or  $\omega_b$ ) in the interval  $0 < \omega < 2$ , S. O. Method will converges much more than Gauss-Seidel Method.

It is proved by Young [12] Theorem 2.1 that

matrix  $A$  satisfies condition (4.21 c) if and only if there exists a vector  $\delta = (\delta_1, \delta_2, \dots, \delta_N)$  with integral components such that if  $a_{1,j} \neq 0$  and  $1 \neq j$  then  $|\delta_1 - \delta_j| = 1$ . According to the Young [2, p. 107] we shall show such an ordering vector  $\delta$  exists by exhibiting two ordering vectors in order to obtain two constant orderings  $a$  and  $c$  denotes the distance of the boundary from the  $x$  axis and  $y$  axis, respectively. We now write the coordinates of the point  $(x_1, y_1)$  in the form

$$(4.22) \quad \begin{cases} x_1 = a + p_1 h \\ y_1 = c + q_1 h \end{cases} \quad i = 1, 2, \dots, N$$

We define two ordering vectors  $\delta^{(1)}$  and  $\delta^{(2)}$  by

$$(4.23) \quad \delta_i^{(1)} = \begin{cases} 1 & \text{if } p_1 + q_1 \text{ is even} \\ 0 & \text{if } p_1 + q_1 \text{ is odd} \end{cases}$$

$$\delta_i^{(2)} = p_1 + q_1$$

Now if  $a_{1,j} \neq 0$  then  $(x_1, y_1)$  and  $(x_j, y_j)$  are adjacent and consequently, either  $|x_1 - x_j| = h$  or  $|y_1 - y_j| = h$ . Hence either  $|p_1 - p_j| = 1$  and  $q_1 = q_j$  or  $|q_1 - q_j| = 1$  and  $p_1 = p_j$ . In either case we have

$$|(p_1 + q_1) - (p_j + q_j)| = |\delta_i^{(1)} - \delta_j^{(1)}| = |\delta_i^{(2)} - \delta_j^{(2)}| = 1;$$

hence it follows that  $A$  satisfies condition (4.21 c).

The ordering vector  $\delta$  of the rows and columns of  $A$  with the above properties describes a consistent

ordering if, whenever  $a_{i,j} \neq 0$  and  $\delta_i > \delta_j$ , the  $i$ th row follows the  $j$ th row under the ordering.

We now wish to show that eigenvalues of  $A$  are real. It is stated in Warlick [9] that if  $P$  is a real but not symmetric  $n \times n$  matrix, and if there exist real weight factors  $s_i > 0$   $i = 1, 2, 3, \dots, n$ , such that

$$(4.24) \quad s_i p_{i,j} = s_j p_{j,i}$$

then the  $n$  eigenvalues of  $P$  are real. To see this, consider the diagonal matrix

$$(4.25) \quad D = \begin{pmatrix} \sqrt{s_1} & & & \\ & \sqrt{s_2} & & \\ & & \ddots & \\ & & & \sqrt{s_n} \end{pmatrix} = \text{dia} (\sqrt{s_1}, \sqrt{s_2}, \dots, \sqrt{s_n})$$

Since  $s_i > 0$  we have  $D^{-1} = (\frac{1}{\sqrt{s_1}}, \frac{1}{\sqrt{s_2}}, \dots, \frac{1}{\sqrt{s_n}})$

This diagonalization method is equivalent with Schmidt process of orthogonalization. Consider now the matrix

$$(4.26) \quad C = D P D^{-1}$$

The general elements  $c_{i,j}$  and  $c_{j,i}$  are respectively,

$$(4.27) \quad c_{i,j} = \sqrt{\frac{s_i}{s_j}} p_{i,j} \quad c_{j,i} = \sqrt{\frac{s_j}{s_i}} p_{j,i}$$

From (4.17) and (4.26) it follows that

$$c_{i,j} = c_{j,i}$$

therefore the matrix  $C$  is real and symmetric, and it has  $n$  real eigenvalues. Since  $D$  is nonsingular, it follows from (4.26) that  $p$  and  $c$  are similar; hence the eigenvalues of  $p$  are real.

