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Dynamic optimal asset allocation with optimal stopping

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DYNAMIC OPTIMAL ASSET ALLOCATION WITH OPTIMAL STOPPING

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ABSTRACT

We develop a model of optimal consumption, labor and portfolio choice with endogenous retirement for an individual's life-cycle decisions. Explicit solutions for finite horizon are derived both for an individual with power utility and for an individual with log utility. There are two distinct phases in the life-cycle, the first being accumulation phase and the second being retirement phase. The individual simultaneously chooses consumption, labor, portfolio and whether to retire so as to maximize the expected utility. We show that the dynamic budget constraint can be reduced to a static budget constraint. For this static optimization problem involving both stochastic optimal control and optimal stopping, we use the convex duality approach to transform it to a pure optimal stopping problem. The value function can be characterized using early exercise premium representation which depends on the optimal retirement boundary. We show that immediate retirement is optimal when a state variable hits the boundary. We derive the backward recursive equation of the boundary parameterized by a multiplier which itself satisfies a nonlinear equation from the static budget constraint. The optimal wealth and the optimal portfolio are derived and they depend on the retirement boundary and the derivative of this boundary with respect to the multiplier. A numerical algorithm is developed for computation of the solutions. We analyze the properties and the structures of the optimal policies and also prove that retirement is optimal when the financial wealth crosses its boundary of which we derive an explicit form.

We study next a model of optimal dividend-contribution, portfolio and liquidation from the viewpoint of a defined benefit pension fund. The sponsor faces a stream of intermediate liability and a terminal liability. The optimization problem stops at the optimal liquidation date rather than continues for the second phase. Preference of the sponsor is defined over net cash flow (dividends or contributions) depending on whether outlay from the fund is higher than or lower than the liability. We analyze the behavior of the optimal policies and identify in the optimal portfolio the hedges against fluctuations in the intermediate liability and in the terminal liability.

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List of Abbreviations

CRRA	....	Constant Relative Risk Aversion
DEP	....	Delayed Exercise Premium
EEP	....	Early Exercise Premium
SPD	....	State Price Density

Chapter 1

Introduction

We study the dynamic optimal asset allocation problem from both an individual's and a pension fund's point of view. For an individual, who faces the tradeoffs of consumption-saving, labor-leisure, risk-reward and working-retirement, we examine the optimal life-cycle policies. For a defined benefit pension fund, of which the asset-liability management involves the choices of dividend-contribution, risk-reward and continuation-liquidation, we investigate the behavior of the optimal decisions.

The scientific literature for the portfolio choice problem begins with one-period mean-variance optimization, notably in Markowitz (1952, 1959). Samuelson (1969) analyzes discrete time multi-period model. Merton (1969, 1971) first develops continuous time diffusion process model, and provides the fundamental insight that optimal portfolio includes intertemporal hedging against fluctuations in investment opportunity set. Cox and Huang (1989, 1991), Karatzas et al. (1987) and Pliska (1986) propose the martingale method, and this technique establishes the correspondence between a dynamic optimization problem and a static problem. Ocone and Karatzas (1991) derive representation formula for the optimal portfolio. Detemple et al. (2003) develop a simulation-based approach for calculating the optimal portfolio.

Bodie et al. (1992, 2004, 2009, 2012) are critical extensions for individuals' optimal life-cycle decision problem, by incorporating essential elements such as flexible labor, habit formation and retirement phase. We extend along these lines to study a model of optimal consumption, labor and portfolio choice with endogenous retirement for an individual's life-cycle decisions. There are two distinct phases in the life-cycle, the first being accumulation phase and the second being retirement phase. At each point in time, the individual simultaneously chooses consumption, labor, portfolio and whether to retire so as to maximize the expected utility. We derive closed form solutions of all optimal policies involved and provide numerical implementation of our model, for an individual with power utility in Chapter 2 and for an individual with log utility in Chapter 3.

Existing literature on optimal retirement decision examines the case of infinite horizon where the individual is infinitely lived, such as Choi and Shim (2006), Farhi and Panageas (2007), Choi et al. (2008), Lim and Shin (2011). When the individual examined has a deadline for retirement, such as in Farhi and Panageas (2007), only approximate optimal policies are provided. Our first contribution is to derive closed-form solutions of the optimal policies for finite horizon case. Our model is especially suitable to capture the feature of reality that there is a retirement phase with a terminal date in life-cycle and it has the advantage that the retirement boundary derived is time dependent, whereas the boundary derived from the infinite horizon case has no time dimension. Our approach of decomposing the value function by early exercise premium representation affords interesting interpretations in terms of net local gains from early retirement or delayed retirement. We show that immediate retirement is optimal when an endogenous state variable, related to the wage and the state price density, hits the boundary. We provide efficient numerical implementation to compute the boundary and the optimal policies. It is notable that this optimal retirement boundary depends on time and an initial Lagrange multiplier, but does not depend on the underlying Brownian motion. We show that a transformation of this boundary gives us the retirement boundary for the liquid wealth. This boundary is more practical and useful because an individual can compare his observed liquid wealth with this boundary and know if it's optimal to retire immediately. Interestingly, we show that the closed form representations of the optimal wealth and the optimal portfolio not only depend on the retirement boundary, but also depend on the derivative of the boundary with respect to the multiplier. The components in the representation of the wealth processes that depend on the derivative, are precisely the discrepancy between an early exercise premium representation of the wealth processes and their true value. We also examine the effects of retirement option to the optimal consumption, the optimal leisure and the total expenditure. We show that the effects can be examined by comparing the initial wealth of an individual who has the retirement option with the initial wealth of an individual who does not have the retirement option. We also investigate the relationship between the optimal policies of an individual with log utility and the corresponding optimal policies of an individual with power utility. We identify the conditions under which the optimal policies of an individual with log utility are the limits of the optimal policies of an individual with power utility as the coefficient of relative risk aversion converges to 1.

In Chapter 4, we study a model of optimal dividend-contribution, portfolio and liquidation from the viewpoint of a defined benefit pension fund. Relatively limited attendation has been devoted to applying the theory of dynamic optimal asset allocation to study asset-liability management, in spite of the large amount of defined benefit pension asset accumulated over the years currently under management. Boulier et al. (1995), Cairns (2000), Rudolf and Ziemba (2004), Detemple and Rindisbacher (2008), Detemple et al. (2010) and Van Binsbergen and Brandt (2012) are examples of recent contributions in the asset-liability management for defined benefit pension fund literature. Our work extends Detemple et al. (2010) to incorporate endogenous optimal liquidation in the asset-liability management model.

For a defined benefit pension fund, its asset-liability management involves the choices of dividend-contribution, risk-reward and continuation-liquidation. The crucial difference between the asset allocation problem for a pension fund and the problem for an individual's life-cycle is that the pension fund faces a stream of intermediate liability and a terminal liability, which are benefit payments to the pension plan participants. The value and the risk of this liability side of the balance sheet of a pension fund, are crucial for its optimal asset allocation of pension asset. In other words, the whole balance sheet of the pension fund should be taken into consideration for optimal policies. An asset allocation strategy focusing exclusively on the pension asset while neglecting either the intermediate liability or the terminal liability, is bound to be suboptimal.

The two problems share similarities. The optimal liquidation date, or the optimal date to switch from a defined benefit pension plan to a defined contribution pension plan, is modeled as an optimal stopping time chosen by the fund sponsor, similar to the optimal retirement date of an individual's life-cycle. However, the liquidation date of the pension fund is the terminal date of the investment horizon, and the sponsor cares about the lump sum amount of dividend or contribution at this date. The optimization problem stops at the liquidation date and does not continue to a second phase as in an individual's life-cycle decisions.

We derive the recursive integral equation of the optimal liquidation boundary for an endogenous variable, which is related to the terminal liability and the state price density. We provide closed-form solutions of the optimal net cash flow (dividends or contributions), the optimal liquid wealth and the optimal portfolio. As in the individuals' life-cycle problem, early exercise premium representation affords interesting interpretations in terms of net local gains from early liquidation or delayed liquidation. Optimal net cash flows are shown to be positive (dividends) when state price density is sufficiently low and be negative (contributions) when state price density is sufficiently high. We also derive the liquidation boundary for the pension assets and identify in the optimal portfolio the hedges against fluctuations in the intermediate liability and in the terminal liability.

The mathematical tools used are those of convex duality theory and optimal stopping theory. We reduce the dynamic budget constraint to a static budget constraint. For this static optimization problem involving both stochastic optimal control and optimal stopping, we use the convex duality approach to transform it to a pure optimal stopping problem in which we can treat the multiplier as a constant. The value function can be characterized using early exercise premium representation which depends on the optimal exercise boundary. The EEP representation corresponds to the Riesz decomposition of the Snell Envelope, which is the smallest supermartingale majorant of the underlying stochastic process. It gives a decomposition of the Snell Envelope to the sum of a martingale and a potential and this potential corresponds to the early exercise premium. By the fact that the Snell Envelope is equal to the underlying process when the state variable is within the exercise region, we derive the backward recursive equation of the boundary parameterized by the multiplier which itself satisfies a nonlinear equation from the static budget constraint. An efficient numerical algorithm is developed for computation of the boundary and the optimal policies.

