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2017-07

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Oliver Lorscheid, Matt Szczesny. June 2018. "Quasicoherent sheaves on projective schemes over." Journal of Pure and Applied Algebra, Volume 222, Issue 6, pp. 1337-1354
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QUASICOHERENT SHEAVES ON PROJECTIVE SCHEMES OVER $\mathbb{F}_1$

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ABSTRACT. Given a graded monoid A with 1, one can construct a projective monoid scheme $\mathrm{MProj}(A)$ analogous to $\mathrm{Proj}(R)$ of a graded ring R. This paper is concerned with the study of quasicoherent sheaves on $\mathrm{MProj}(A)$, and we prove several basic results regarding these. We show that:

(1) every quasicoherent sheaf $\mathcal{F}$ on $\mathrm{MProj}(A)$ can be constructed from a graded A–set in analogy with the construction of quasicoherent sheaves on $\mathrm{Proj}(R)$ from graded R–modules
(2) if $\mathcal{F}$ is coherent on $\mathrm{MProj}(A)$, then $\mathcal{F}(n)$ is globally generated for large enough n, and consequently, that $\mathcal{F}$ is a quotient of a finite direct sum of invertible sheaves
(3) if $\mathcal{F}$ is coherent on $\mathrm{MProj}(A)$, then $\Gamma(\mathrm{MProj}(A), \mathcal{F})$ is finitely generated over A_0 (and hence a finite set if $A_0 = \{0, 1\}$).

The last part of the paper is devoted to classifying coherent sheaves on $\mathbb{P}^1$ in terms of certain directed graphs and gluing data. The classification of these over $\mathbb{F}_1$ is shown to be much richer and combinatorially interesting than in the case of ordinary $\mathbb{P}^1$, and several new phenomena emerge.

1. INTRODUCTION

The last twenty years have seen the development of several notions of "algebraic geometry over $\mathbb{F}_1$". This quest is motivated by a variety of questions in arithmetic, representation theory, algebraic geometry, and combinatorics. Since several surveys [12, 15, 23] of the motivating ideas exist, we will not attempt to sketch them here. One of the simplest approaches to algebraic geometry over $\mathbb{F}_1$ is via the theory of monoid schemes, originally developed by Kato [17], Deitmar [4], and Connes-Consani [1]. Here, the idea is to replace prime spectra of commutative rings which are the local building blocks of ordinary schemes, by prime spectra of commutative unital monoids with 0. One then obtains a topological space X with a structure sheaf of commutative monoids $\mathcal{O}_X$. In this setting, one can define a notion of *quasicoherent sheaf* on $(X, \mathcal{O}_X)$, as a sheaf of pointed sets carrying an action of $\mathcal{O}_X$, which for each affine $U \subset X$ is described by an $\mathcal{O}_X(U)$–set. Imposing a condition of local finite generation yields a notion of *coherent sheaf*.

This paper is devoted to the study of quasicoherent and coherent sheaves on projective monoid schemes. Given a graded commutative unital monoid with 0, $A = \bigoplus_{n \in \mathbb{N}} A_n$, one can form a monoid scheme $\mathrm{MProj}(A)$ in a manner analogous to the Proj construction in the setting of graded rings. We call such a monoid scheme *projective*. In analogy with the setting of ordinary schemes, we construct a functor:

$$grA - Mod \to Qcoh(X)$$
$$M \to \widetilde{M}$$

from the category of graded (set-theoretic) A–modules to the category of quasicoherent sheaves on $\mathrm{MProj}(A)$. It sends M to the quasicoherent sheaf $\widetilde{M}$ whose sections over the affine open $\mathrm{MProj}(A_f)$ are $M_{(f)}$ - the degree zero elements of the localization of M with respect to the multiplicative subset generated by f. We prove that every quasicoherent sheaf on $\mathrm{MProj}(A)$ arises via this construction, and that as in the case of ordinary schemes, there is a canonical representative:

Theorem (6.0.10)**.** Let A be a graded monoid finitely generated by A_1 over A_0, and let $X = \mathrm{MProj}(A)$. Given a quasicoherent sheaf $\mathcal{F}$ on X, there exists a natural isomorphism $\beta : \widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}$, where $\Gamma_*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \Gamma(\mathrm{MProj}(A), \mathcal{F}(n))$.

This result allows a purely combinatorial classification of quasicoherent sheaves on $\mathrm{MProj}(A)$ in terms of graded A–sets. We use this to obtain $\mathbb{F}_1$–analogs of other key foundational results on quasicoherent and coherent sheaves on projective schemes, such as the following regarding global finite generation:

Theorem (6.1.3)**.** Let A be a graded monoid finitely generated by A_1 over A_0, and $\mathcal{F}$ a coherent sheaf on $X = \mathrm{MProj}(A)$. Then there exists n_0 such that $\mathcal{F}(n)$ is generated by finitely many global sections for all $n \geq n_0$.

As a corollary, we obtain:

Corollary (6.1.4)**.** With the hypotheses of Theorem 6.1.3, there exist integers $m \in \mathbb{Z}, k \geq 0$, such that $\mathcal{F}$ is a quotient of $\mathcal{O}_X(m)^{\oplus k}$.

One of the key properties of a coherent sheaf on a projective scheme over a field k is the finite-dimensionality of the space of global sections. We obtain the following $\mathbb{F}_1$–analog:

Theorem (6.2.1)**.** Let A_0 be a finitely generated monoid, A a graded monoid finitely generated by A_1 over A_0, and $\mathcal{F}$ a coherent sheaf on $X = \mathrm{MProj}(A)$. Then $\Gamma(X, \mathcal{F})$ is a finitely generated A_0–module. In particular, when $A_0 = \{0, 1\}$, $\Gamma(X, \mathcal{F})$ is a finite pointed set.

The last section is devoted to the study and classification of coherent sheaves on the simplest non-trivial projective scheme - $\mathbb{P}^1$. By viewing $\mathbb{P}^1$ as two copies of $\mathbb{A}^1$ glued together as $\mathbb{A}^1_0 \cup \mathbb{A}^1_\infty$, we give a combinatorial description of the indecomposable coherent sheaves in terms of certain directed graphs and gluing data along the intersection $\mathbb{A}^1_0 \cap \mathbb{A}^1_\infty$. The classification is much richer than in the case of $\mathbb{P}^1_k$ for k a field, as base change to $\mathrm{Spec}(k)$ identifies many coherent sheaves non-isomorphic over $\mathbb{F}_1$. Example 4 exhibits an unusual phenomenon possible over $\mathbb{F}_1$ but impossible over $\mathrm{Spec}(k)$: a pair of coherent sheaves $\mathcal{F}, \mathcal{F}'$ with infinitely many non-isomorphic extensions of $\mathcal{F}$ by $\mathcal{F}'$ (though these yield a finite-dimensional space of extensions upon based-change to $\mathrm{Spec}(k)$).

It is natural to ask why quasi-coherent sheaves over monoid schemes are interesting and about possible applications of these ideas. One such motivation comes from the study of Hall algebras. Given an abelian category $\mathcal{C}$ with strong finiteness properties, one may define an algebra $\mathbb{H}(\mathcal{C})$ with basis the isomorphism classes of objects in $\mathcal{C}$, and whose structure constants count the number of extensions between objects (for an excellent introduction, see for instance [18]). Classical examples of $\mathcal{C}$ include categories of representations of a quiver over a finite field $\mathbb{F}_q$ as well as the category of coherent sheaves on a projective variety X over $\mathbb{F}_q$. In these cases $\mathbb{H}(\mathcal{C})$ recovers various quantum groups and other algebras important in representation theory. In the classical setting however, very little is known in the case when $\mathcal{C} = Coh(X)$ and $dim(X) > 1$. The case of curves (i.e. $dim(X) = 1$) is already very rich and intimately connected with the theory of automorphic forms over function fields. Results of the second author ([20]) suggest that Hall algebras of coherent sheaves on monoid schemes may be viewed as a $q \to 1$ limit of the finite field case, and that computations "over $\mathbb{F}_1$" may be used to study the more complicated situation over $\mathbb{F}_q$, especially when $dim(X) > 2$ and the latter appears out of reach. This will be taken up in future papers.

Coherent sheaves over monoid schemes have also been studied in [7], with a view towards calculating Picard and class groups. In that paper, the authors obtain general results relating these groups to those of the associated toric variety.

Acknowledgements: M.S. gratefully acknowledges the support of a Simons Foundation Collaboration Grant during the writing of this paper.