From the equation of (2.11) we can see that  $A$  is real, nonsymmetric matrix. We now exhibit Warlick [9] the weight factors  $s_i$ . In determining the  $s_i$ , we need to be concerned only with the elements  $a_{i,j} \neq 0$  as given in the equation of (4.14). From section II we see that we may set  $y_i = f h$ , where  $f$  takes on positive integral values. If  $a_{i,j} = -1$   
 $|x_i - x_j| = h$  and  $y_i = y_j = f h$ .  
 It follows immediately that  $a_{i,j} = -1$  in that case, hence  $s_i = s_j = q_m$ . It is also clear that for all points  $(x_j, y_j)$  such that  $y_j = m h$ , we have  $s_j = q_m$ . If we set  $q_{m0} = 1$ , where  $m_0 = \frac{c-h}{h}$ , then the factors  $s_i$  are defined such that  $s_i > 0$  for  $i = 1, 2, \dots, N$ . We showed that the matrix  $A = (a_{i,j})$  satisfies condition (4.20), and possesses real eigenvalues; hence we may conclude that Successive Over-relaxation Method converge for consistent orderings and for all  $\omega$  such that  $0 < \omega < 2$ . This existence of the convergence in each iterative method under the given condition, we

now discuss the convergence rate of those iterative  
method mentioned above.

## V Convergence Rate

### A) Group one method

We now examine the spectrum of eigenvalue of the transformation T to be the maximum magnitude of the absolute values of the eigenvalues of T. We denote the spectrum eigenvalue of transformation T as  $K^*$ .

Now we consider the spectrum eigenvalue  $K^*$  which is

$$(5.1) \quad K^* = \text{Max}_{(r,s)} |K_{(r,s)}|$$

Since by the equation of (3.8) the eigenvalues of  $K_{(r,s)}$  is

$$(4.8) \quad K_{(r,s)} = 1 + \alpha L_{(r,s)}$$

be in the range

$$(5.2) \quad K_m = 1 + \alpha L_m \leq K_{(r,s)} \leq 1 + \alpha L_0 = K_0$$

and the equality sign holds for  $(r,s) = (1,1)$  and  $(p-1, q-1)$ ,  $K^*$  is determined by

$$(5.3) \quad K^* = \text{Max} \left\{ |K_m|, |K_0| \right\}$$

As  $\alpha$  increases from zero,  $K_0$  drops slowly from unity,  $K_m$  drops rapidly from unity. Thus  $K^* = K_0 > 0$  so long as  $K_0 \geq -K_m$ . For the greater values of  $\alpha$ ,  $K^* = -K_m$ , hence  $K^*$  rises with increasing  $\alpha$ . The most rapid

convergence is occurring where

$$(5.4a) \quad K_0 = 1 + \alpha(L_0) = -K_m = -(1 + \alpha(L_m))$$

and so

$$(5.4b) \quad \alpha = -\frac{2}{L_0 + L_m}$$

From the equation for  $L_0$  (4.6) and  $L_m$  (4.7) we obtain

$$(5.5) \quad L_0 + L_m = -\frac{2}{p^2 + q^2} - 8 + \frac{2}{p^2 + q^2} = -8$$

Hence for

$$(5.6) \quad \alpha = \frac{1}{4}$$

To substitute  $\alpha = \frac{1}{4}$  to the equation (4.8)

$$(5.7) \quad K(r,s) = 1 + \frac{1}{4}L(r,s)$$

The eigenvalues  $L(r,s)$  of Laplace's operator  $L$ , and the function

$$(5.8) \quad t = \cos(\pi r/p) + \cos(\pi s/q)$$

which is defined in the equation (2.7) of Frankel [1], satisfies the relation

$$(5.9) \quad L(r,s) = -4 + 2t$$

Hence the equation (5.7) becomes

$$(5.10) \quad K_{(r,s)} = \frac{1}{2}t$$

Now we denote  $\bar{t}$  is the spectrum eigenvalue of  $K$ , transformation of the Richardson Method at the optimum  $\alpha = \frac{1}{2}$ , so that the spectrum eigenvalue  $\bar{t}$  in optimum value  $\alpha = \frac{1}{2}$  is defined as

$$(5.11) \quad \begin{aligned} \bar{t} = K^* &= \frac{1}{2} t_{\max} \\ &= \frac{1}{2} \left[ \cos(\pi/p) - \cos(\pi/q) \right] \\ &\approx 1 - \frac{h^2(p^{-2} + q^{-2})}{4} \end{aligned}$$