Part I

Optimal consumption, labor, portfolio and retirement

Chapter 2

Optimal consumption, labor, portfolio and retirement for individuals with power utility

2.1 Introduction

We develop a model of optimal consumption, labor and portfolio choice with endogenous retirement for an individual's life-cycle decisions. There are two distinct phases in the life-cycle, the first being accumulation phase and the second being retirement phase. At each point in time before retirement, the individual simultaneously chooses consumption, labor, portfolio and whether to retire so as to maximize the expected utility. After retirement, the individual chooses only consumption and portfolio as labor supply becomes zero.

Existing literature on optimal retirement decision examines the case of infinite horizon where the individual is infinitely lived. When the individual examined has a deadline for retirement, only approximate optimal policies are provided. Our first contribution is to derive closed-form solutions of the optimal policies for finite horizon case. Our model is especially suitable to capture the feature of reality that there is a retirement phase with a terminal date in life-cycle and it has the advantage that the retirement boundary derived is time dependent, whereas the boundary derived from the infinite horizon case has no time dimension. Our approach of decomposing the value function by early exercise premium representation affords interesting interpretations in terms of net local gains from early retirement or delayed retirement. We show that immediate retirement is optimal when an endogenous state variable, related to the wage and the state price density, hits the boundary. We provide efficient numerical implementation to compute the boundary and the optimal policies. It is notable that this optimal retirement boundary depends on time and an initial Lagrange multiplier, but does not depend on the underlying Brownian motion. We show that a transformation of this boundary gives us the retirement boundary for the liquid wealth. This boundary is more practical and useful because an individual can compare his observed liquid wealth with this boundary and know if it's optimal to retire immediately. Interestingly, we show that the closed form representations of the optimal wealth and the optimal portfolio not only depend on the retirement boundary, but also depend on the derivative of the boundary with respect to the multiplier. The components in the representation of the wealth processes that depend on the derivative, are precisely the discrepancy between an early exercise premium representation of the wealth processes and their true value. We also examine the effects of retirement option to the optimal consumption, the optimal leisure and the total expenditure. We show that the effects can be examined by comparing the initial wealth of an individual who has the retirement option with the initial wealth of an individual who does not have the retirement option. We also investigate the relationship between the optimal policies of an individual with log utility and the corresponding optimal policies of an individual with power utility. We identify the conditions under which the optimal policies of an individual with log utility are the limits of the optimal policies of an individual with power utility as the coefficient of relative risk aversion converges to 1.

The procedure to derive the solutions is as follows. We reduce the dynamic budget constraint to a static budget constraint. For this static optimization problem involving both stochastic optimal control and optimal stopping, we use the convex duality approach to transform it to a pure optimal stopping problem in which we can treat the multiplier as a constant. The value function can be characterized using early exercise premium representation which depends on the optimal retirement boundary. We derive the backward recursive equation of the boundary parameterized by the multiplier which itself satisfies a nonlinear equation from the static budget constraint. A numerical algorithm is developed for computation of the retirement boundary, the optimal consumption, labor, wealth and portfolio. We analyze the properties and the structures of the optimal policies.

The scientific literature for the portfolio choice problem begins with one-period mean-variance optimization, notably in Markowitz (1952, 1959). Samuelson (1969) analyzes discrete time multi-period model. Merton (1969, 1971) first develops continuous time diffusion process model, and provides the fundamental insight that optimal portfolio includes intertemporal hedging against fluctuations in investment opportunity set. Cox and Huang (1989, 1991), Karatzas et al. (1987) and Pliska (1986) propose the martingale method, and this technique establishes the correspondence between a dynamic optimization problem and a static problem. Ocone and Karatzas (1991) derive representation formula for the optimal portfolio. Detemple et al. (2003) develop a simulation-based approach for calculating the optimal portfolio.

Bodie et al. (1992) provide critical extensions where they study the effect of labor flexibility on consumption and portfolio choice and find that human capital is crucial to explaining optimal behavior. Basak (1999) analyzes the effect of human capital in a general equilibrium model. Bodie et al. (2004) examine consumption and portfolio choice with habit formation and they consider the effect of retirement by incorporating two distinct phases, the accumulation phase and the retirement phase. Bodie et al. (2009) review recent literature on theoretical life-cycle models and related empirical literature and they discuss the implications of the life-cycle models for design of pension plans. Bodie et al. (2012) study the property of the optimal pension contract that finances consumption during the retirement phase. Our model extends along the lines of Bodie et al. (1992, 2004, 2009, 2012) to consider endogenous retirement decision.

A number of papers examine optimal policies with endogenous retirement. Karatzas and Wang (2000) first develop the convex duality approach to study the problem with mixed optimal control and optimal stopping. We extend the result to the case with two distinct phases in life-cycle where consumption and investment continue after retirement to a fixed terminal date and with endogenous labor before retirement for individuals with constant relative risk aversion. Sundaresan and Zapatero (1997) examine the effect of pension plan on the retirement incentives. Liu and Neis (2004) study the interaction of optimal portfolio and optimal retirement, without a structural retirement phase. Dybvig and Liu (2010) examine the effect of retirement flexibility and borrowing constraint on life-cycle policies, but do not consider a deadline for retirement. A number of papers study optimal retirement close to the model in Bodie et al (1992), for the case of infinite horizon where the individual is infinitely lived, such as Choi and Shim (2006), Farhi and Panageas (2007), Choi et al. (2008) and Lim and Shin (2011). Choi and Shim (2006) derive optimal policies for an individual who suffers a utility loss from labor using dynamic programming approach. Lim and Shin (2011) study the effect of borrowing constraints, also for an individual who suffers utility loss from labor. In Choi et al. (2008), preference of the individual examined is defined by a constant elasticity of substitution utility function. A limitation for the infinite horizon case in these papers is that the retirement boundaries derived have no time dimension. Our approach using early exercise premium representation is especially suitable for the finite horizon case and retirement boundaries we derived are time dependent. Farhi and Panageas (2007) study optimal retirement for an infinitely lived individual, both when the retirement date has a deadline and when it does not. When the individual examined has a deadline for retirement in Farhi and Panageas (2007), only approximate optimal policies are derived using ideas proposed by Barone-Adesi and Whaley (1987) to approximate a partial differential equation by an ordinary differential equation. We derive closed-form solutions which are exact expressions of the retirement boundary for the finite horizon case using early exercise premium representation of the value function, both for an individual with power utility and for an individual with log utility. Early exercise premium (EEP) representation was introduced by Kim (1990), Jacka (1991), Carr et al. (1992) for pricing American options and generalized by Rutkowski (1994) to semimartingale payoffs. The EEP representation corresponds to the Riesz decomposition of the Snell Envelope to the sum of a martingale and a potential, and this potential corresponds to the early exercise premium.

2.2 The model

$(\Omega, \mathcal{F}, P)$ is a complete probability space. W_t, $t \in [0, T]$ is a Brownian motion on the probability space. The flow of information $\mathcal{F}_t$, $t \in [0, T]$ is the filtration generated by W_t.

The market consists of a riskless asset and a risky asset. Riskless asset is a money market account with a constant interest rate $r > 0$. Risky asset has instantaneous return dR_t which satisfies $dR_t = \mu dt + \sigma dW_t$. μ is the expected rate of return, σ is the volatility of the return. μ and σ are both constants and positive. The market price of risk is $\theta = (\mu - r) / \sigma$. The state price density process is $\xi_t = \exp\left(-rt - \frac{1}{2}\theta^2 t - \theta W_t\right)$. This structure implies that the Brownian motion risk is hedgeable, thus the market is complete and there are no arbitrage opportunities.

There are two distinct phases in the individual's life-cycle, the first being accumulation phase and the second being retirement phase. At each point in time before retirement, the individual has liquid wealth or financial wealth X_t, and determines the consumption c_t, the leisure l_t, and the dollar amount π_t invested in the risky asset. Consumption process c_t and leisure process l_t are nonnegative, progressively measurable, and they satisfy $\int_0^T c_t dt < \infty$ a.s. and $\int_0^T l_t dt < \infty$ a.s., where time 0 is the beginning of working life and time T is terminal date of life. Portfolio process π_t is progressively measurable and satisfies $\int_0^T \pi_t^2 dt < \infty$ a.s. The individual is endowed with a maximal amount of labor $\overline{h}$. $h_t = \overline{h} - l_t$ is amount of labor supplied by the individual for which the individual earns wages w_t. We normalize the maximal work capacity $\overline{h}$ to be 1. The wage process w_t is stochastic and satisfies $dw_t = w_t \left(\mu_w dt + \sigma_w dW_t\right)$, where μ_w is the expected growth rate and σ_w is the volatility of the growth rate. μ_w and σ_w are both constants. The liquid wealth X_t satisfies

$$\begin{aligned} dX_t &= \left(X_t - \pi_t\right) r dt - c_t dt + w_t h_t dt + \pi_t dR_t \\ &= \left(rX_t - c_t + w_t - w_t l_t\right) dt + \pi_t \sigma \left(\theta dt + dW_t\right), \end{aligned} \tag{2.1}$$

starting from $X_0 = x$. After retirement, $l_t = 1$ and the liquid wealth X_t satisfies

$$\begin{aligned} dX_t &= \left(X_t - \pi_t\right) r dt - c_t dt + \pi_t dR_t \\ &= \left(rX_t - c_t\right) dv + \pi_t \sigma \left(\theta dt + dW_t\right). \end{aligned}$$

Preference ordering for the individual is represented by von Neumann-Morgenstein expected utility

$$\mathcal{U} = E\left[\int_0^{\tau} a_v u^a \left(c_v, l_v\right) dv + \phi \int_{\tau}^{T} a_v u^r \left(c_v, 1\right) dv\right],$$

where $u^a\left(c_v, l_v\right) = \frac{\left(c_v^{\eta} l_v^{1-\eta}\right)^{1-R}}{\eta(1-R)}$, $u^r\left(c_v, 1\right) = \frac{c_v^{1-R}}{1-R}$ and $a_v = \exp\left(-\beta v\right)$. Intermediate utility during accumulation phase $u^a\left(c_v, l_v\right)$ is Cobb-Douglas utility of consumption and leisure, and intermediate utility $u^r\left(c_v, 1\right)$ during retirement phase is power utility of consumption. η is a measure of relative weight of consumption and leisure. R is the coefficient of relative risk aversion. β is the subjective discount rate. ϕ is a coefficient that measures the relative weight of the retirement phase. τ is the retirement date chosen by the individual. $\tau \in \mathcal{S}$, where $\mathcal{S}$ is the collection of stopping times with values in $[0, T]$.