2. MONOID SCHEMES

In this section, we briefly recall the notion of a monoid scheme following [2,4], which we will use as our model for algebraic geometry over $\mathbb{F}_1$. This is essentially equivalent to the notion of $\mathfrak{M}_0$-scheme in the sense of [1]. For other (some much more general) approaches to schemes over $\mathbb{F}_1$, see [6,10,12,15,19,24]. Recall that ordinary schemes are ringed spaces locally modeled on affine schemes, which are spectra of commutative rings. A monoid scheme is locally modeled on an affine monoid scheme, which is the spectrum of a commutative unital monoid with 0. In the following, we will denote monoid multiplication by juxtaposition or "$\cdot$". In greater detail:

A *monoid* A will be a commutative associative monoid with identity 1_A and zero 0_A (i.e. the absorbing element). We require

$$1_A \cdot a = a \cdot 1_A = a \qquad 0_A \cdot a = a \cdot 0_A = 0_A \qquad \forall a \in A$$

Maps of monoids are required to respect the multiplication as well as the special elements $1_A, 0_A$. An *ideal* of A is a nonempty subset $\mathfrak{a} \subset A$ such that $A \cdot \mathfrak{a} \subset \mathfrak{a}$. An proper ideal $\mathfrak{p} \subset A$ is *prime* if $xy \in \mathfrak{p}$ implies either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Given a monoid A, $\mathrm{MSpec}(A)$ is defined to be the topological space with underlying set

$$\mathrm{MSpec}(A) := \{\mathfrak{p} | \mathfrak{p} \subset A \text{ is a prime ideal }\},$$

equipped with the Zariski topology, whose closed sets are of the form

$$V(\mathfrak{a}) := \{\mathfrak{p} | \mathfrak{a} \subset \mathfrak{p}, \mathfrak{p} \text{ prime }\},$$

where $\mathfrak{a}$ ranges over all ideals of A. Given a multiplicatively closed subset $S \subset A$, the *localization of A by S*, denoted $S^{-1}A$, is defined to be the monoid consisting of symbols $\{\frac{a}{s} | a \in A, s \in S\}$, with the equivalence relation

$$\frac{a}{s} = \frac{a'}{s'} \iff \exists\, s'' \in S \text{ such that } as's'' = a'ss'',$$

and multiplication is given by $\frac{a}{s} \times \frac{a'}{s'} = \frac{aa'}{ss'}$.

For $f \in A$, let S_f denote the multiplicatively closed subset $\{1, f, f^2, f^3, \cdots, \}$. We denote by A_f the localization $S_f^{-1}A$, and by $D(f)$ the open set $\mathrm{MSpec}(A) \backslash V(f) \simeq \mathrm{MSpec}\, A_f$, where $V(f) := \{\mathfrak{p} \in \mathrm{MSpec}(A) | f \in \mathfrak{p}\}$. The open sets $D(f)$ cover $\mathrm{MSpec}(A)$. $\mathrm{MSpec}(A)$ is equipped with a *structure sheaf* of monoids $\mathcal{O}_{\mathrm{MSpec}(A)}$, satisfying the property $\Gamma(D(f), \mathcal{O}_{\mathrm{MSpec}(A)}) = A_f$. Its stalk at $\mathfrak{p} \in \mathrm{MSpec}\, A$ is $A_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}A$, where $S_{\mathfrak{p}} = A \backslash \mathfrak{p}$.

A unital homomorphism of monoids $\phi : A \to B$ is *local* if $\phi^{-1}(B^{\times}) \subset A^{\times}$, where $A^{\times}$ (resp. $B^{\times}$) denotes the invertible elements in A (resp. B). A *monoid space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of monoids. A *morphism of monoid spaces* is a pair $(f, f^{\#})$ where $f : X \to Y$ is a continuous map, and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of monoids, such that the induced morphism on stalks $f^{\#}_{\mathfrak{p}} : \mathcal{O}_{Y,f(\mathfrak{p})} \to f_*\mathcal{O}_{X,\mathfrak{p}}$ is local. An *affine monoid scheme* is a monoid space isomorphic to $(\mathrm{MSpec}(A), \mathcal{O}_{\mathrm{MSpec}(A)})$. Thus, the category of affine monoid schemes is opposite to the category of monoids. A monoid space $(X, \mathcal{O}_X)$ is called a *monoid scheme*, if for every point $x \in X$ there is an open neighborhood $U_x \subset X$ containing x such that $(U_x, \mathcal{O}_X|_{U_x})$ is an affine monoid scheme. We denote by $\mathcal{M}sch$ the category of monoid schemes.

Example 1. Denote by $\langle t \rangle$ the free commutative unital monoid with zero generated by t, i.e.

$$\langle t \rangle := \{0, 1, t, t^2, t^3, \cdots, t^n, \cdots\},$$

and let $\mathbb{A}^1 := \mathrm{MSpec}(\langle t \rangle)$ - the monoid affine line. Let $\langle t, t^{-1} \rangle$ denote the monoid

$$\langle t, t^{-1} \rangle := \{\cdots, t^{-2}, t^{-1}, 1, 0, t, t^2, t^3, \cdots\}.$$

We obtain the following diagram of inclusions

$$\langle t \rangle \hookrightarrow \langle t, t^{-1} \rangle \hookleftarrow \langle t^{-1} \rangle.$$

Taking spectra, and denoting by $U_0 = \text{MSpec}(\langle t\rangle), U_\infty = \text{MSpec}(\langle t^{-1}\rangle)$, we obtain

$$\mathbb{A}^1 \simeq U_0 \hookleftarrow U_0 \cap U_\infty \hookrightarrow U_\infty \simeq \mathbb{A}^1$$

We define $\mathbb{P}^1$, the monoid projective line, to be the monoid scheme obtained by gluing two copies of $\mathbb{A}^1$ according to this diagram. It has three points - two closed points $0 \in U_0$, $\infty \in U_\infty$, and the generic point η. Denote by $\iota_0 : U_0 \hookrightarrow \mathbb{P}^1$, $\iota_\infty : U_\infty \hookrightarrow \mathbb{P}^1$ the corresponding inclusions.

2.1. **Base Change.** Given a commutative ring R, there exists a base-change functor

$$\begin{aligned} \mathcal{M}sch &\to \text{Sch} \,/\, \text{Spec}\, R \\ X &\to X_R \end{aligned}$$

It is defined on affine schemes by

$$(\text{MSpec}\, A)_R = \text{Spec}\, R[A]$$

where $R[A]$ is the monoid algebra:

$$R[A] := \left\{\sum r_i a_i | a_i \in A, a_i \neq 0, r_i \in R\right\}$$

with multiplication induced from the monoid multiplication. For a general monoid scheme X, X_R is defined by gluing the open affine subfunctors of X.

Remark 2.1.1. The base change construction may be extended to the case where R is a semiring, yielding a semiring scheme X_R in the sense of [14]. The case when $R = \mathbb{T}$, the tropical semifield, is of interest in tropical geometry, see [8].

3. A-MODULES

In this section, we briefly recall the properties of set-theoretic A-modules following [3].

Let A be a monoid. An *A–module* is a pointed set $(M, *_M)$ together with an action

$$\begin{aligned} \mu : A \times M &\to M \\ (a, m) &\to a \cdot m \end{aligned}$$

which is compatible with the monoid multiplication (i.e. $1_A \cdot m = m$, $a \cdot (b \cdot m) = (a \cdot b) \cdot m$) for $a, b \in A$, and $0_A \cdot m = *_M$ $\forall m \in M$). We will refer to elements of $M \backslash *_M$ as *nonzero* elements.

A *morphism of A–modules* $f : (M, *_M) \to (N, *_N)$ is a map of pointed sets (i.e. we require $f(*_M) = *_M$) compatible with the action of A, i.e. $f(a \cdot m) = a \cdot f(m)$.

A pointed subset $(M', *_M) \subset (M, *_M)$ is called an *A–sub-module* if $A \cdot M' \subset M'$. In this case we may form the quotient module M/M', where $M/M' := M \backslash (M' \backslash *_M)$, $*_{M/M'} = *_M$, and the action of A is defined by setting

$$a \cdot \overline{m} = \begin{cases} \overline{a \cdot m} & \text{if } a \cdot m \notin M' \\ *_{M/M'} & \text{if } a \cdot m \in M' \end{cases}$$

where $\overline{m}$ denotes m viewed as an element of M/M'. If M is finite, we define $|M| = \#M - 1$, i.e. the number of non-zero elements.