Richardson Method explains the rate of convergence in terms of the spectrum eigenvalue. We will discuss the rate of convergence as the form of the rate of convergence of a convergent transform. Since we mentioned the condition for convergence as  $|t| < 1$  at section IV (A),  $K$  is convergence transformation within  $|\bar{t}| < 1$ . According to Geiringer [2] we can define the rate of convergence of a convergent transformation as

$$(5.12) \quad R[K] = -\log \bar{t}$$

In the next we can consider the Liebmann Method. This method decreases the number of iteration by using

new value whenever it is ready for use. Therefore, this method converges faster than Simultaneous Substitution Method which is the Richardson Method. We now denote  $\bar{\zeta}(\frac{1}{2})$  to be the spectrum eigenvalue of  $K(\frac{1}{2})$ , the transformation of the Liebmann Method at optimum value  $\lambda = \frac{1}{2}$ . If  $\bar{t}$  is spectrum eigenvalue of  $K$ , then  $\bar{t}^2$  is the spectrum eigenvalue of  $K(\frac{1}{2})$ ; if  $\bar{\zeta}(\frac{1}{2})$  is spectrum eigenvalue of  $K(\frac{1}{2})$ , then  $\bar{t}$  is spectrum eigenvalue of  $K$  [12]. Therefore,  $\bar{\zeta}$  is obtained in terms of  $\bar{t}^2$ , and the relation becomes

$$\begin{aligned}
 (5.13) \quad \bar{\zeta} = \bar{t}^2 &= \left[ \frac{1}{2} t_{\max} \right]^2 \\
 &= \frac{1}{2} \left[ \cos(\pi/p) + \cos(\pi/q) \right] \\
 &= \left[ 1 - \frac{\pi^2(p^{-2} + q^{-2})}{4} \right]^2
 \end{aligned}$$

Since  $K(\frac{1}{2})$  is convergent transformation within  $|\bar{t}| < 1$ , the rate of convergence of Liebmann Method will be

$$(5.14) \quad R\left[K\left(\frac{1}{2}\right)\right] = -\log \bar{\zeta}\left(\frac{1}{2}\right) = -2\log \bar{t}$$

This formula in form is that the rates of convergence of  $K(\frac{1}{2})$  is twice the rate of convergence of  $K$  in Richardson Method. Further more we can find an advantage in number of iteration required to achieve a given accuracy.

Warlick [9] mentioned that the number of iterations is approximated by inversely proportional to the rate of convergence. Therefore, Liebmann Method requires half of the number of iterations than using Liebmann Method in order to get the specified accuracy.

In Frankel [1] it is shown that the optimum  $\alpha$  is determined by the equation (35)[1]

$$(5.15) \quad \alpha^2 t_{\max}^2 - 4\alpha + 1 = 0$$

for the rectangular region,  $t_{\max} = \cos(\pi/p) - \cos(\pi/q)$ . In Young [12] it is better to overestimate the optimum  $\alpha$  than to underestimate it. Therefore, by calculating proper value of  $\alpha$  from (5.15) we can see the gain the rate of convergence and the decreasing number of iteration.

In order to get an optimum convergence factor (relaxation factor)  $\omega$  we will consider the following. From Young [11] equation (29) shows that relationship between  $\alpha$  in Extrapolated Liebmann Method and relaxation factor  $\omega$  is that

$$(5.16) \quad \omega = 4\alpha$$

We denote the optimum value of  $\alpha$  by  $\alpha_0$ . Then we can construct optimum  $\alpha_0$  from equation (5.15) and Young [12 section] as

$$(5.17) \quad \alpha_b = \left\{ \frac{1}{2} + \left[ \frac{\bar{t}}{1 + (1 - \bar{t}^2)^{\frac{1}{2}}} \right]^2 \right\}$$

where  $\bar{t}$  is the spectrum eigenvalue which defined above (5.11).