2.3 Convex duality and pure optimal stopping problem

We first reduce the dynamic budget constraint (2.1) to a static budget constraint.

The policy of consumption, labor and portfolio choice (c, l, π) is said to be admissible: $(c, l, \pi) \in A$, if the no-bankruptcy condition is satisfied, i.e., the total wealth is nonnegative. Before retirement, the total wealth N_t is the sum of the liquid wealth X_t and the human capital H_t, where the human capital H_t is the present value of the maximal earnings during the accumulation phase $E_t\left[\int_t^\tau \xi_{t,v} w_v dv\right]$. Thus (c, l, π) is admissible if $X_t + E_t\left[\int_t^\tau \xi_{t,v} w_v dv\right] \geq 0$. After retirement, the human capital is exhausted and the total wealth is equal to the liquid wealth, thus (c, l, π) is admissible if $X_t \geq 0$.

Static budget constraint is

$$E\left[\int_0^\tau \xi_v \left(c_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv\right] \leq x + E\left[\int_0^\tau \xi_v w_v dv\right]. \tag{2.2}$$

It states that initial total wealth (the sum of initial liquid wealth x and initial human capital $E\left[\int_0^\tau \xi_v w_v dv\right]$) is sufficiently large to finance consumption and leisure during accumulation phase, and consumption during retirement phase.

Lemma 1. *If* (c, l, π) *is admissible, then* (c, l) *satisfies the static budget constraint* (2.2)*. If* (c, l) *satisfies the static budget constraint* (2.2)*, then* $\exists\ \pi$*, such that* (c, l, π) *is admissible.*

Proof. See appendix. ♦

The lemma shows that the optimization problem can be reduced to optimizing the expected utility subject to the static budget constraint (2.2). The maximization problem for the individual is

$$V(x) = \sup_{\tau \in \mathcal{S},\ (c,l,\pi)\in A} E\left[\int_0^{\tau} a_v u^a(c_v, l_v)\,dv + \phi \int_{\tau}^{T} a_v u^r(c_v, 1)\,dv\right] \tag{2.3}$$

subject to

$$E\left[\int_0^{\tau} \xi_v (c_v + l_v w_v)\,dv\right] - E\left[\int_0^{\tau} \xi_v w_v dv\right] + E\left[\int_{\tau}^{T} \xi_v c_v dv\right] \le x.$$

Next we transform the combined stochastic optimal control and optimal stopping problem (2.3) to a pure optimal stopping problem. We first introduce the Legendre-Fenchel transform of utility function.

Utility function $U : (0,\infty) \to \mathcal{R}$ is strictly increasing, strictly concave and continuously differentiable. The inverse of $U'(\cdot)$ is denoted as $I(\cdot)$. Legendre-Fenchel transform of utility function $U(x_0)$ is defined as

$$\widetilde{U}(y_0) \triangleq \max_{x_0>0} [U(x_0) - x_0 y_0] = U(I(y_0)) - y_0 I(y_0), \quad 0 < y_0 < \infty.$$

Its derivative $\widetilde{U}'(y_0) = -I(y_0)$. $-I(y_0)$ is negative and increasing, therefore the function $\widetilde{U}(\cdot)$ is strictly decreasing and strictly convex. We have

$$U(x_0) \le \widetilde{U}(y_0) + x_0 y_0, \text{ for } 0 < x_0 < \infty,\ 0 < y_0 < \infty. \tag{2.4}$$

For wo variables, define

$$\widetilde{U}(y_1, y_2) \triangleq \max_{x_1>0,\ x_2>0} [U(x_1, x_2) - x_1 y_1 - x_2 y_2], \quad 0 < y_1 < \infty,\ 0 < y_2 < \infty.$$

The maximum is attained when the first order conditions are satisfied, i.e., $U_{x_1}(x_1, x_2) =$

y_1, and $U_{x_2}(x_1, x_2) = y_2$. We have

$$U(x_1, x_2) \leq \widetilde{U}(y_1, y_2) + x_1 y_1 + x_2 y_2, \tag{2.5}$$

$$\text{for } 0 < x_1 < \infty,\ 0 < x_2 < \infty,\ 0 < y_1 < \infty,\ 0 < y_2 < \infty.$$

Choose $x_0 = c_v$, $y_0 = y a_v^{-1} \xi_v$ and $U(x_0) = \phi u^r\,(c_v, 1)$ in (2.4), we have

$$\phi u^r\,(c_v, 1) \leq \widetilde{U}^r(y a_v^{-1} \xi_v) + y a_v^{-1} \xi_v c_v,$$

or

$$\phi a_v u^r\,(c_v, 1) \leq a_v \widetilde{U}^r(y a_v^{-1} \xi_v) + y \xi_v c_v.$$

Choose $x_1 = c_v$, $x_2 = l_v$, $y_1 = y a_v^{-1} \xi_v$, $y_2 = y a_v^{-1} \xi_v w_v$ and $U(x_1, x_2) = u^a\,(c_v, l_v)$ in (2.5), we have

$$u^a\,(c_v, l_v) \leq \widetilde{U}^a(y a_v^{-1} \xi_v, y a_v^{-1} \xi_v w_v) + y a_v^{-1} \xi_v c_v + y a_v^{-1} \xi_v w_v l_v,$$

or

$$a_v u^a\,(c_v, l_v) \leq a_v \widetilde{U}^a(y a_v^{-1} \xi_v, y a_v^{-1} \xi_v w_v) + y \xi_v c_v + y \xi_v w_v l_v.$$

Therefore,

$$
\begin{aligned}
& E\left[\int_0^{\tau} a_v u^a\left(c_v, l_v\right) dv + \phi \int_{\tau}^{T} a_v u^r\left(c_v, 1\right) dv\right] \\
\leq\ & E\left[\int_0^{\tau} a_v \widetilde{U}^a(y a_v^{-1}\xi_v, y a_v^{-1}\xi_v w_v) dv + \int_{\tau}^{T} a_v \widetilde{U}^r(y a_v^{-1}\xi_v) dv\right] \\
& + yE\left[\int_0^{\tau}\left(\xi_v c_v + \xi_v w_v l_v\right) dv + \int_{\tau}^{T} \xi_v c_v dv\right] \\
=\ & E\left[\int_0^{\tau} a_v \widetilde{U}^a(y a_v^{-1}\xi_v, y a_v^{-1}\xi_v w_v) dv + y\int_0^{\tau} \xi_v w_v dv + \int_{\tau}^{T} a_v \widetilde{U}^r(y a_v^{-1}\xi_v) dv\right] \\
& + yE\left[\int_0^{\tau}\left(\xi_v c_v + \xi_v w_v l_v\right) dv - \int_0^{\tau} \xi_v w_v dv + \int_{\tau}^{T} \xi_v c_v dv\right] \\
\leq\ & E\left[\int_0^{\tau} a_v \widetilde{U}^a(y a_v^{-1}\xi_v, y a_v^{-1}\xi_v w_v) dv + y\int_0^{\tau} \xi_v w_v dv + \int_{\tau}^{T} a_v \widetilde{U}^r(y a_v^{-1}\xi_v) dv\right] \\
& + yx.
\end{aligned}
$$

Define

$$
\begin{aligned}
& \widetilde{J}\left(y;\tau\right) \\
\triangleq\ & E\left[\int_0^{\tau} a_v \widetilde{U}^a(y a_v^{-1}\xi_v, y a_v^{-1}\xi_v w_v) dv + y\int_0^{\tau} \xi_v w_v dv + \int_{\tau}^{T} a_v \widetilde{U}^r(y a_v^{-1}\xi_v) dv\right].
\end{aligned}
$$

Thus we have

$$
E\left[\int_0^{\tau} a_v u^a\left(c_v, l_v\right) dv + \phi \int_{\tau}^{T} a_v u^r\left(c_v, 1\right) dv\right] \leq \inf_{y>0}\left[\widetilde{J}\left(y;\tau\right) + yx\right],
$$

and

$$
V\left(x\right) \leq \sup_{\tau\in\mathcal{S}}\inf_{y>0}\left[\widetilde{J}\left(y;\tau\right) + yx\right].
$$

Define

$$
\begin{aligned}
& \widetilde{V}(y) \\
\triangleq\ & \sup_{\tau\in\mathcal{S}} \widetilde{J}(y;\tau) \\
=\ & \sup_{\tau\in\mathcal{S}} E\left[\int_0^{\tau} a_v \widetilde{U}^a(ya_v^{-1}\xi_v, ya_v^{-1}\xi_v w_v)dv + y\int_0^{\tau}\xi_v w_v dv + \int_{\tau}^{T} a_v\widetilde{U}^r(ya_v^{-1}\xi_v)dv\right],
\end{aligned}
$$

and denote the stopping time that attains the supremum by τ_y^*, i.e., $\widetilde{V}(y) = \widetilde{J}\left(y;\tau_y^*\right)$. Therefore, we have

$$
V(x) \le \sup_{\tau\in\mathcal{S}}\inf_{y>0}\left[\widetilde{J}(y;\tau)+yx\right] \le \inf_{y>0}\left[\sup_{\tau\in\mathcal{S}}\widetilde{J}(y;\tau)+yx\right] = \inf_{y>0}\left[\widetilde{V}(y)+yx\right]. \tag{2.6}
$$

The next proposition extends the convex duality approach proposed by Karatzas and Wang (2000) to two distinct phases where consumption and investment continue after retirement to a fixed terminal date with endogenous labor before retirement for power utility. It shows that the inequality in (2.6) is indeed an equality, thus we can first solve the pure optimal stopping problem of $\widetilde{V}(y)$, while treating the multiplier y as a constant.