Denote by A$-$*mod* the category of A–modules. It has the following properties:

(1) A$-$*mod* has a zero object $0 = \{*\}$ - the one-element pointed set.
(2) A morphism $f : (M, *_M) \to (N, *_N)$ has a kernel $(f^{-1}(*_N), *_M)$ and a cokernel $N/\operatorname{Im}(f)$.
(3) A$-$*mod* has sums

$$M \oplus N := M \vee N := M \sqcup N / *_M \sim *_N.$$

(4) A$-$*mod* has products

$$M \times N := M \times N,$$

with basepoint $*_{M \times N} = (*_M, *_N)$ and diagonal A–action.
(5) If $R \subset M \oplus N$ is an A–submodule, then $R = (R \cap M) \oplus (R \cap N)$.
(6) A$-$*mod* has a symmetric monoidal structure

$$M \otimes_A N := M \times N / \sim,$$

where $\sim$ is the equivalence relation generated by

$$(a \cdot m, n) \sim (m, a \cdot n), \; a \in A,$$

with identity object $\{A\}$.
(7) $\oplus, \otimes$ satisfy the usual associativity and distributivity properties.

$M \in$ A$-mod$ is *finitely generated* if there exists a surjection $\oplus_{i=1}^{n} A \to M$ of A–modules for some n. Explicitly, this means that there are $m_1, \cdots, m_n \in M$ such that for every $m \in M$, $m = a \cdot m_i$ for some $1 \leq i \leq n$, and we refer to the m_i as *generators*. M is said to be *free of rank n* if $M \simeq \oplus_{i=1}^{n} A$. For an element $m \in M$, define

$$Ann_A(m) := \{a \in A | a \cdot m = *_M\}.$$

Obviously, $0_A \subset Ann_A(m)\ \forall m \in M$. An element $m \in M$ is *torsion* if $Ann_A(m) \neq \{0_A\}$. The subset of all torsion elements in M forms an A–submodule, called the *torsion submodule* of M, and denoted M_{tor}. An A–module is *torsion-free* if $M_{tor} = \{*_M\}$ and *torsion* if $M_{tor} = M$. We define the *length* of a torsion module M to be $|M|$.

4. Quasicoherent sheaves

In this section, we briefly recall the definitions and properties of quasicoherent and coherent sheaves over monoid schemes. We refer the reader to [3] for details.

Given a multiplicatively closed subset $S \subset A$ and an A–module M, we may form the $S^{-1}A$–module $S^{-1}M$, where

$$S^{-1}M := \{\frac{m}{s} |\ m \in M, s \in S\}$$

with the equivalence relation

$$\frac{m}{s} = \frac{m'}{s'} \iff \exists\ s'' \in S \text{ such that } s's'' \cdot m = ss'' \cdot m',$$

where the $S^{-1}A$–module structure is given by $\frac{a}{s} \cdot \frac{m}{s'} := \frac{a \cdot m}{ss'}$. For $f \in A$, we define M_f to be $S_f^{-1}M$.

Let X be a topological space, and $\mathcal{A}$ a sheaf of monoids on X. We say that a sheaf of pointed sets $\mathcal{M}$ is an *$\mathcal{A}$–module* if for every open set $U \subset X$, $\mathcal{M}(U)$ has the structure of an $\mathcal{A}(U)$–module with the usual compatibilities. In particular, given a monoid A and an A–module M, there is a sheaf of $\mathcal{O}_{\mathrm{MSpec}(A)}$–modules $\widetilde{M}$ on MSpec(A), defined on basic affine sets $D(f)$ by $\widetilde{M}(D(f)) := M_f$. For a monoid scheme X, a sheaf of $\mathcal{O}_X$–modules $\mathcal{F}$ is said to be *quasicoherent* if for every $x \in X$ there exists an open affine $U_x \subset X$ containing x and an $O_X(U_x)$–module M such that $\mathcal{F}|_{U_x} \simeq \widetilde{M}$. $\mathcal{F}$ is said to be *coherent* if M can always be taken to be finitely generated, and *locally free* if M can be taken to be free. Please note that here our conventions are different from [4]. If X is connected, we can define the *rank* of a locally free sheaf $\mathcal{F}$ to be the rank of the stalk $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$–module for any $x \in X$. A locally free sheaf of rank one will be called *invertible*.

Remark 4.0.1. The notion of coherent sheaf on ordinary schemes is well-behaved only for schemes that are locally Noetherian. The corresponding notion for monoid schemes is being of *finite type*, and is introduced at the end of section 5. We will consider coherent sheaves only on monoid schemes satisfying this property.

For a monoid A, there is an equivalence of categories between the category of quasicoherent sheaves on $\operatorname{Spec} A$ and the category of A–modules, given by $\Gamma(\operatorname{Spec} A, \cdot)$. A quasicoherent sheaf $\mathcal{F}$ on X is *torsion* (resp. *torsion-free*) if $\mathcal{F}(U)$ is a torsion $\mathcal{O}_X(U)$–module (resp. torsion-free $\mathcal{O}_X(U)$–module) for every open affine $U \subset X$.

The operations $\oplus, \otimes$ induce analogous ones on $\mathcal{O}_X$–modules. More precisely, for $\mathcal{O}_X$–modules $\mathcal{F}, \mathcal{G}$ and an open subset $U \subset X$ we define $\mathcal{F} \oplus \mathcal{G}(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$ with the obvious $\mathcal{O}_X(U)$-structure. $\mathcal{F} \otimes_{O_X} \mathcal{G}$ is defined to be the sheaf associated to the presheaf $U \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. If $X = \operatorname{MSpec}(A)$ is affine, and M, N are A–modules, we have $\widetilde{M \oplus N} = \widetilde{M} \oplus \widetilde{N}$ and $\widetilde{M \otimes_A N} = \widetilde{M} \otimes_{\mathcal{O}_{\operatorname{MSpec}(A)}} \widetilde{N}$. This implies that on an arbitrary monoid scheme X, quasicoherent and coherent sheaves are closed under $\oplus$ and $\otimes$.

Remark 4.0.2. It follows from property (5) of the category A$-$*mod* that if $\mathcal{F}, \mathcal{F}'$ are quasicoherent $\mathcal{O}_X$–modules, and $\mathcal{G} \subset \mathcal{F} \oplus \mathcal{F}'$ is an $\mathcal{O}_X$–submodule, then $\mathcal{G} = (\mathcal{G} \cap \mathcal{F}) \oplus (\mathcal{G} \cap \mathcal{F}')$, where for an open subset $U \subset X$,

$$(\mathcal{G} \cap \mathcal{F})(U) := \mathcal{G}(U) \cap \mathcal{F}(U). \tag{1}$$

If X is a monoid scheme, we will denote by $QCoh(X)$ (resp. $Coh(X)$) the category of quasicoherent (resp. coherent) $\mathcal{O}_X$–modules on X. It follows from the properties of the category A$-$*mod* listed in section 3 that $QCoh(X)$ possesses a zero object $\emptyset$ (defined as the zero module $\emptyset$ on each open affine $\operatorname{MSpec} A \subset X$), kernels and co-kernels, as well as monoidal structures $\oplus$ and $\otimes$ (see [3] for more details on $QCoh(X)$). We may therefore talk about exact sequences in $QCoh(X)$. More precisely, a sequence

$$\mathcal{F} \overset{f}{\mapsto} \mathcal{G} \overset{g}{\mapsto} \mathcal{H} \tag{2}$$

is *exact* in the middle if for every $x \in X$, the sequence of stalks (viewed in $\mathcal{O}_{X,x}-$ $-mod$).

$$\mathcal{F}_x \overset{f_x}{\mapsto} \mathcal{G}_x \overset{g_x}{\mapsto} \mathcal{H}_x$$

has the property $Im(f_x) = Ker(g_x)$. As shown in [3], (2) is exact in the middle iff for every open affine $U \subset X$,

$$\mathcal{F}(U) \overset{f(U)}{\mapsto} \mathcal{G}(U) \overset{g(U)}{\mapsto} \mathcal{H}(U)$$

is exact as a sequence in $\mathcal{O}_X(U)$–mod. This means in particular that

$$\mathcal{F} \overset{f}{\mapsto} \mathcal{G} \mapsto 0$$

is exact in the middle iff for every open affine $U \subset X$,

$$\mathcal{F}(U) \overset{f(U)}{\mapsto} \mathcal{G}(U)$$

is surjective. We note that if $\mathcal{L}$ is locally free, and

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{3}$$

is a short exact sequence, then

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{H} \otimes_{\mathcal{O}_X} \mathcal{L} \to 0$$

is also short exact. It suffices to establish this locally on affines $U = \mathrm{MSpec}(A)$ where $\mathcal{L} \cong \mathcal{O}_X^{\oplus n}$ and (3) is given by a short exact sequence in A$-mod$

$$0 \to M \to N \to P \to 0$$

The statement now follows from the isomorphism $M \otimes_A A \cong M$.