Now we will denote  $\bar{\zeta}(\alpha)$  to be the spectrum eigenvalue of the transformation  $K(\alpha)$  of Extrapolated Liebmann Method where  $\alpha$  is the relaxation factor. Since  $K(\alpha)$  is the convergence transformation  $|\bar{t}| < 1$ , we can define the spectrum eigenvalue as

$$(5.18) \quad \bar{\zeta}(\alpha) = (\alpha - \frac{1}{2})$$

If  $\alpha_b \leq \alpha \leq \frac{1}{2}$  by Young [12], Theorem (3.2)

$$(5.19) \quad R[K(\alpha)] < R[K(\alpha_b)]$$

for all real  $\alpha$  such that  $\alpha \neq \alpha_b$ . Hence the rate of convergence of  $K(\alpha_b)$  is given as

$$(5.20) \quad R[K(\alpha_b)] = -\log \bar{\zeta}(\alpha_b) = -\log(\alpha_b - \frac{1}{2}) \\ = -2 \log \frac{\bar{t}}{1 + (1 - \bar{t}^2)^{\frac{1}{2}}}$$

where spectral norm

$$(5.21) \quad \bar{\zeta}(\alpha_b) = (\alpha_b - \frac{1}{2}) = \left[ \frac{\bar{t}}{1 + (1 - \bar{t}^2)^{\frac{1}{2}}} \right]^2$$

$$\approx 1 - \sqrt{2\pi} (p^{-2} + q^{-2})^{\frac{1}{2}}$$

With this procedure and with this optimum values of  $\alpha$  (relaxation factor) the number of iterations require to produce a substantial improvement in a trial solution. The convergence rate of  $K(\alpha)$  is asymptotically equal to twice of the square root of the rate of convergence of  $K(\frac{1}{2})$ . Since the number of iterations is inversely proportional to the rate of convergence, the number of iteration of the Extrapolated Liebmann Method is less than the number of iterations in the Liebmann Method. Young [10] has mentioned that the number of iterations required to solve the Laplace equation which is asymptotically proportional to  $h^{-1}$  while Liebmann Method require  $h^{-2}$  number of iterations.

The Second-order Richardson Method which has iteration formula (3.17) appears that two lines of starting values are necessary. It is not clear what relation they must bear to one another or how this relation will affect later iterations. Therefore, this is not very applicable. Now we will denote the  $\xi(\alpha, \beta)$  to be the spectrum eigenvalue of  $K(\alpha, \beta)$ , the transformation of the Second-order Richardson Method. Since  $\alpha_p$  in the optimum value of  $\alpha$  in the same sense as above, we can describe the  $\xi(\alpha_p, \beta)$  as

$$(5.22) \quad \bar{\xi}(\alpha, \beta) = \frac{\bar{\xi}}{1 + (1 - \bar{\xi}^2)^{\frac{1}{2}}}$$

$$\cong 1 - \frac{K(p^{-2} + q^{-2})^{\frac{1}{2}}}{\sqrt{2}}$$

Since the transformation  $K(\alpha, \beta)$  is convergent (section IV,A) it can define the rate of convergence as

$$(5.23) \quad R[K(\alpha, \beta)] = -\log \bar{\xi}(\alpha, \beta)$$

$$= -\log \left( \frac{\bar{\xi}}{1 + (1 - \bar{\xi}^2)^{\frac{1}{2}}} \right)$$

where  $\bar{\xi}$  has mentioned above.

Now let us consider the convergence rate of this method and the Extrapolated Liebmann Method. We denote the  $\bar{\xi}(\alpha, \beta)$  to be the spectrum eigenvalue of  $K(\alpha, \beta)$ , the transformation of the Second-order Richardson Method. Then we have the convergence rate relationship between this method and the Extrapolated Liebmann Method [1]

$$(5.24) \quad R[K(\alpha, \beta)] = \frac{1}{2} R[K(\alpha, \beta)]$$

We now consider the convergence rate in the Improved Second-order Richardson Method. Since this iterative method's error formula is resemble to the Extrapolated Liebmann Method, the rate of convergence of this method

is equal to the Extrapolated Liebmann Method. This was the purpose to construct the iterative formula of the Improved Second-order Richardson from the Second-order Richardson Method [5]. In The next we will discuss the rate of convergence for the second group method.