Proposition 2.

$$
V(x) = \inf_{y>0}\left[\widetilde{V}(y)+yx\right].
$$

Proof. Before retirement, $a_v u^a(c_v,l_v) = a_v\frac{\left(c_v^{\eta}l_v^{1-\eta}\right)^{1-R}}{\eta(1-R)}$. First order conditions are $a_v u_c^a(c_v,l_v) = y\xi_v$, $a_v u_l^a(c_v,l_v) = y\xi_v w_v$. Thus,

$$
a_v u_c^a(c_v,l_v) = a_v\left(c_v^{\eta}l_v^{1-\eta}\right)^{-R} c_v^{\eta-1}l_v^{1-\eta} = y\xi_v, \tag{2.7}
$$

$$
a_v u_l^a(c_v,l_v) = a_v\left(c_v^{\eta}l_v^{1-\eta}\right)^{-R}\frac{1-\eta}{\eta}c_v^{\eta}l_v^{-\eta} = y\xi_v w_v. \tag{2.8}
$$

(2.8) divided (2.7), we get $\frac{1-\eta}{\eta}\frac{c_v}{l_v} = w_v$. Thus $e_v \equiv c_v + l_v w_v = \frac{c_v}{\eta}$. For (2.7), both sides multiply by c_v, we get $a_v\left(c_v^{\eta}l_v^{1-\eta}\right)^{1-R} = y\xi_v c_v$. Thus $a_v u^a(c_v,l_v) = a_v\frac{\left(c_v^{\eta}l_v^{1-\eta}\right)^{1-R}}{\eta(1-R)} =$

$\frac{y\xi_v c_v}{\eta(1-R)} = \frac{y\xi_v e_v}{1-R}$. Calculation gives the following solutions which solve (2.7) and (2.8),

$$c_v^* = \eta \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho} f,$$

$$l_v^* = (1-\eta) \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho} f \frac{1}{w_v},$$

$$e_v^* = \frac{c_v^*}{\eta} = \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho} f,$$

where $\rho = 1 - \frac{1}{R}$ and $f = \frac{1}{\eta}\left(\frac{1-\eta}{\eta}\right)^{-(1-\eta)\rho}$. During retirement, $a_v u^r(c_v, 1) = a_v \frac{c_v^{1-R}}{1-R}$. First order condition is $\phi a_v u_c^r(c_v, 1) = y\xi_v$. Thus

$$\phi a_v u_c^r(c_v, 1) = \phi a_v c_v^{-R} = y\xi_v.$$

Both sides multiply by c_v gives $\phi a_v c_v^{1-R} = y\xi_v c_v$, thus $\phi a_v u^r(c_v, 1) = \phi a_v \frac{c_v^{1-R}}{1-R} = \frac{y\xi_v c_v}{1-R}$. Calculation gives the following solution

$$c_v^* = \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} \phi^{\frac{1}{R}}.$$

Now we have

$$\begin{aligned} a_v\widetilde{U}^a(ya_v^{-1}\xi_v, ya_v^{-1}\xi_v w_v) &= a_v u^a(c_v^*, l_v^*) - y\xi_v c_v^* - y\xi_v w_v l_v^* \\ &= \frac{R}{\eta(1-R)} y\xi_v c_v^* \\ &= \frac{R}{1-R} f a_v^{\frac{1}{R}} (y\xi_v)^\rho w_v^{(1-\eta)\rho}. \end{aligned}$$

$a_v\widetilde{U}^a(ya_v^{-1}\xi_v, ya_v^{-1}\xi_v w_v)$ is a convex function of y. During retirement,

$$\begin{aligned} a_v\widetilde{U}^r(ya_v^{-1}\xi_v) &= \phi a_v u^r\left(c_v^*, 1\right) - y\xi_v c_v^* \\ &= \frac{R}{1-R} y\xi_v c_v^* \\ &= \frac{R}{1-R}\phi^{\frac{1}{R}} a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}. \end{aligned}$$

$a_v\widetilde{U}^r(ya_v^{-1}\xi_v)$ is a convex function of y. Now we have

$$\begin{aligned} &\widetilde{J}\left(y;\tau\right) \\ = \; & E\left[\int_0^{\tau} a_v\widetilde{U}^a(ya_v^{-1}\xi_v, ya_v^{-1}\xi_v w_v)dv + y\int_0^{\tau}\xi_v w_v dv + \int_{\tau}^{T} a_v\widetilde{U}^r(ya_v^{-1}\xi_v)dv\right] \\ = \; & E\left[\int_0^{\tau}\frac{R}{1-R} f a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho}dv + y\int_0^{\tau}\xi_v w_v dv\right. \\ & \left. + \int_{\tau}^{T}\frac{R}{1-R}\phi^{\frac{1}{R}} a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho} dv\right]. \end{aligned}$$

$\widetilde{J}\left(y;\tau\right)$, as the sum of the three parts above, is also a convex function of y. For $0 < y_1 < y_2$, $0 < s < 1$, $y_0 \triangleq sy_1 + (1-s)\,y_2$,

$$\widetilde{V}(y_0) = \widetilde{J}\left(y_0;\tau_{y_0}^*\right) \le s\widetilde{J}\left(y_1;\tau_{y_0}^*\right) + (1-s)\,\widetilde{J}\left(y_2;\tau_{y_0}^*\right) \le s\widetilde{V}(y_1) + (1-s)\,\widetilde{V}(y_2).$$

Therefore $\widetilde{V}(y)$ is convex. For any given $\tau \in \mathcal{S}$ and a given y, define

$$x_{\tau}\left(y\right) \triangleq E\left[\int_0^{\tau}\left(\xi_v c_v^* + \xi_v w_v l_v^*\right)dv - \int_0^{\tau}\xi_v w_v dv + \int_{\tau}^{T}\xi_v c_v^* dv\right].$$

For power utility functions, we have

$$x_{\tau}\left(y\right) = E\left[\int_0^{\tau} f y^{-\frac{1}{R}} a_v^{\frac{1}{R}}\xi_v^{\rho} w_v^{(1-\eta)\rho}dv - \int_0^{\tau}\xi_v w_v dv + \int_{\tau}^{T}\phi^{\frac{1}{R}} y^{-\frac{1}{R}} a_v^{\frac{1}{R}}\xi_v^{\rho}dv\right].$$

Now because $\frac{R}{1-R}y^{1-\frac{1}{R}}$ is a convex function of y, we have

$$\frac{R}{1-R}y_2^{1-\frac{1}{R}} - \frac{R}{1-R}y_1^{1-\frac{1}{R}} \geq (y_2 - y_1)\frac{d\left(\frac{R}{1-R}y_1^{1-\frac{1}{R}}\right)}{dy_1} = -(y_2 - y_1)\, y_1^{-\frac{1}{R}}.$$

Thus,

$$\begin{aligned}
& \widetilde{V}(y_2) - \widetilde{V}(y_1) \\
= \;& \widetilde{V}(y_2) - \widetilde{J}\left(y_1; \tau_{y_1}^*\right) \\
\geq \;& \widetilde{J}\left(y_2; \tau_{y_1}^*\right) - \widetilde{J}\left(y_1; \tau_{y_1}^*\right) \\
= \;& E\left[\int_0^{\tau_{y_1}^*} \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y_2 \xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + y_2 \int_0^{\tau_{y_1}^*} \xi_v w_v dv \right. \\
& \left. + \int_{\tau_{y_1}^*}^{T} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y_2 \xi_v\right)^{\rho} dv \right] \\
& - E\left[\int_0^{\tau_{y_1}^*} \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y_1 \xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + y_1 \int_0^{\tau_{y_1}^*} \xi_v w_v dv \right. \\
& \left. + \int_{\tau_{y_1}^*}^{T} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y_1 \xi_v\right)^{\rho} dv \right] \\
\geq \;& -(y_2 - y_1)\, E\left[\int_0^{\tau_{y_1}^*} f y_1^{-\frac{1}{R}} a_v^{\frac{1}{R}} \xi_v^{\rho} w_v^{(1-\eta)\rho} dv - \int_0^{\tau_{y_1}^*} \xi_v w_v dv \right. \\
& \left. + \int_{\tau_{y_1}^*}^{T} \phi^{\frac{1}{R}} y_1^{-\frac{1}{R}} a_v^{\frac{1}{R}} \xi_v^{\rho} dv \right] \\
= \;& -(y_2 - y_1)\, x_{\tau_{y_1}^*}(y_1).
\end{aligned}$$