A short exact sequence isomorphic to one of the form

$$0 \to \mathcal{F} \to \mathcal{F} \oplus \mathcal{G} \to \mathcal{G} \to 0$$

is called *split*. A coherent sheaf $\mathcal{F}$ which cannot be written as $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ for non-zero coherent sheaves is called *indecomposable*. A coherent sheaf containing no non-zero proper sub-sheaves is called *simple*.

4.1. **Base Change.** Given an A–module M, and a commutative ring R, let

$$R[M] := \left\{\sum r_i m_i | m_i \in M, m_i \neq *, r_i \in R\right\}$$

$R[M]$ naturally inherits the structure of an $R[A]$–module. We may use this to define a base extension functor

$$QCoh(X) \to QCoh(X_R)$$
$$\mathcal{F} \to \mathcal{F}_R$$

It is defined on affines by assigning to $\widetilde{M}$ on MSpec A the quasicoherent sheaf

$$\widetilde{M}_R := \widetilde{R[M]}$$

on $\text{MSpec}(A)_R = \text{Spec}(R[A])$, and for a general monoid scheme by gluing in the obvious way.

Given a monoid scheme X and $\mathcal{F} \in QCoh(X)$, we have for each open $U \subset X$ a map

$$\phi_R(U) : R[\Gamma(U, \mathcal{F})] \to \Gamma(U_R, \mathcal{F}_R) \tag{4}$$

defined as the unique R–linear map with the property that $\phi_R(U)(s) = s \ \forall s \in \Gamma(U, \mathcal{F})$. When U is understood, we will refer to this map simply as ϕ_R.

Remark 4.1.1. As in Remark 2.1.1, this construction may be generalized to the case when R is a semiring, and yields a quasicoherent sheaf $\mathcal{F}_R$ over the semiring scheme X_R in the sense of [11, 14]. When K is a valued field with a valuation $\nu : K \mapsto R$ to an idempotent semiring, and $Y \subset X_K$ is closed subscheme, the construction in [8] realizes the tropicalization of Y as certain sub-scheme $\widetilde{Y} \subset X_R$. One may then produce a quasicoherent sheaf on $\widetilde{Y}$ by restricting $\mathcal{F}_R$.

5. Projective schemes over $\mathbb{F}_1$

In this section, we briefly recall the construction of the projective monoid scheme MProj(A) attached to a graded monoid A, following [2] (see also [22] and [13] for a more general construction in the context of blueprints). It is a straightforward analogue of the Proj construction for graded commutative rings.

Let $A = \oplus_{i=0}^{\infty} A_i$ be an $\mathbb{N}$–graded monoid (i.e. $A_i \cdot A_j \subset A_{i+j}$). $A_{\geq 1} = \oplus_{i \geq 1} A_i$ is therefore an ideal, and the map $A \to A/A_{\geq 1} \simeq A_0$ induces a map $\text{MSpec}(A_0) \to \text{MSpec}(A)$, whose image consists of all the prime ideals of A containing $A_{\geq 1}$. Let

MProj(A) denote the topological space MSpec$(A) \setminus$ MSpec(A_0) with the induced Zariski topology.

Given a multiplicatively closed subset S of A, the localization $S^{-1}A$ inherits a natural $\mathbb{Z}$–grading by $deg(\frac{a}{b}) = deg(a) - deg(b)$. For an element $f \in A$, let $A_{(f)}$ denote the elements of degree 0 in the localization A_f. Similarly, given a prime ideal $\mathfrak{p} \in A$, let $A_{(\mathfrak{p})}$ denote the elements of degree 0 in $A_{\mathfrak{p}}$. For $f \in A$, we may, as in the case or ordinary schemes, identify $D_+(f) := \text{MSpec}(A_{(f)})$ with the open subset $\{\mathfrak{p} \in \text{MProj}(A) | f \notin \mathfrak{p}\}$ - these cover MProj(A). Finally, we equip $X = \text{MProj}(A)$ with a monoid structure sheaf $\mathcal{O}_X$ defined by the property that $\mathcal{O}_X|_{D_+(f)} \simeq \widetilde{A_{(f)}}$. $(X, \mathcal{O}_X)$ thus acquires the structure of a monoid scheme locally isomorphic to $(\text{MSpec}(A_{(f)}), \mathcal{O}_{A_{(f)}})$. The stalk of $\mathcal{O}_{X,\mathfrak{p}}$ at $\mathfrak{p} \in \text{MProj}(A)$ is $A_{(\mathfrak{p})}$.

Remark 5.0.1. Note that MProj(A) is naturally a monoid scheme over MSpec(A_0).

Definition 5.0.2. Let B be a monoid. Let $A = B\langle x_1, x_2, \cdots, x_m\rangle$ denote the graded monoid with $A_0 = B$ and $A_n = \{bx_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}\}$ where $b \in B$, $i_j \geq 0$, and $i_1 + i_2 + \cdots + i_m = n$ (i.e. the x_i each have degree 1), with multiplication

$$(bx_1^{i_1}x_2^{i_2} \cdots x_m^{i_m}) \cdot (b'x_1^{j_1}x_2^{j_2} \cdots x_m^{j_m}) = bb'x_1^{i_1+j_1}x_2^{i_2+j_2} \cdots x_m^{i_m+j_m}.$$

When $B = \{0, 1\}$, we write $B\langle x_1, x_2, \cdots, x_m\rangle$ simply as $\langle x_1, x_2, \cdots, x_m\rangle$.

Example 2. Let $A = \langle t_0, t_1\rangle$. Then MProj$(A) \simeq \mathbb{P}^1$. More generally, we define $\mathbb{P}^n$ to be MProj$(\langle t_0, t_1, \cdots, t_n\rangle)$.

Given a graded monoid A and commutative ring R, the monoid algebra $R[A]$ acquires the structure of a graded R–algebra (by assigning elements of R degree 0). As shown in [2], we have

Lemma 5.0.3. MProj$(A)_R \simeq$ Proj$(R[A])$

As indicated in Remark 4.0.1, coherent sheaves are well-behaved only on monoid schemes satisfying a local finiteness condition. We recall this property following [2],[3].

Definition 5.0.4. A monoid scheme X is of *finite type* if every $x \in X$ has an open affine neighborhood $U_x = \text{MSpec}(A_x)$ with A_x a finitely generated monoid.

The following lemma is immediate:

Lemma 5.0.5. (1) *Let A be a graded monoid. If A is finitely generated, then* $\mathrm{MProj}(A)$ *is of finite type.*
(2) *Let B be a finitely generated monoid. Then for every $m \geq 0$,* $\mathrm{MProj}(B\langle x_1, x_2, \cdots, x_m\rangle)$ *is of finite type.*

6. QUASICOHERENT SHEAVES ON PROJECTIVE SCHEMES OVER $\mathbb{F}_1$

Let A be an $\mathbb{N}$–graded monoid, $X = \mathrm{MProj}(A)$, and $M = \oplus_{i\in\mathbb{Z}} M_i$ a $\mathbb{Z}$–graded A–module (an A–module such that $A_i \cdot M_j \subset M_{i+j}$). For a multiplicatively closed subset $S \subset A$, the localization $S^{-1}M$ inherits a natural grading by $deg(\frac{m}{s}) = deg(m) - deg(s)$, and as in the previous section we use the notation $M_{(f)}$ and $M_{(\mathfrak{p})}$ to denote the degree zero elements in $S_f^{-1}M$ and $S_{\mathfrak{p}}^{-1}M$ respectively. We may associate to M a quasi-coherent sheaf $\widetilde{M}$ on X such that for an open subset $U \subset X$, $\widetilde{M}(U)$ consists of functions $U \to \sqcup_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$ which are locally induced by fractions of the form $\frac{m}{s}$. As for ordinary Proj, one readily checks that $\widetilde{M}|_{D_+(f)} \simeq \widetilde{M_{(f)}}$, and that $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$. If A and M are finitely generated (as a monoid and A–module respectively), then $\widetilde{M}$ is coherent.

Remark 6.0.1. Note that the space of global secitons $\Gamma(X, \widetilde{M})$ is naturally an A_0–module.