#### B) Group one method

We define the spectral norm of a transformation  $T$  to be the maximum of the absolute values of eigenvalues of  $T$ , we denote the spectral norm of  $T$  as  $\bar{\lambda}$ . The rate of convergence of a convergent transform is defined by

$$(5.25) \quad R(T) = -\log \bar{\lambda}$$

It is well known that  $T$  is a convergent transformation if and only if the characteristic circle of  $T$  lies within the unit circle, i.e. if and only if  $0 < \bar{\lambda} < 1$ . This is proved by Oldenburger. The relationship between the rate of convergence and number of iteration is approximately inversely proportional to rate of convergence. Therefore increase in the rate of convergence cause to reduce the number of iterations.

Let  $\bar{A}$  denote the spectral norm of  $K$ , the transformation of Jacobi Method. In the equation (4.25) we already know about that  $K$  is a linear transformation whose matrix is  $(b_{i,j})$ . Geiringer [2] showed that

if the matrix A satisfies condition (4.21 a) and (4.21b), then  $\bar{\mu} < 1$ , therefore, reduced matrix B from matrix A also satisfies the condition (4.21 a) and (4.21b). According to Geiringer K converges within  $\bar{\mu} < 1$ , so that the rate of convergence of Jacobi Method is given by

$$(5.26) \quad R(K) = -\log \bar{\mu}$$

In the next to consider the Gauss-Seidel Method we may realize large gains in the convergence rate with a proportional decrease in the number of iterations required to achieve a given accuracy. In Young [12] it is shown that consistent ordering of the rows and columns of A, and if  $\mu$  is an eigenvalue of K, then  $\lambda = \mu^2$  is the eigenvalue of  $L_{\sigma,1}$ . In the equation (3.31)

$L_{\sigma,1}$  is a linear transformation for the order  $\sigma$ .

The spectral norm  $\bar{\lambda}(1)$  of  $L_{\sigma,1}$  is thus given by  $\bar{\lambda}(1) = \bar{\mu}^2$  where  $\bar{\mu}$  is defined above. Hence the rate of convergence of  $L_{\sigma,1}$  is given by

$$(5.27) \quad R(L_{\sigma,1}) = -\log \bar{\lambda}(1) = -2 \log \bar{\mu}$$

We can be able to notice that the rate of convergence of  $L_{\sigma,1}$  is twice the rate of convergence of K. This relationship is also defined in Young. [12] corollary (2.1), p 100. In this method another advantage is that the half of the number of iterations is required to achieve a specified accuracy using this method instead of Jacobi Method. From the error formula of Successive

over-relaxation Iterative Method is defined in section III of equation (3.37), and  $L_{\sigma, \omega}$  is a linear transformation of consistent order  $\sigma$  and relaxation factor  $\omega$ . In the Young [12] theorem 3.3, it is shown that a gain is realized by using in Successive over relaxation method the proper choice of the relaxation factor  $\omega$ . As shown in Young [12] the optimum choice of  $\omega$  is given by

$$(5.28) \quad \omega_b = 1 + \left[ \frac{\bar{\mu}}{1 + (1 - \mu^2)} \right]^2$$

where  $\bar{\mu}$  is the spectral norm of  $K$ .

The spectral norm of  $L_{\sigma, \omega}$  is  $\bar{\lambda}(\omega)$  and from Young [12, theorem 3.2, p 102] we see that the all  $0 < \omega < 2$  such that  $\omega \neq \omega_b$ , we have

$$R(L_{\sigma, \omega}) < R(L_{\sigma, \omega_b})$$

If  $\omega_b < \omega < 2$ , then

$$(5.29) \quad \lambda(\omega) = \omega - 1$$

Now we denote  $\bar{\lambda}(\omega_b)$  to be the spectral norm of  $L_{\sigma, \omega_b}$ , then the rate of convergence in Successive over relaxation method is given by

$$(5.30) \quad \begin{aligned} R[L_{\omega, b}] &= -\log \bar{\lambda}(\omega_b) \\ &= -\log (\omega_b - 1) \\ &= -2 \log \frac{\bar{\mu}}{1 + (1 - \mu^2)} \end{aligned}$$

where  $(\omega_b) = \omega_b - 1$ .