Let y_2 approaches y_1, we get

$$\lim_{y_2\uparrow y_1}\frac{\widetilde{V}(y_2)-\widetilde{V}(y_1)}{y_2-y_1}\leq -x_{\tau^*_{y_1}}(y_1)\leq \lim_{y_2\downarrow y_1}\frac{\widetilde{V}(y_2)-\widetilde{V}(y_1)}{y_2-y_1}.$$

Thus $\widetilde{V}'(y)=-x_{\tau^*_y}(y)$. For initial wealth x, suppose that we have $x=x_{\tau^*_{\overline{y}}}(\overline{y})$ for some $\overline{y}$. Because $\widetilde{V}(y)$ is convex and at $\overline{y}$, $\widetilde{V}'(\overline{y})=-x$, we have $\widetilde{V}(y)-\widetilde{V}(\overline{y})\geq(-x)(y-\overline{y})$, or $\widetilde{V}(y)+yx\geq\widetilde{V}(\overline{y})+\overline{y}x$, for any y. Therefore, it shows that if we solve the optimal stopping problem of $\widetilde{V}(\overline{y})$, we obtain

$$\begin{aligned}
& V(x) \\
=\; & \sup_{\tau\in\mathcal{S},\ (c,l,\pi)\in A} E\left[\int_0^{\tau} a_v u^a(c_v,l_v)\,dv+\phi\int_{\tau}^{T} a_v u^r(c_v,1)\,dv\right] \\
\geq\; & E\left[\int_0^{\tau^*_{\overline{y}}} a_v u^a(c^*_v(\overline{y}),l^*_v(\overline{y}))\,dv+\phi\int_{\tau^*_{\overline{y}}}^{T} a_v u^r(c^*_v(\overline{y}),1)\,dv\right] \\
=\; & E\left[\int_0^{\tau^*_{\overline{y}}}\frac{R}{1-R} f a_v^{\frac{1}{R}}(\overline{y}\xi_v)^{\rho} w_v^{(1-\eta)\rho}dv+\overline{y}\int_0^{\tau^*_{\overline{y}}}\xi_v w_v dv\right. \\
& \left.+\int_{\tau^*_{\overline{y}}}^{T}\frac{R}{1-R}\phi^{\frac{1}{R}} a_v^{\frac{1}{R}}(\overline{y}\xi_v)^{\rho}\,dv\right]+\overline{y}x \\
=\; & \widetilde{V}(\overline{y})+\overline{y}x \\
=\; & \inf_{y>0}\left[\widetilde{V}(y)+yx\right],
\end{aligned}$$

i.e., $V(x)\geq\inf_{y>0}\left[\widetilde{V}(y)+yx\right]$. We also know that $V(x)\leq\inf_{y>0}\left[\widetilde{V}(y)+yx\right]$, therefore

$$V(x)=\inf_{y>0}\left[\widetilde{V}(y)+yx\right].\ \blacklozenge$$

2.4 Early exercise premium representation and optimal retirement boundary

Now the problem is a pure optimal stopping problem

$$
\begin{aligned}
& \widetilde{V}(y) \\
= \ & \sup_{\tau\in\mathcal{S}} E\left[\int_0^\tau a_v \widetilde{U}^a(ya_v^{-1}\xi_v, ya_v^{-1}\xi_v w_v)dv + y\int_0^\tau \xi_v w_v dv + \int_\tau^T a_v \widetilde{U}^r(ya_v^{-1}\xi_v)dv\right] \\
= \ & \sup_{\tau\in\mathcal{S}} E\left[\int_0^\tau \left(a_v u^a\left(c_v^*, l_v^*\right) - y\xi_v e_v^*\right)dv + \int_0^\tau y\xi_v w_v dv\right. \\
& \left. + \int_\tau^T \left(\phi a_v u^r\left(c_v^*, 1\right) - y\xi_v c_v^*\right)dv\right].
\end{aligned}
$$

Let

$$
\begin{aligned}
& D_t \\
\triangleq \ & \int_0^t \left(a_v u^a\left(c_v^*, l_v^*\right) - y\xi_v e_v^*\right)dv + \int_0^t y\xi_v w_v dv + E_t\left[\int_t^T \left(\phi a_v u^r\left(c_v^*, 1\right) - y\xi_v c_v^*\right)dv\right],
\end{aligned}
$$

and

$$
\begin{aligned}
& J_t \\
\triangleq \ & \sup_{\tau\in\mathcal{S}} E_t\left[D_\tau\right] \\
= \ & \sup_{\tau\in\mathcal{S}} E_t\left[\int_0^\tau \left(a_v u^a\left(c_v^*, l_v^*\right) - y\xi_v e_v^*\right)dv + \int_0^\tau y\xi_v w_v dv\right. \\
& \left. + \int_\tau^T \left(\phi a_v u^r\left(c_v^*, 1\right) - y\xi_v c_v^*\right)dv\right].
\end{aligned}
$$

J_t is known as the Snell Envelope of D_t and it is the smallest supermartingale majorant of D_t[1]. The optimal stopping time is $\tau_t^* = \inf\{s \in [t,T] : J_s = D_s\}$. The next proposition gives the early exercise premium (EEP) representation of J_t. The EEP representation corresponds to the Riesz decomposition of the Snell Envelope which was proposed by El Karoui and Karatzas (1991) and applied by Myneni (1992) for pricing American options. It decomposes the Snell Envelope to the sum of a martingale and a potential[2].

Proposition 3.

$$J_t = J_t^n + J_t^a,$$

where

$$\begin{aligned} J_t^n &= \int_0^t a_v u^a(c_v^*, l_v^*)\, dv + y\left[\int_0^t \xi_v w_v dv - \int_0^t \xi_v e_v^* dv\right] \\ &+ E_t\left[\int_t^T a_v u^a(c_v^*, l_v^*)\, dv\right] + y\left[E_t\left[\int_t^T \xi_v w_v dv\right] - E_t\left[\int_t^T \xi_v e_v^* dv\right]\right], \end{aligned}$$

$$\begin{aligned} J_t^a &= -E_t\left[\int_t^T a_v u^a(c_v^*, l_v^*)\, 1_{\mathcal{R}(v)} dv\right] + E_t\left[\int_t^T \phi a_v u^r(c_v^*, 1)\, 1_{\mathcal{R}(v)} dv\right] \\ &+ y\left[-E_t\left[\int_t^T \xi_v w_v 1_{\mathcal{R}(v)} dv\right] + E_t\left[\int_t^T \xi_v e_v^* 1_{\mathcal{R}(v)} dv\right]\right. \\ &\left.-E_t\left[\int_t^T \xi_v c_v^* 1_{\mathcal{R}(v)} dv\right]\right], \end{aligned}$$

and $R(v) \equiv \{v = \tau_v^*\}$ is the immediate retirement region at time v.

Proof. On $\{t < \tau_t^*\}$, we have $J_s = E_s\left[D_{\tau_t^*}\right]$ for $t \le s < \tau_t^*$, thus J_t is a martingale

[1] $E_t[J_s] = E_t\left[E_s\left[D_{\tau_s^*}\right]\right] = E_t\left[D_{\tau_s^*}\right] \le E_t\left[D_{\tau_t^*}\right] = J_t$, $\forall s \ge t$. And $J_t = \sup_{\tau \in \mathcal{S}} E_t[D_\tau] \ge D_t$. Suppose $\overline{J}_t$ is a supermartingale majorant of D_t, we have $J_t = E_t\left[D_{\tau_t^*}\right] \le E_t\left[\overline{J}_{\tau_t^*}\right] \le \overline{J}_t$.

[2] A potential is a right-continuous, nonnegative supermartingale with expected value converging to 0 as time goes to infinity.

and $dJ_t = \psi_t dW_t$ for some ψ_t, $E\left[\int_0^T \psi_t^2 dt\right] < \infty$; on $\{t = \tau_t^*\}$, $J_t = D_t$ and is a supermartingale. Doob-Meyer decomposition gives $J_t = J_0 + M_t - A_t$, where M_t is an RCLL martingale and A_t is continuous and nondecreasing. RCLL martingale of the Brownian filtration is continuous[3], M_t is continuous, thus J_t is continuous.

$$\begin{aligned} J_T - J_t &= \int_t^T dJ_v \\ &= \int_t^T 1_{\{v<\tau_v^*\}} dJ_v + \int_t^T 1_{\{v=\tau_v^*\}} dJ_v, \end{aligned}$$

and

$$\begin{aligned} E_t\left[J_T - J_t\right] &= E_t\left[\int_t^T 1_{\{v<\tau_v^*\}} dJ_v + \int_t^T 1_{\{v=\tau_v^*\}} dJ_v\right] \\ &= E_t\left[\int_t^T 1_{\{v<\tau_v^*\}} \psi_v dW_v + \int_t^T 1_{\{v=\tau_v^*\}} dD_v\right] \\ &= E_t\left[\int_t^T 1_{\{v=\tau_v^*\}} dD_v\right]. \end{aligned}$$