Let $grA - mod$ denote the category of graded A–modules whose morphisms are grading-preserving maps of A–modules. The assignment $M \to \widetilde{M}$ defines a functor $grA - mod \to Qcoh(X)$.

Given a $\mathbb{Z}$–graded A–module M and $n \in \mathbb{Z}$, let $M(n)$ denote the graded A–module defined by $M(n)_i := M_{i+n}$.

Definition 6.0.2. For $n \in \mathbb{Z}$, denote by $\mathcal{O}_X(n)$ the sheaf $\widetilde{A(n)}$ of $\mathcal{O}_X$–modules on $X = \mathrm{MProj}(A)$. More generally, for a sheaf $\mathcal{F}$ of $\mathcal{O}_X$–modules, denote $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ by $\mathcal{F}(n)$.

Let us now assume that A is generated by A_1 over A_0, and that A_1 is finite. $X = \mathrm{MProj}(A)$ then has a finite affine cover of the form $\{D_+(f)\}, f \in A_1$, with $\mathcal{O}_X|_{D_+(f)} \simeq \widetilde{A_{(f)}}$.

Lemma 6.0.3. *Let* $X = \mathrm{MProj}(A)$*.*

(1) *The sheaf* $\mathcal{O}_X(n)$ *is locally free.*
(2) *For graded* A*–modules* M *and* N*,* $\widetilde{M \otimes_A N} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$*.*
(3) $\widetilde{M}(n) \simeq \widetilde{M(n)}$*. In particular,* $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \simeq \mathcal{O}_X(m+n)$

Proof. (1) Let $f \in A_1$. We have $\mathcal{O}_X(n)|_{D_+(f)} = \widetilde{A(n)_{(f)}}$, with $A(n)_{(f)} = \{\frac{a}{f^d}\}$, where $a \in A(n)_d = A(n+d)$. Since f is invertible in $A_{(f)}$, division by f^n defines an isomorphism of $A_{(f)}$–modules from $A(n)_{(f)}$ to $A_{(f)}$, inducing an isomorphism of $\mathcal{O}_X$ modules

$$\mathcal{O}_X(n)|_{D_+(f)} \to \widetilde{A_{(f)}} = \mathcal{O}_X|_{D_+(f)}.$$

Since A_1 generates A over A_0, the $D_+(f)$ cover X, and so the claim follows.

For (2), we have $\widetilde{M \otimes_A N}|_{D_+(f)} = M \widetilde{\otimes_A N}_{(f)}$, and $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}|_{D_+(f)} = M_{(f)} \widetilde{\otimes_{A_{(f)}}} N_{(f)}$. Given $m \in M_i, n \in N_j$, we can send $\frac{m \otimes n}{f^{i+j}} \in M \otimes_A N_{(f)}$ to $\frac{m}{f^i} \otimes \frac{n}{f^j} \in M_{(f)} \otimes_{A_{(f)}} N_{(f)}$. This defines an isomorphism of $A_{(f)}$–modules. These are easily seen to glue, yielding the desired isomorphism of quasicoherent sheaves.

(3) This follows from (2). For the first part, take $N = A(n)$. The second follows by further specializing to $M = A(m)$.

□

Remark 6.0.4. In [7], the authors study the Picard and class groups of monoid schemes. It shown, among other things, that the Picard group of a monoid scheme agrees with that of its associated toric variety. Thus for instance, $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$.

Definition 6.0.5. Let A be a graded monoid, $X = \mathrm{MProj}(A)$ and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$–modules. Let $\Gamma_*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$. $\Gamma_*(\mathcal{F})$ has the structure of a $\mathbb{Z}$–graded A–module by placing $\Gamma(X, \mathcal{F}(n))$ in degree n, and defining the action

$$A_i \times \Gamma(X, \mathcal{F}(n)) \to \Gamma(X, \mathcal{F}(n+i))$$

by identifying A_i with global sections of $\mathcal{O}_X(i)$ and using the isomorphism $\mathcal{F}(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(i) \simeq \mathcal{F}(n+i)$.

Theorem 6.0.6. *Let B be a monoid, and $A = B\langle x_0, \cdots, x_r\rangle$, $r \geq 1$. Let $X = \mathrm{MProj}(A)$. Then $\Gamma(X, \mathcal{O}_X(n)) = A_n$.*

Proof. We have $\Gamma(D_+(x_i), \mathcal{O}_X(n)) = A(n)_{(x_i)}$, which is the set of degree n elements in the localization A_{x_i}. Thus a global section of $\mathcal{O}_X(n)$ consists of a tuple $(t_0, \cdots, t_r), t_i \in A_{x_i}, deg(t_i) = n$, such that $t_i|_{D_+(x_ix_j)} = t_j|_{D_+(x_ix_j)}$ for all $i \neq j$. This is equivalent to requiring that $t_i = t_j$ in $A_{x_ix_j}$. For $g = x_0^{k_0} \cdots x_r^{k_r}$, natural map $A_g \to A_{gx_i}$ is an inclusion for each i. A_{x_i} and $A_{x_ix_j}$ are therefore naturally submonoids of $A_{x_0\cdots x_r}$. It follows that $\Gamma(X, \mathcal{O}_X(n)) = \{h \in \cap_{i=0}^r A_{x_i} | deg(h) = n\}$. An element of $A_{x_0\cdots x_n}$ can be written uniquely in the form $bx_0^{k_0} \cdots x_r^{k_r}, b \in B, k_j \in \mathbb{Z}$, and lies in A_{x_i} if and only if $k_j \geq 0$ for all $j \neq i$. The result follows.

□

Corollary 6.0.7. *Let $X = \mathrm{MProj}(B\langle x_0, \cdots, x_r\rangle)$. Then $\Gamma_*(X, \mathcal{O}_X(n)) = A(n)$.*

Proof. $\Gamma_*(X, \mathcal{O}_X(n)) = \oplus_{m\in\mathbb{Z}}\Gamma(X, \mathcal{O}_X(m+n)) = \oplus_{m\in\mathbb{Z}} A_{m+n}$, where A_{m+n} occurs in degree m. The result follows. □

Lemma 6.0.8. *Let A be a monoid, $\mathcal{F}$ a quasicoherent sheaf on $X = \mathrm{MSpec}(A)$, and $f \in A$.*

(1) *If $s_1, s_2 \in \Gamma(X, \mathcal{F})$ are global sections such that $s_1|_{D(f)} = s_2|_{D(f)}$, then $f^n s_1 = f^n s_2 \in \Gamma(X, \mathcal{F})$ for some positive integer n.*
(2) *If $s \in \Gamma(D(f), \mathcal{F})$ then $f^n s$ extends to a global section in $\Gamma(X, \mathcal{F})$ for some positive integer n.*

Proof. (1) Since $\mathcal{F}$ is quasicoherent, $\mathcal{F} = \widetilde{M}$ for some A–module M, and s_1, s_2 can be identified with elements of M. $s_i|_{D(f)} = \frac{s_i}{1} \in M_f$. The hypothesis then implies that $f^n s_1 = f^n s_2 \in M$ by the definition of the localized module M_f.

(2) We can identify $s \in \Gamma(D(f), \mathcal{F})$ with an element of the form $\frac{m}{f^l} \in M_f$ for some $l \in \mathbb{Z}_{\geq 0}$. Then $f^l s = m \in M = \Gamma(X, \mathcal{F})$. □

Lemma 6.0.9. *Let A be a graded monoid, finitely generated by A_1 over A_0, $f \in A_1$, and $\mathcal{F}$ a quasicoherent sheaf on $X = \mathrm{MProj}(A)$.*

(1) *If* $s_1, s_2 \in \Gamma(X, \mathcal{F})$ *are global sections such that* $s_1|_{D_+(f)} = s_2|_{D_+(f)}$*, then there is a positive integer* n *such that* $f^n s_1 = f^n s_2$ *viewed as global sections in* $\Gamma(X, \mathcal{F}(n))$.

(2) *Given a section* $s \in \Gamma(D_+(f), \mathcal{F})$*, there is a positive integer* m *such that* $f^m s$ *extends to a global section of* $\mathcal{F}(m)$

Proof. (1) Denote by $x_1, \cdots, x_r$ the elements of A_1. The sets $D_+(x_i)$ then form a finite open affine cover of X. Let $u_i = \frac{f}{x_i} \in A_{(x_i)}$. The intersection $D_+(f) \cap D_+(x_i) = D_+(fx_i)$, viewed as a subset of $D_+(x_i)$ is then the distinguished open $D(u_i)$. We have $s_1 = s_2$ on $D(u_i)$, and by part (1) of lemma 6.0.8, there is a positive integer n_i such that $u_i^{n_i} s_1 = u_i^{n_i} s_2$ on $D_+(x_i)$. Take $n \geq n_i$ for $i = 1, \cdots, r$.