In Young [12] theorem 3.3, it is shown that

$$(5.31) \quad R(L_{\sigma}, \omega_b) \sim 2 \left[ R(L_{\sigma}, 1) \right]^{\frac{1}{2}}$$

as  $R(L_{\sigma}, 1)$  tends asymptotically to zero. Evidently, a large gain in the rate of convergence and a consequent decrease in iterations is to be realized when  $R(L_{\sigma}, 1)$  is small. The required number of iteration is the order of  $h^{-2}$  using  $L_{\sigma}, 1$  and only of the order of  $h^{-1}$  using  $L_{\sigma}, \omega_b$ . Successive over-relaxation method proves to be the most rapidly convergent of the methods we are considering in this paper. As we have seen, the most rapid convergence of this method is obtained when  $\omega = \omega_b$ , where  $\omega_b$  is the function of  $\bar{\mu}$  expressed in (5.27).

It is interesting to note here that as shown in Young [12, theorem 3.4], if  $\omega_b$  and  $\omega$  satisfy

$$(5.32) \quad \omega_b^2 \bar{\mu}^2 - 4(\omega_b - 1) = 0 \quad \omega_b < 2.$$

with  $\bar{\mu} = \bar{\mu}$  and  $\bar{\mu} = \bar{\mu}'$  respectively where

$$(5.33) \quad (1 - \bar{\mu}') = \theta(1 - \bar{\mu}) \quad (0 < \theta \leq 1)$$

then as  $\bar{\mu} \rightarrow 1$  - we have

$$(5.34) \quad R(L_{\sigma}, \omega) \sim \theta^{\frac{1}{2}} R(L_{\sigma}, \omega_b)$$

Thus a relative decrease in  $R(L_{\sigma}, \omega)$ , corresponding to an overestimation of  $\bar{\mu}$  is not very serious. On the other hand, an underestimation of  $\bar{\mu}$  causes a much larger relative decrease in the rate of convergence. In the next we will discuss the  $\bar{\mu}$  value which can be obtained by the overestimation, and relationship with  $\bar{t}$  in the part (A).

VI Eigenvalue  $\bar{\mu}$  and  $\bar{\tau}$ 

If we enclose a given rectangle, then spectral norm of  $\bar{\mu}$  for the region will be less than or equal to the spectral norm for the rectangle. Hence, when we replace a given region by a circumscribing rectangle and compute  $\bar{\mu}$  for the rectangle. The slight overestimation of  $\bar{\mu}$  for the given region does not seriously affect the convergence rate  $R(L_\sigma, \omega_j)$ .

Warlick [9] considers the computation of  $\bar{\mu}$  for the rectangular region. In his special case ( $k = 0$ , or Laplace equation) expression for  $\bar{\mu}$  is given as: the exact value of  $\bar{\mu}$ .

$$(6.1) \quad \bar{\mu}_N = \frac{1}{2} \left[ \cos\left(\frac{\pi h}{b-a}\right) + \cos\left(\frac{\pi h}{d-c}\right) \right] \quad \begin{cases} p = \frac{1}{h}(b-a) \\ q = \frac{1}{h}(d-c) \end{cases}$$

and approximated value of  $\bar{\mu}$

$$(6.2) \quad \bar{\mu}_T \cong 1 - \frac{h^2}{4} \left[ \left(\frac{\pi}{b-a}\right)^2 + \left(\frac{\pi}{d-c}\right)^2 \right] \quad \begin{cases} p = \frac{1}{h}(b-a) \\ q = \frac{1}{h}(d-c) \end{cases}$$

In this equation, when  $h$  is small the approximated value of  $\bar{\mu}$  equals to the exact value of  $\bar{\mu}$ . This formula has the complete agreement with the spectrum eigenvalue  $\bar{\tau}$  in section V (A) which form is

the exact value

$$(6.3) \quad \bar{\tau} = \frac{1}{2} \left[ \cos (\pi/p) - \cos (\pi/q) \right]$$

and the approximated value

$$(6.4) \quad \bar{\tau} \approx 1 - \frac{\pi^2 (p^{-2} + q^{-2})}{4}$$

Hence, we can assume that spectral norm of  $\bar{A}$  and spectrum eigenvalue  $\bar{\tau}$  has equal value.