Thus

$$\begin{aligned} J_t &= E_t\left[J_T\right] - E_t\left[\int_t^T 1_{\{v=\tau_v^*\}} dD_v\right] \\ &= J_t^n + J_t^a, \end{aligned}$$

[3]See Karatzas and Shreve (1991), chapter 3, problem 4.16.

where

$$\begin{aligned}
J_t^n &= E_t\left[J_T\right] \\
&= E_t\left[\int\limits_0^T a_v u^a\left(c_v^*, l_v^*\right) dv\right] + y\left[E_t\left[\int\limits_0^T \xi_v w_v dv\right] - E_t\left[\int\limits_0^T \xi_v e_v^* dv\right]\right] \\
&= \int\limits_0^t a_v u^a\left(c_v^*, l_v^*\right) dv + y\left[\int\limits_0^t \xi_v w_v dv - \int\limits_0^t \xi_v e_v^* dv\right] \\
&\quad + E_t\left[\int\limits_t^T a_v u^a\left(c_v^*, l_v^*\right) dv\right] + y\left[E_t\left[\int\limits_t^T \xi_v w_v dv\right] - E_t\left[\int\limits_t^T \xi_v e_v^* dv\right]\right],
\end{aligned}$$

and

$$\begin{aligned}
J_t^a &= -E_t\left[\int\limits_t^T 1_{\{v=\tau_v^*\}} dD_v\right] \\
&= -E_t\left[\int\limits_t^T a_v u^a\left(c_v^*, l_v^*\right) 1_{\mathcal{R}(v)} dv\right] + E_t\left[\int\limits_t^T \phi a_v u^r\left(c_v^*, 1\right) 1_{\mathcal{R}(v)} dv\right] \\
&\quad + y\left[-E_t\left[\int\limits_t^T \xi_v w_v 1_{\mathcal{R}(v)} dv\right] + E_t\left[\int\limits_t^T \xi_v e_v^* 1_{\mathcal{R}(v)} dv\right]\right. \\
&\quad \left. - E_t\left[\int\limits_t^T \xi_v c_v^* 1_{\mathcal{R}(v)} dv\right]\right]. \blacklozenge
\end{aligned}$$

J_t^n is the value of J_t when the stopping time τ is equal to T. J_t^n is a martingale. The early retirement premium J_t^a is a supermartingale which converges to 0 as t approaches T and it corresponds to the potential in the Riesz decomposition. The net local gains from early retirement is $\phi a_v u^r\left(c_v^*, 1\right) - a_v u^a\left(c_v^*, l_v^*\right) + y\xi_v e_v^* - y\xi_v w_v - y\xi_v c_v^*$. The first component $\phi a_v u^r\left(c_v^*, 1\right)$ is instantaneous utility gain from immediately retiring. The second component $-a_v u^a\left(c_v^*, l_v^*\right)$ represents the instantaneous utility loss incurred upon retiring. The next three components are transformations from monetary terms to utility terms by the marginal cost $y\xi_v$. $y\xi_v e_v^*$ represents the utility loss from total expenditure avoided by the individual from early retiring. $-y\xi_v w_v$ represents the utility gain from wages forgone by early retiring. $-y\xi_v c_v^*$ represents the utility loss incurred from retirement consumption in the next increment of time. Immediate retirement is optimal when the net local gain is sufficiently large. An immediate corollary is that an individual will never retire prior to T if $\phi a_v u^r (c_v^*, 1) - a_v u^a (c_v^*, l_v^*) + y\xi_v e_v^* - y\xi_v w_v - y\xi_v c_v^* \leq 0$ for all $v \in [0, T]$. A counterpart of the EEP representation is the delayed exercise premium (DEP) representation that emphasizes the local gains from delaying retirement.

Proposition 4.

$$J_t = D_t + J_t^d,$$

where D_t is the immediate retirement value function and the DEP is

$$\begin{aligned} J_t^d &= E_t \left[\int_t^T a_v u^a (c_v^*, l_v^*) 1_{\{v<\tau_v^*\}} dv \right] - E_t \left[\int_t^T \phi a_v u^r (c_v^*, 1) 1_{\{v<\tau_v^*\}} dv \right] \\ &\quad + y \left[E_t \left[\int_t^T \xi_v w_v 1_{\{v<\tau_v^*\}} dv \right] - E_t \left[\int_t^T \xi_v e_v^* 1_{\{v<\tau_v^*\}} dv \right] \right. \\ &\quad \left. + E_t \left[\int_t^T \xi_v c_v^* 1_{\{v<\tau_v^*\}} dv \right] \right]. \end{aligned}$$

Proof.

$$J_t = \sup_{\tau \in \mathcal{S}} E_t [D_\tau] = E_t \left[D_{\tau_t^*} \right],$$

where τ_t^* is the optimal stopping time at time t. We have

$$\begin{aligned}
J_t &= D_t + E_t\left[D_{\tau_t^*} - D_t\right] \\
&= D_t + E_t\left[\int_t^{\tau_t^*} a_v u^a\left(c_v^*, l_v^*\right) dv\right] - E_t\left[\int_t^{\tau_t^*} \phi a_v u^r\left(c_v^*, 1\right) dv\right] \\
&\quad + y\left[E_t\left[\int_t^{\tau_t^*} \xi_v w_v dv\right] - E_t\left[\int_t^{\tau_t^*} \xi_v e_v^* dv\right]\right. \\
&\quad \left. + E_t\left[\int_t^{\tau_t^*} \xi_v c_v^* dv\right]\right] \\
&= D_t + J_t^d. \blacklozenge
\end{aligned}$$

In the DEP representation, net local gains from delaying retirement is $-\phi a_v u^r\left(c_v^*, 1\right) + a_v u^a\left(c_v^*, l_v^*\right) - y\xi_v e_v^* + y\xi_v w_v + y\xi_v c_v^*$. $-\phi a_v u^r\left(c_v^*, 1\right)$ represents the instantaneous utility loss from delaying retiring, but the individual benefits from the instantaneous utility gain $a_v u^a\left(c_v^*, l_v^*\right)$ before retirement. $-y\xi_v e_v^*$ represents the utility loss from total expenditure incurred by the individual from delaying retiring. $y\xi_v w_v$ represents the utility gain from wages enjoyed by delaying retiring. $y\xi_v c_v^*$ represents the utility loss from retirement consumption avoided by the individual from delaying retiring.

When $\tau_t^* = t$, immediate retirement value function is

$$\begin{aligned}
D_t &= \int_0^t a_v u^a\left(c_v^*, l_v^*\right) dv + E_t\left[\int_t^T \phi a_v u^r\left(c_v^*, 1\right) dv\right] \\
&\quad + y\left[\int_0^t \xi_v w_v dv - \int_0^t \xi_v e_v^* dv - E_t\left[\int_t^T \xi_v c_v^* dv\right]\right].
\end{aligned}$$

In the immediate retirement region $\mathcal{R}(t)$, $J_t = J_t^n + J_t^a = D_t$. This gives the equation satisfied by the state variable $x_t \equiv \left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} w_t^{(1-\eta)\rho-1}$ in the retirement region.

Proposition 5. *In the immediate retirement region $R(t)$, we have*

$$\begin{aligned} & E_t\left[\int_t^T a_v u^a\left(c_v^*, l_v^*\right) 1_{\mathcal{A}(v)} dv\right] + yE_t\left[\int_t^T \xi_v w_v 1_{\mathcal{A}(v)} dv\right] \\ & -yE_t\left[\int_t^T \xi_v e_v^* 1_{\mathcal{A}(v)} dv\right] - E_t\left[\int_t^T \phi a_v u^r\left(c_v^*, 1\right) 1_{\mathcal{A}(v)} dv\right] \\ & +yE_t\left[\int_t^T \xi_v c_v^* 1_{\mathcal{A}(v)} dv\right] \\ = \; & 0. \end{aligned} \tag{2.9}$$

For c_v^*, l_v^* and e_v^* in power utility, we have

$$\begin{aligned} & \frac{R}{1-R} f x_t G\left(t, T; \beta, \rho, \eta, \mathcal{A}\right) + G\left(t, T; 0, 1, 0, \mathcal{A}\right) \\ & -\frac{R}{1-R} \phi^{\frac{1}{R}} z_t G\left(t, T; \beta, \rho, 1, \mathcal{A}\right) = 0, \end{aligned}$$

where

$$x_t \equiv \left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} w_t^{(1-\eta)\rho-1}, \quad z_t \equiv \left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} w_t^{-1},$$

$$G\left(t, T; \beta, \rho, \eta\right) \equiv E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}} \xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho} dv\right],$$

$$G\left(t, T; \beta, \rho, \eta, \mathcal{A}\right) \equiv E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}} \xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho} 1_{\mathcal{A}(v)} dv\right],$$

and $A(v) \equiv \{v < \tau_v^*\}$ is the continuation region at time v.