Viewing $\frac{1}{x_i}$ as a section in $\Gamma(D_+(x_i), \mathcal{O}_X(-1))$, let $\rho_i : \mathcal{F}(n)|_{D_+(x_i)} \to \mathcal{F}|_{D_+(x_i)}$ be the isomorphism of "dividing by x_i^n". I.e. $\rho_{i,n}(s) = s \otimes \frac{1}{x_i^n}$ for $s \in \Gamma(D_+(x_i), \mathcal{F}(n))$. Viewing $f^n s_i = f^n \otimes s$ as elements in $\Gamma(X, \mathcal{F}(n))$, we have $\rho_{i,n}(f^n s_1) = f^n s_1 \otimes \frac{1}{x_i^n} = u_i^n s_1 = u_i^n s_2 = \rho_{i,n}(f^n s_2)$ for every i. It follows that $f^n s_1 = f^n s_2$ in $\Gamma(X, \mathcal{F}(n))$.

(2) Consider $s|_{D_+(x_i) \cap D_+(f)}$. By part (2) of lemma 6.0.8, there is a positive integer n_i such that $u_i^{n_i} s = q_i|_{D_+(x_i) \cap D_+(f)}$ for some $q_i \in \Gamma(D_+(x_i), \mathcal{F})$. Take $n \geq n_i$ for $i = 1, \cdots, r$, and let $t_i = \rho_{i,n}^{-1}(q_i) \in \Gamma(D_+(x_i), \mathcal{F}(n))$. We have $t_i|_{D_+(f) \cap D_+(x_i)} = f^n s$, thus $t_i = t_j$ on $D_+(x_i) \cap D_+(x_j) \cap D_+(f)$. By part (1), there is a positive integer k_{ij} such that $f^{k_{ij}} t_i = f^{k_{ij}} t_j$ in $\Gamma(D_+(x_i) \cap D_+(x_j), \mathcal{F}(n + k_{ij}))$. Taking $k \geq k_{ij}$ for all pairs i, j, we see that the sections $f^k t_i \in \Gamma(D_+(x_i), \mathcal{F}(n+k))$ glue to yield a global section whose restriction to $D_+(f)$ is $f^{n+k} s$. The result follows taking $m = n + k$. □

Theorem 6.0.10. *Let* A *be a graded monoid finitely generated by* A_1 *over* A_0*, and let* $X = \underline{\mathrm{MProj}}(A)$*. Given a quasicoherent sheaf* $\mathcal{F}$ *on X, there exists a natural isomorphism* $\beta : \widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}$.

Proof. Let $f_1, \cdots, f_r \in A_1$ be a set of generators for A over A_0. Since $\widetilde{\Gamma_*(\mathcal{F})}$ is quasicoherent, it suffices to specify isomorphisms of $A_{(f_i)}$–modules

$$\beta_i : \Gamma(D_+(f_i), \widetilde{\Gamma_*(\mathcal{F})}) \to \Gamma(D_+(f_i), \mathcal{F}),$$

and check these glue. A section on the left is represented by a fraction of the form $\frac{t}{f_i^d}$, where $t \in \Gamma(X, \mathcal{F}(d))$. Let $\beta_i(\frac{t}{f_i^d}) = t \otimes f_i^{-d}$, where f_i^{-d} is viewed as a section of $\mathcal{O}_X(-d)$, and we use the isomorphism $\mathcal{F}(d) \otimes \mathcal{O}_X(-d) \simeq \mathcal{F}$. It is immediate that $\beta_i = \beta_j$ on $D_+(f_i f_j) = D_+(f_i) \cap D_+(f_j)$.

We now verify that β_i is an isomorphism for all i. Let $s \in \Gamma(D_+(f_i), \mathcal{F})$. By the second part of 6.0.9, $f_i^n s$ extends to a section of $\Gamma(X, \mathcal{F}(n))$ for some $n > 0$. We have that $\beta_i(f_i^n s) = s$, so β_i is surjective. To show injectivity, suppose $\beta_i(\frac{t_1}{f_i^d}) = \beta_i(\frac{t_2}{f_i^d})$. By the first part of 6.0.9, there is an $n > 0$ such that $f_i^{n-d} t_1 = f_i^{n-d} t_2$ as global sections of $\mathcal{F}(n)$. It follows that $\frac{t_1}{f_i^d} = \frac{t_2}{f_i^d}$ in $\Gamma(D_+(f_i), \widetilde{\Gamma_*(\mathcal{F})})$.

□

Remark 6.0.11. We note that just as in the case of ordinary schemes, the graded A–module M giving rise to $\mathcal{F} = \widetilde{M}$ is not unique. Let $M_{\geq d} = \oplus_{i \geq d} M_i$. This is a graded A–submodule of M. Define an equivalence relation $\sim$ on graded A–modules by declaring $M \sim M'$ if there exists an integer d such that $M_{\geq d} \simeq M'_{\geq d}$ as graded A–modules. We then have the following result, proved exactly as for ordinary schemes.

Lemma 6.0.12. *Let M, M' be two graded A–modules such that $M \sim M'$. Then $\widetilde{M} \simeq \widetilde{M'}$ as quasicoherent $\mathcal{O}_X$–modules.*

6.1. **Global generation of twists.**

Definition 6.1.1. Let X be a monoid scheme and $\mathcal{F}$ and $\mathcal{O}_X$–module. We say that $\mathcal{F}$ is *generated by* $\{s_i\}_{i \in I} \in \Gamma(X, \mathcal{F})$ if for each $x \in X$, the stalk $\mathcal{F}_x$ is generated by $\{s_{i,x}\}_{i \in I}$ as an $\mathcal{O}_{X,x}$–module.

Remark 6.1.2. If $\mathcal{F}$ is a coherent sheaf on an affine monoid scheme $X = \operatorname{MSpec} A$ of finite type, then $\mathcal{F} = \widetilde{M}$ for a finitely generated A–module $M = \Gamma(X, \mathcal{F})$. $\mathcal{F}$ is thus generated by finitely many global sections (the generators of M).

Theorem 6.1.3. *Let A_0 be a finitely generated monoid, A a graded monoid finitely generated by A_1 over A_0, and $\mathcal{F}$ a coherent sheaf on $X = \operatorname{MProj}(A)$. Then there exists n_0 such that $\mathcal{F}(n)$ is generated by finitely many global sections for all $n \geq n_0$.*

Proof. Let $f_1, \cdots, f_r \in A_1$ be a set of generators for A over A_0. Since $\mathcal{F}$ is coherent, there is for each $i = 1, \cdots, r$ an $m_i \geq 0$ and $s_{i1}, \cdots, s_{im_i} \in \Gamma((D_+(f_i), \mathcal{F})$ which generate $\mathcal{F}|_{D_+(f_i)}$. By Lemma 6.0.9, there is an $n_0 \geq 0$ such that $f^{n_0} s_{ij}$ extend to global sections in $\Gamma(X, \mathcal{F}(n_0))$ for $1 \leq i \leq r, 0 \leq j \leq m_i$. Since

$$\Gamma((D_+(f_i), \mathcal{F}(n)) = f_i^n \otimes \Gamma(D_+(f_i), \mathcal{F}),$$

it follows that $f^n s_{ij}$ generate $\mathcal{F}(n)$ for all $n \geq n_0$. □

Corollary 6.1.4. *With the hypotheses of Theorem 6.1.3, there exist integers $m \in \mathbb{Z}, k \geq 0$, such that $\mathcal{F}$ is a quotient of $\mathcal{O}_X(m)^{\oplus k}$*

Proof. By Theorem 6.1.3, for large enough n, $\mathcal{F}(n)$ is generated by global sections, say k of them $s_1, \cdots s_k$. We thus have a surjection of $\mathcal{O}_X$–modules

$$\mathcal{O}_X^{\oplus k} \to \mathcal{F}(n) \to 0$$

As explained in section 4, Tensoring this sequence with the invertible sheaf $\mathcal{O}_X(-n)$ preserves surjectivity, and we obtain

$$\mathcal{O}_X(-n)^{\oplus k} \to \mathcal{F} \to 0,$$

proving the result. □

6.2. **Finiteness of global sections of coherent sheaves.** One of the key results about coherent sheaves on projective schemes is the finite-dimensionality of the space of global sections. In this section, we proceed to prove the $\mathbb{F}_1$–analog of this.