Now we will compare the convergence rate of group one method and group 2 method according to the above assumption.

## VII Comparison of the rate of convergence

We now know how much the rate of convergence varies among the methods which we have described. Also noticeable difference in the number of iterations has caused by the rate of convergence, because of the relationship between the rate of convergence and the number of iterations is inversely proportional to each other. Thus the total calculation time is considerably affected by the rate of convergence and the number of iterations.

In the consideration of elementary methods, the Richardson Method in the group one and Jacobi Method in the group two have greater similarity. When the Richardson Method with optimum value  $\alpha = \frac{1}{2}$ , the rate of convergence is same as the Jacobi Method since  $\bar{\epsilon} = \bar{\mu}$  showed in section VI. The formula of the rate of convergence defined as

$$(7.1) \quad R[K] = \frac{1}{\log \bar{\mu}} = -\log \bar{\kappa}$$

where  $\bar{\mu}$  is spectral norm of the convergence transformation  $K$ . Therefore the number of iteration and total calculation time is same.

The slight difference in the iterative process causes the gain of the rate of convergence. The Liebmann Method at the optimum value  $\alpha = \frac{1}{2}$  and Gauss-Seidel Method are constructed by the same iterative formula of Richardson Method and Jacobi Method with the successive displacement.

From using this successive displacement technique there is a greater gain in the rate of convergence which is twice as more than the rate of Richardson and Jacobi methods. Since  $\bar{\tau} = \bar{\mu}$ , the rate of convergence of L of the equation (5.14) and the rate of convergence of Gauss-Seidel Method of the equation (5.27) are equal, and we could define

$$(7.2) \quad R \left[ K\left(\frac{1}{2}\right) \right] = R \left[ L_{GS} \right] = -2 \log \bar{\mu}$$

where  $\bar{\mu}$  is the spectral norm of the convergence transformation  $L_{GS}$ . From this gain of convergence rate the number of iterations decrease the half of the number of the iterations required in the Richardson and Jacobi methods, and also the total calculation time is reduced half in order to get the specified accuracy.

In the next the most recommendable methods which are the Extrapolated Liebmann Method and Successive Overrelaxation Method, are introduced by the using of convergence factor. This iterative method is based on the Liebmann and Gauss-Seidel Methods and multiple of  $\alpha$  and  $\omega$  respectively, where  $\alpha$  and  $\omega$  are relaxation factor. From the choosing of optimum value  $\omega$  of  $\omega_0$  and  $\alpha$  of  $\alpha_0$ , we can gain the twice the square root of the rate of convergence of Liebmann and Gauss-Seidel methods as the latter tend to zero. Since the equation (5.20)

agree with the equation (5.30) we could have the relationship

$$(7.3) \quad R[K(\alpha)] = R[L_{\sigma, \omega_b}] = -2 \log \frac{\bar{u}}{1 + (1 - \bar{u}^2)^{\frac{1}{2}}}$$

Young [10] the required number of iteration is of the order of  $h^{-2}$  in Liebmann and Gauss-Seidel methods and the only of the order of  $h^{-1}$  using the Extrapolated Liebmann and Successive Overrelaxation Methods, and the time also diminishes inversely proportional to the rate of convergence.

We Now will discuss the improvement in the rate of convergence of Second-order Richardson from the Richardson Method. The improvement is made from the multiple of  $\alpha$  and  $\beta$  of convergence factor and is twice of the rate of convergence of Richardson Method. Difficulty here is the Relaxation factor to be different in each iterations Young [13]. Therefore from using this method there are the complication in computation, and the required time is relatively more than the necessary in that convergence rate.

The Improved Second-order relaxation method which is defined at the equation (3.23) has increased twice of the rate of convergence of the Second-order Richardson

Method and no computational difficulty is there.  
This has the same rate of convergence as Extrapolated  
Liebmann Method or Successive Overrelaxation Method.

Therefore, when we apply the Extrapolated  
Liebmann Method , Improved Second-order Richardson  
Method and Successive Overrelaxation Method for solving  
Laplace's Equation, there exist the great advantage  
in the total calculating time.

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