Proof. Plugin the representations of J_t^n, J_t^a and Y_t and simplify, we get

$$\begin{aligned}
& E_t\left[\int_t^T a_v u^a\left(c_v^*, l_v^*\right) dv\right] + y\left[E_t\left[\int_t^T \xi_v w_v dv\right] - E_t\left[\int_t^T \xi_v e_v^* dv\right]\right] \\
& -E_t\left[\int_t^T a_v u^a\left(c_v^*, l_v^*\right) 1_{\mathcal{R}(v)} dv\right] + E_t\left[\int_t^T \phi a_v u^r\left(c_v^*, 1\right) 1_{\mathcal{R}(v)} dv\right] \\
& +y\left[-E_t\left[\int_t^T \xi_v w_v 1_{\mathcal{R}(v)} dv\right] + E_t\left[\int_t^T \xi_v e_v^* 1_{\mathcal{R}(v)} dv\right]\right. \\
& \left.-E_t\left[\int_t^T \xi_v c_v^* 1_{\mathcal{R}(v)} dv\right]\right] \\
= \; & E_t\left[\int_t^T \phi a_v u^r\left(c_v^*, 1\right) dv\right] - yE_t\left[\int_t^T \xi_v c_v^* dv\right],
\end{aligned}$$

or

$$\begin{aligned}
& E_t\left[\int_t^T a_v u^a\left(c_v^*, l_v^*\right) 1_{\mathcal{A}(v)} dv\right] + yE_t\left[\int_t^T \xi_v w_v 1_{\mathcal{A}(v)} dv\right] \\
& -yE_t\left[\int_t^T \xi_v e_v^* 1_{\mathcal{A}(v)} dv\right] - E_t\left[\int_t^T \phi a_v u^r\left(c_v^*, 1\right) 1_{\mathcal{A}(v)} dv\right] \\
& +yE_t\left[\int_t^T \xi_v c_v^* 1_{\mathcal{A}(v)} dv\right] \\
= \; & 0.
\end{aligned}$$

For power utility, we have

$$E_t\left[\int_t^T a_v u^a\left(c_v^*, l_v^*\right)dv\right] = E_t\left[\int_t^T \frac{y\xi_v e_v^*}{1-R}dv\right]$$
$$= \frac{y\xi_t e_v^*}{1-R}E_t\left[\int_t^T \xi_{t,v} e_{t,v}^* dv\right] = \frac{y\xi_t e_v^*}{1-R}E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho}dv\right]$$
$$= \frac{y\xi_t e_v^*}{1-R}G\left(t,T;\beta,\rho,\eta\right),$$

$$E_t\left[\int_t^T \phi a_v u^r\left(c_v^*, 1\right)dv\right] = E_t\left[\int_t^T \frac{y\xi_v c_v^*}{1-R}dv\right]$$
$$= \frac{y\xi_t c_v^*}{1-R}E_t\left[\int_t^T \xi_{t,v} c_{t,v}^* dv\right] = \frac{y\xi_t c_v^*}{1-R}E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}dv\right]$$
$$= \frac{y\xi_t c_v^*}{1-R}G\left(t,T;\beta,\rho,1\right),$$

$$E_t\left[\int_t^T \xi_v e_v^* dv\right] = \xi_t e_v^* E_t\left[\int_t^T \xi_{t,v} e_{t,v}^* dv\right]$$
$$= \xi_t e_v^* E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho}dv\right] = \xi_t e_v^* G\left(t,T;\beta,\rho,\eta\right),$$

$$E_t\left[\int_t^T \xi_v w_v dv\right] = \xi_t w_t E_t\left[\int_t^T \xi_{t,v} w_{t,v} dv\right] = \xi_t w_t G\left(t,T;0,1,0\right),$$

$$E_t\left[\int_t^T \xi_v c_v^* dv\right] = \xi_t c_v^* E_t\left[\int_t^T \xi_{t,v} c_{t,v}^* dv\right]$$
$$= \xi_t c_v^* E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}dv\right] = \xi_t c_v^* G\left(t,T;\beta,\rho,1\right).$$

Therefore,

$$\begin{aligned}
& \frac{y\xi_t e_v^*}{1-R} G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) + y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\
& - y\xi_t e_v^* G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) - \frac{y\xi_t c_v^*}{1-R} G\left(t,T;\beta,\rho,1,\mathcal{A}\right) \\
& + y\xi_t c_v^* G\left(t,T;\beta,\rho,1,\mathcal{A}\right) \\
= \; & 0,
\end{aligned}$$

or

$$\begin{aligned}
& \frac{R}{1-R} y\xi_t e_v^* G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) + y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\
& - \frac{R}{1-R} y\xi_t c_v^* G\left(t,T;\beta,\rho,1,\mathcal{A}\right) \\
= \; & 0.
\end{aligned}$$

Plugin the optimal values of e_t^* and c_v^*, we have

$$\begin{aligned}
& \frac{R}{1-R} f\left(a_t\right)^{\frac{1}{R}} \left(y\xi_t\right)^{\rho} w_t^{(1-\eta)\rho} G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) + y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\
& - \frac{R}{1-R} \phi^{\frac{1}{R}} \left(a_t\right)^{\frac{1}{R}} \left(y\xi_t\right)^{\rho} G\left(t,T;\beta,\rho,1,\mathcal{A}\right) \\
= \; & 0,
\end{aligned}$$

or

$$\begin{aligned}
& \frac{R}{1-R} f x_t G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) + G\left(t,T;0,1,0,\mathcal{A}\right) \\
& - \frac{R}{1-R} \phi^{\frac{1}{R}} z_t G\left(t,T;\beta,\rho,1,\mathcal{A}\right) \\
= \; & 0. \; \blacklozenge
\end{aligned}$$

How do we calculate the terms in equation (2.9) which are conditional expectations of random variables restricted to the event of retirement? The next lemma gives closed form expressions for conditional expectations of random variables restricted to the event of retirement and their derivatives with respect to the multiplier y. These expressions are used for the representation of the optimal retirement boundary, the optimal wealth and the optimal portfolio. First we specify a standing assumption that the immediate retirement region $\mathcal{R}(t)$ is up connected for the state variable x_t, then we can calculate the conditional expectations involving the event $\{x_t \geq B_t\}$.

Assumption. *The immediate retirement region $R(t)$ is up connected for the state variable x_t.*

Sufficient conditions to satisfy this assumption are $R > 1$ *and* $\sigma_w \leq \theta/R$. Under these conditions, we can show that if x_t is in the retirement region, then $(\lambda x_t, t)$ is also in the retirement region, $\forall \lambda \geq 1$. See Proposition A1 and its proof in Appendix.

Lemma 6. *Define*

$$C(\rho, \eta) = \rho\left(\theta - (1-\eta)\sigma_w\right),$$

$$A(\beta, \rho, \eta) = \frac{\beta}{R} + \rho\left(r + \frac{1}{2}\theta^2\right) - (1-\eta)\rho\left(\mu_w - \frac{1}{2}\sigma_w^2\right) - \frac{1}{2}C(\rho, \eta)^2,$$

and

$$d(x_t, B_v, v; C(\rho, \eta)) = \frac{1}{\sigma_x\sqrt{v-t}}\left[\log\left(\frac{x_t}{B_v}\right) + \left(\mu_x - \frac{1}{2}\sigma_x^2 - \sigma_x C(\rho, \eta)\right)(v-t)\right].$$

Then we have

$$\begin{aligned} & G(t, T; \beta, \rho, \eta) \\ = \; & E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}} \xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho} dv\right] \\ = \; & \int_t^T \exp\left(-A(\beta, \rho, \eta)(v-t)\right) dv, \end{aligned}$$

$$
\begin{aligned}
& G\left(t,T;\beta,\rho,\eta,\mathcal{R}\right) \\
= \; & E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}1_{\mathcal{R}(v)}dv\right] \\
= \; & \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right)N\left(d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)dv,
\end{aligned}
$$

$$
\begin{aligned}
& G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) \\
= \; & E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}1_{\mathcal{A}(v)}dv\right] \\
= \; & \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right)N\left(-d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)dv,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial y}G\left(t,T;\beta,\rho,\eta,\mathcal{R}\right) \\
= \; & \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right)n\left(d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right) \\
& \times\frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial x_t}{\partial y}}{x_t}-\frac{\frac{\partial B_v}{\partial y}}{B_v}\right)dv,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial y}G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) \\
= \; & \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right)n\left(-d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right) \\
& \times\frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v}-\frac{\frac{\partial x_t}{\partial y}}{x_t}\right)dv,
\end{aligned}
$$

where $N(\cdot)$ is the cumulative distribution function of standard normal distribution and $n(\cdot)$ is the probability density function of standard normal distribution.

Proof. See appendix. ♦

Now we give the backward recursive equation satisfied by the retirement boundary B_t of the state variable x_t. The boundary B_t separates the retirement region $\mathcal{R}(t)$ and the continuation region $\mathcal{A}(t)$, and the individual is optimal to retire when x_t in $\mathcal{A}(t)$ first crosses B_t.