Theorem 6.2.1. *Let A_0 be a finitely generated monoid, A a graded monoid finitely generated by A_1 over A_0, and $\mathcal{F}$ a coherent sheaf on $X = \mathrm{MProj}(A)$. Then $\Gamma(X, \mathcal{F})$ is a finitely generated A_0–module. In particular, when $A_0 = \{0, 1\}$, $\Gamma(X, \mathcal{F})$ is a finite pointed set.*

Proof. Let K be a field. Base changing to K yields a coherent sheaf $\mathcal{F}_K$ over the Noetherian projective scheme X_K, and so $\Gamma(X_K, \mathcal{F}_K) = H^0(X_K, \mathcal{F}_K)$ is a finitely generated $K[A_0]$–module, hence Noetherian. Consider the base-change map on global sections

$$\phi_K : K[\Gamma(X, \mathcal{F})] \to \Gamma(X_K, \mathcal{F}_K)$$

This is a $K[A_0]$–module homomorphism. The codomain of ϕ_K is Noetherian, and thus so is

$$K[\Gamma(X, \mathcal{F})]/Ker(\phi_K).$$

Let $f_1, \cdots, f_r \in A_1$ be a set of generators for A. $D_+(f_i)$ therefore form a finite affine cover of X. Let $T_i \subset \Gamma(D_+(f_i), \mathcal{F})$ denote the image of the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(D_+(f_i), \mathcal{F}), i = 1 \cdots r$. We have

$$\Gamma(X, \mathcal{F}) \subset T_1 \times T_2 \times \cdots \times T_r.$$

Restriction to $D_+(f_i)$ induces a surjective homomorphism of $K[A_0]$–modules

$$K[\Gamma(X, \mathcal{F})]/Ker(\phi_K) \to K[T_i],$$

which implies that $K[T_i]$, being a quotient of a Noetherian $K[A_0]$–module, is Noetherian as well. Now, given a A_0–module T, $K[T]$ is finitely generated over $K[A_0]$ if and only if T is finitely generated over A_0. This shows that T_i is finitely generated over A_0 for each i, and thus so is the product $T_1 \times T_2 \times \cdots \times T_r$. $\Gamma(X, \mathcal{F})$, being A_0–submodule of a finitely generated A_0–module is therefore finitely generated itself. □

Remark 6.2.2. As demonstrated by Example (3), $Ker(\phi_K)$ is non-zero in general.

7. Classification of coherent sheaves on $\mathbb{P}^1$

In this section, we undertake the classification of coherent sheaves on $\mathbb{P}^1 = \mathrm{MProj}(\langle t_0, t_1 \rangle)$, the simplest projective monoid scheme. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$ is obtained by gluing coherent sheaves on two copies of $\mathbb{A}^1$ along their intersection, so we begin there.

Remark 7.0.1. In [20], a certain subcategory of *normal sheaves* of $QCoh(\mathbb{P}^1)$ was considered, and used to define the Hall algebra of $\mathbb{P}^1$.

7.1. Coherent sheaves on $\mathbb{A}^1$.

A coherent sheaf $\mathcal{F}$ on $\mathbb{A}^1$ can be described uniquely as $\mathcal{F} = \widetilde{M}$, where M is a finitely generated $\langle t \rangle$–module. We may associate to M a directed graph Γ_M whose vertices are the underlying set of $M \backslash *_M$, with directed edges from m to $t \cdot m$ for every $m \in M \backslash *_M$. Γ_M thus completely describes the isomorphism class of M. We note that every vertex of Γ_M has at most one out-going edge, and call a vertex a *leaf* if it has no incoming edges, and a *root* if it has no outgoing edges. It follows that elements of M corresponding to leaves of Γ_M form a minimal system of generators for M as a $\langle t \rangle$–module. If M, N are $\langle t \rangle$–modules, then $\Gamma_{M \oplus N} = \Gamma_M \amalg \Gamma_N$ - i.e. direct sums of $\langle t\rangle$–modules (or equivalently coherent sheaves on $\mathbb{A}^1$ correspond to disjoint unions of graphs). In view of these observations, the following lemma is obvious:

Lemma 7.1.1. *Let M be a finitely generated $\langle t\rangle$–module, and $\mathcal{F} = \widetilde{M}$ the corresponding coherent sheaf on $\mathbb{A}^1$. Then*

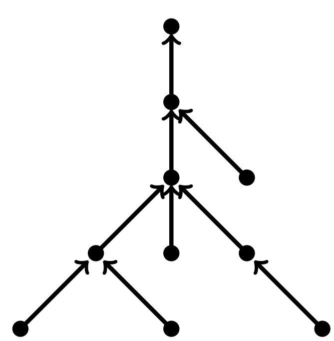
1. *$\mathcal{F}$ is indecomposable iff Γ_M is connected.*
2. *Γ_M has finitely many leaves.*

The classification of coherent sheaves on $\mathbb{A}^1$ amounts to the classification of the isomorphism classes of the graphs Γ_M (up to isomorphism of directed graphs), which was undertaken in [21]. Since every finitely generated $\langle t\rangle$–module can be uniquely expressed as a finite direct sum of indecomposable ones (up to reordering), it suffices to classify the latter.

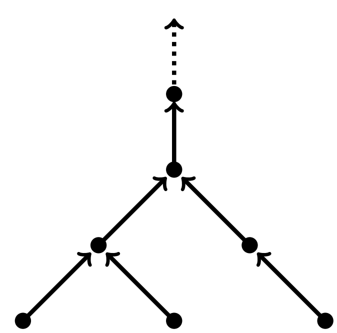
Definition 7.1.2. Let Γ be a connected directed graph with finitely many leaves, and with each vertex having at most one out-going edge. We say that

1. Γ is of *t*ype 1 if it is a rooted tree - i.e. the underlying undirected graph of Γ is a tree possessing a unique root, such there is a unique directed path from every vertex to the root.
2. Γ is of *t*ype 2 if it is obtained by joining a rooted tree to the initial vertex of $\Gamma_{\langle t\rangle}$.
3. Γ is of *type 3* if it is obtained from a directed cycle by attaching rooted trees to an oriented cycle.

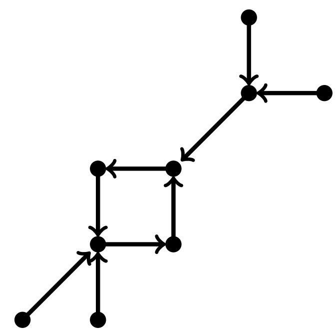
Examples of each type are given below:



Type 1



Type 2



Type 3

We then have the following classification result:

Theorem 7.1.3. *Let M be a non-trivial finitely generated indecomposable $\langle t\rangle$–module. Then Γ_M is of type 1, 2, or 3.*

Proof. When Γ_M is a finite graph, It is shown in [21] that it must be of type 1 or 3. We may therefore assume that Γ_M is infinite. It is proven in [21] that Γ_M contains at most one cycle - necessarily oriented. However, it is clear that if Γ_M has a cycle and finitely many leaves, it must be finite. Γ_M is therefore an infinite tree. If $m, m' \in M$ are elements corresponding to leaves of Γ_M (and are therefore members of a minimal generating set of M), there are n, n', such that $t^n \cdot m = t^{n'} \cdot m'$. Consequently, there is a vertex v of Γ_M such that every directed path starting at a leaf eventually passes through v. Γ_M is then of type 2. □

Note that a finitely generated $\langle t\rangle$–module M is torsion iff every connected component of Γ_M is of type 1, and torsion-free iff every connected component is of type 2 or 3.

Remark 7.1.4. While a type 3 sheaf $\mathcal{F}$ is torsion-free over $\mathbb{F}_1$, its base-change $\mathcal{F}_k$ to a field k with sufficiently many roots of unity (in particular, if k is algebraically closed) is a torsion sheaf, supported at 0 and roots of unity.

7.2. **Coherent sheaves on $\mathbb{P}^1$.** Specifying a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$ amounts to specifying coherent sheaves $\mathcal{F}', \mathcal{F}''$ on

$$U_1 = \mathrm{MSpec}(\langle t\rangle) \simeq \mathbb{A}^1 \text{ and } U_2 = \mathrm{MSpec}(\langle t^{-1}\rangle) \simeq \mathbb{A}^1$$

respectively, together with a gluing isomorphism

$$\varphi : \mathcal{F}'|_{U_1 \cap U_2} \simeq \mathcal{F}''|_{U_1 \cap U_2}$$

on $U_1 \cap U_2 = \mathrm{MSpec}(\langle t, t^{-1} \rangle)$. We denote the defining tripe of $\mathcal{F}$ by $(\mathcal{F}', \mathcal{F}'', \phi)$.

Note that $\langle t, t^{-1} \rangle$ is the infinite cyclic group $\mathbb{Z}$ with a zero element adjoined. Coherent sheaves on $\mathrm{MSpec}(\langle t, t^{-1} \rangle)$ therefore correspond to finitely generated $\mathbb{Z}$–sets, and indecomposable coherent sheaves to $\mathbb{Z}$–orbits. We therefore have:

Lemma 7.2.1. *The indecomposable coherent sheaves on* $U_1 \cap U_2 = \mathrm{MSpec}(\langle t, t^{-1} \rangle)$ *are of the form* $\widetilde{N}$*, where* N *is a* $\langle t, t^{-1} \rangle$*–module of the form* $\langle t, t^{-1} \rangle$ *or* $\langle t, t^{-1} \rangle / \langle t^k, t^{-k} \rangle$*. We denote by* $\mathcal{L}$ *and* $\mathcal{C}_k$ *the corresponding coherent sheaves.*

The following result regarding the restriction of coherent sheaves from $\mathbb{A}^1$ to $U_1 \cap U_2$ is immediate:

Lemma 7.2.2. *Suppose* $\mathcal{F} = \widetilde{M}$ *an indecomposable coherent sheaf on* $\mathbb{A}^1$*. Then*

(1) *If* Γ_M *is of type 1, then* $\mathcal{F}|_{U_1 \cap U_2} \simeq 0$.
(2) *If* Γ_M *is of type 2, then* $\mathcal{F}|_{U_1 \cap U_2} \simeq \mathcal{L}$.
(3) *if* Γ_M *is of type 3 with an oriented cycle of length* k*, then* $\mathcal{F}|_{U_1 \cap U_2} \simeq \mathcal{C}_k$.

Note that:

- the automorphism group of $\mathcal{L}$ on $U_1 \cap U_2$ is $\mathbb{Z}$. We denote by $\phi_n : \mathcal{L} \to \mathcal{L}$ the automorphism of L induced by multiplication by t^n on $\langle t, t^{-1} \rangle$.
- the automorphism group of $\mathcal{C}_k$ is $\mathbb{Z}/k\mathbb{Z}$.. We denote by $\psi_m : \mathcal{C}_k \to \mathcal{C}_k$ the automorphism of $\mathcal{C}_k$ induced by multiplication by t^m on $\langle t, t^{-1} \rangle / \langle t^k, t^{-k} \rangle$ (note that ψ_m only depends on m $(mod\ k)$).

We thus come to our main result in this section.

Theorem 7.2.3. *Let* $\mathcal{F}$ *be an indecomposable coherent sheaf on* $\mathbb{P}^1$*. Then* F *is described by one of the following defining triples* $(\mathcal{F}', \mathcal{F}'', \varphi)$*, where* $\mathcal{F}'$ *and* $\mathcal{F}''$ *are indecomposable coherent sheaves on* U_1 *and* U_2 *respectively, and*

$$\varphi : \mathcal{F}'|_{U_1 \cap U_2} \simeq \mathcal{F}''|_{U_1 \cap U_2}$$

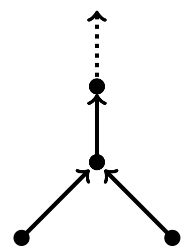
is a gluing isomorphism.

(1) $(\mathcal{F}', 0, 0)$*, where* $\mathcal{F}'$ *is of type 1.*

(2) $(0, \mathcal{F}'', 0)$, *where* $\mathcal{F}'$ *is of type 1.*
(3) $(\mathcal{F}', \mathcal{F}'', \phi)$ *where* $\mathcal{F}'$ *and* $\mathcal{F}''$ *are of type 2. After choosing isomorphisms* $\mathcal{F}'_{U_1 \cap U_2} \simeq \mathcal{L}$, $\mathcal{F}''_{U_1 \cap U_2} \simeq \mathcal{L}$, ϕ *may be identified with* ϕ_n *for some* $n \in \mathbb{Z}$.
(4) $(\mathcal{F}', \mathcal{F}'', \psi)$ *where* $\mathcal{F}'$ *and* $\mathcal{F}''$ *are of type 3. After choosing isomorphisms* $\mathcal{F}'_{U_1 \cap U_2} \simeq \mathcal{C}_k$, $\mathcal{F}''_{U_1 \cap U_2} \simeq \mathcal{C}_k$, ψ *may be identified with* ψ_m *for some* $m \in \mathbb{Z}/k\mathbb{Z}$.

$\mathcal{F}$ *is torsion in the first two cases, and torsion-free in the last two.*

Example 3. Let M be the $\langle t \rangle$–module on two generators with Γ_M as shown:



Example 3

Let $\mathcal{F} = \widetilde{M}$ be the corresponding coherent sheaf on $\mathbb{A}^1$, and $\mathcal{F}_1, \mathcal{F}_2$ sheaves on U_1 and U_2 respectively isomorphic to $\mathcal{F}$. Denote the generators on U_1 by a_0, b_0 and those on U_2 by a_∞, b_∞. Consider the coherent sheaf $(\mathcal{F}_1, \mathcal{F}_2, \psi)$, where $\psi : \mathcal{F}_1 \to \mathcal{F}_2$ identifies the images of a_0, b_0 with a_∞, b_∞ over $U_1 \cap U_2$. Given a field K, we have

$$\mathcal{F}_K \cong \mathcal{O}_{\mathbb{P}^1} \oplus K_0 \oplus K_\infty$$

where K_0 and K_∞ are torsion sheaves supported at 0 and ∞ isomorphic in local coordinates to $K[t]/(t)$ and $K[t^{-1}]/(t^{-1})$ respectively. Denoting $s \in \Gamma(\mathbb{P}^1, \mathcal{F})$ by the pair $(s|_{U_1}, s|_{U_2})$, the set of global sections consists of 4 non-zero elements

$$(a_0, a_\infty), (a_0, b_\infty), (b_0, a_\infty), (b_0, b_\infty),$$

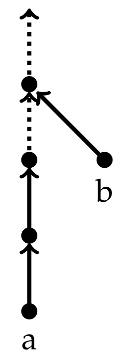
whereas $H^0(\mathbb{P}^1_K, \mathcal{F}_K)$ is 3-dimensional. The base change map

$$\phi_K : K[\Gamma(\mathbb{P}^1, \mathcal{F})] \to \Gamma(\mathbb{P}^1_K, \mathcal{F}_K)$$

is easily seen to be surjective, with kernel spanned by

$$(a_0, a_\infty) - (a_0, b_\infty) - (b_0, a_\infty) + (b_0, b_\infty).$$

Example 4. Let P_n be the $\langle t \rangle$–module on two generators with Γ_{P_n} consisting of an infinite ladder with one additional vertex attached via an incoming edge to the n-th vertex from the bottom. I.e. we have $t^n \cdot a = t \cdot b$ as shown:



Example 4

Let $\mathcal{F}_1 = \widetilde{P_n}$ on U_1 with generators labeled a_0, b_0 and $\mathcal{F}_2 = \widetilde{\langle t^{-1} \rangle}$ on U_2, with generator labeled c_∞. Consider the coherent sheaf $\mathcal{G}_n = (\mathcal{F}_1, \mathcal{F}_2, \rho)$, where ρ identifies a_0 with c_∞ over $U_1 \cap U_2$. The global section (a_0, c_∞) generates a sub-module of $\mathcal{G}_n$ isomorphic to $\mathcal{O}_{\mathbb{P}^1}$, with $\mathcal{G}_n / \mathcal{O}_{\mathbb{P}^1}$ isomorphic to the torsion sheaf $\mathcal{T} = \langle t \rangle / (t)$ (note that the round bracket denotes the *ideal* generated by t). We thus obtain infinitely many non-isomorphic non-split extensions

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{G}_n \to \mathcal{T} \to 0$$

Upon base-change to a field K, $(\mathcal{G}_n)_K \cong \mathcal{O}_{\mathbb{P}^1} \oplus K_0$, and all of these short exact sequences become isomorphic to the split extension

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1} \oplus K_0 \to K_0 \to 0.$$

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