Theorem 7. *The retirement boundary B_t satisfies the following backward recursive equation*

$$\begin{aligned} & \frac{R}{1-R} f B_t \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right) N\left(-d\left(B_t, B_v, v; C\left(\rho,\eta\right)\right)\right) dv \\ & + \int_t^T \exp\left(-A\left(0,1,0\right)\left(v-t\right)\right) N\left(-d\left(B_t, B_v, v; C\left(1,0\right)\right)\right) dv \\ & - \frac{R}{1-R} \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp\left(\delta t\right) B_t^{\gamma} \\ & \times \int_t^T \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right) N\left(-d\left(B_t, B_v, v; C\left(\rho,1\right)\right)\right) dv \\ = \; & 0, \end{aligned} \tag{2.10}$$

with limiting condition of the boundary B_T that satisfies

$$f B_T - \phi^{\frac{1}{R}} z_0 \left(\frac{B_T}{x_0}\right)^{\gamma} \exp\left(\delta T\right) = 1 - \frac{1}{R},$$

where

$$\gamma = \frac{\sigma_z}{\sigma_x}, \;\; \delta = \mu_z - \frac{1}{2}\sigma_z^2 - \gamma\left(\mu_x - \frac{1}{2}\sigma_x^2\right) \;\; \textit{and} \;\; z_t = \left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} w_t^{-1}.$$

Proof. We have shown that in the retirement region,

$$\begin{aligned} & \frac{R}{1-R} f x_t G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) + G\left(t,T;0,1,0,\mathcal{A}\right) \\ & - \frac{R}{1-R} \phi^{\frac{1}{R}} z_t G\left(t,T;\beta,\rho,1,\mathcal{A}\right) = 0, \end{aligned}$$

where $x_v = \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho-1}$ and $z_v = \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{-1}$. For x_v and z_v we have

$$\mu_x = -\frac{\beta}{R} + \frac{1}{R}\left(r + \frac{1}{2}\theta^2\right) + ((1-\eta)\rho - 1)\left(\mu_w - \frac{1}{2}\sigma_w^2\right) + \frac{1}{2}\sigma_x^2,$$

$$\sigma_x = \frac{\theta}{R} + ((1-\eta)\rho - 1)\sigma_w,$$

$$\mu_z = -\frac{\beta}{R} + \frac{1}{R}\left(r + \frac{1}{2}\theta^2\right) - \left(\mu_w - \frac{1}{2}\sigma_w^2\right) + \frac{1}{2}\sigma_z^2,$$

$$\sigma_z = \frac{\theta}{R} - \sigma_w.$$

z_v is a transform of x_v,

$$z_v = z_0\left(\frac{x_v}{x_0}\right)^{\gamma}\exp(\delta v),$$

where $\gamma = \frac{\sigma_z}{\sigma_x}$, $\delta = \mu_z - \frac{1}{2}\sigma_z^2 - \gamma\left(\mu_x - \frac{1}{2}\sigma_x^2\right)$. Therefore we have in the retirement region,

$$\begin{aligned} & \frac{R}{1-R} f x_t \int_t^T \exp(-A(\beta,\rho,\eta)(v-t))\, N(-d(x_t, B_v, v; C(\rho,\eta)))\, dv \\ & + \int_t^T \exp(-A(0,1,0)(v-t))\, N(-d(x_t, B_v, v; C(1,0)))\, dv \\ & - \frac{R}{1-R}\phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta t)\, x_t^{\gamma} \\ & \times \int_t^T \exp(-A(\beta,\rho,1)(v-t))\, N(-d(x_t, B_v, v; C(\rho,1)))\, dv \\ = {} & 0. \end{aligned}$$

Substituting x_t using B_t, the boundary B_t satisfies

$$\begin{aligned}
&\frac{R}{1-R} f B_t \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right) N\left(-d\left(B_t, B_v, v; C\left(\rho,\eta\right)\right)\right) dv \\
&+ \int_t^T \exp\left(-A\left(0,1,0\right)\left(v-t\right)\right) N\left(-d\left(B_t, B_v, v; C\left(1,0\right)\right)\right) dv \\
&- \frac{R}{1-R} \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp\left(\delta t\right) B_t^{\gamma} \\
&\times \int_t^T \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right) N\left(-d\left(B_t, B_v, v; C\left(\rho,1\right)\right)\right) dv \\
= \;& 0.
\end{aligned}$$

Because terminal date T is considered as the deadline for retirement, at time T, retirement must happen. The boundary condition for the backward recursive equation should be that the instantaneous gain minus instantaneous loss in the early exercise premium is equal to 0. In the early exercise premium representation, the instantaneous gain minus instantaneous loss is

$$\begin{aligned}
& -a_v u^a\left(c_v^*, l_v\right) + \phi a_v u^r\left(c_v^*, 1\right) - y\xi_v w_v + y\xi_v e_v^* - y\xi_v c_v^* \\
= \;& -\frac{y\xi_v e_v^*}{1-R} + \frac{y\xi_v c_v^*}{1-R} - y\xi_v w_v + y\xi_v e_v^* - y\xi_v c_v^* \\
= \;& -\frac{R}{1-R} y\xi_v e_v^* + \frac{R}{1-R} y\xi_v c_v^* - y\xi_v w_v.
\end{aligned}$$

Let $-\frac{R}{1-R} y\xi_v e_v^* + \frac{R}{1-R} y\xi_v c_v^* - y\xi_v w_v$ be equal to 0, we have

$$\begin{aligned}
-\frac{R}{1-R} y\xi_v e_v^* + \frac{R}{1-R} y\xi_v c_v^* - y\xi_v w_v &= 0 \\
-\frac{R}{1-R}\left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho-1} f + \frac{R}{1-R}\left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{-1} \phi^{\frac{1}{R}} - 1 &= 0.
\end{aligned}$$

Using the expressions of x_v and z_v, we have

$$f x_v - \phi^{\frac{1}{R}} z_v = 1 - \frac{1}{R},$$

or

$$f x_v - \phi^{\frac{1}{R}} z_0 \left(\frac{x_v}{x_0}\right)^{\gamma} \exp(\delta v) = 1 - \frac{1}{R}.$$

Thus at time T, the limiting condition for B_T is

$$f B_T - \phi^{\frac{1}{R}} z_0 \left(\frac{B_T}{x_0}\right)^{\gamma} \exp(\delta T) = 1 - \frac{1}{R}. \blacklozenge$$

Equation (2.10) is a backward recursive equation. To compute the solution B_t, we use a recursive algorithm. Divide the interval $[0, T]$ into N subintervals $[t_{i-1}, t_i]$, for $i = 1, \ldots, N$, with equal length $\Delta t = T/N$. The terminal condition is $B_{t_N} = B_T$. We use trapezoidal rule to discretize the integrals in equation (2.10)[4]. For the first integral we have

$$\begin{aligned} & \int_t^T \exp(-A(\beta, \rho, \eta)(v - t)) N(-d(B_t, B_v, v; C(\rho, \eta))) \, dv \\ \approx & \sum_{n=i+1}^{N-1} \exp(-A(\beta, \rho, \eta)(n - i)\Delta t) N(-d(B_{t_i}, B_{t_n}, t_n; C(\rho, \eta))) \Delta t \\ & + N(-d(B_{t_i}, B_{t_i}, t_i; C(\rho, \eta))) \Delta t/2 \\ & + \exp(-A(\beta, \rho, \eta)(N - i)\Delta t) N(-d(B_{t_i}, B_{t_N}, t_N; C(\rho, \eta))) \Delta t/2. \end{aligned}$$

We do the same discretization for the other two integrals. Suppose we have solved B_{t_j} for all $j > i$, then the discretized equation (2.10) becomes a nonlinear equation of B_{t_i}. At each time t_i, $i = 0, \ldots, N-1$, we solve this nonlinear equation recursively. The true retirement boundary is the limit of the solution of the discretized equation (2.10) when

[4]See Detemple (2006) chapter 8.2 for numerical procedures for solving backward recursive equations derived from problems of pricing American options.

N goes to infinity. To solve the nonlinear equations at each step i, we implement an efficient procedure using Newton-Raphson iteration developed by Kallast and Kivinukk (2003) for pricing American options. We denote the left hand side of the discretized equation (2.10) by $F(B_{t_i})$. The iterative values are determined by

$$B_{t_i,k+1} = B_{t_i,k} - F(B_{t_i,k})/F'(B_{t_i,k}), \quad \text{for } k = 0, 1, 2, \ldots$$

where $F'(B_{t_i,k})$ is the derivative of $F(B_{t_i,k})$ with respect to $B_{t_i,k}$ and the initial value $B_{t_i,0} = B_{t_{i+1}}$. The iteration is run until the difference between two successive iterates is sufficiently small. The convergence of the iteration is very fast since the point in the boundary immediately after t_i is chosen as the initial value at each step, thus is very close to the true value. Figure 2.1 gives the optimal retirement boundary for state variable x_t when the initial liquid wealth is 0 with parameter values in Table 2.1[5]. Note that this optimal retirement boundary depends on time and an initial Lagrange multiplier but does not depend on the underlying Brownian motion. The individual is optimal to retire the moment that the state variable x_t hits the boundary. Of course, the state variable x_t is not directly observable. The reason we choose to construct the retirement boundary for x_t is to facilitate the computation of equation (2.10). Later we will construct an optimal retirement boundary for the liquid wealth X_t. Because X_t is directly observable, the boundary of X_t is more practical and useful and an individual can compare his observed liquid wealth with this boundary and know if it's optimal to retire immediately.

Next proposition gives the equation satisfied by the derivative of the retirement boundary with respect to the multiplier y. $\frac{\partial B_t}{\partial y}$ is needed for the expressions and computations of the optimal wealth and the optimal portfolio.

[5] Parameter values are the same as in the numerical studies of Bodie et al. (2012) which solves the model for the case of deterministic retirement date.

Table 2.1: Parameter values.

Interest rate r	0.02
Volatility of the market return σ	0.2
Market price of risk θ	0.3
Maximal amount of labor $\overline{h}$	1
Expected growth rate of wage μ_w	0.01
Volatility of growth rate of wage σ_w	0.03
Initial value of wage w_0	10^5
Measure of relative weight of consumption and labor η	2/3
Coefficient of relative risk aversion R	4
Subjective discount rate β	0
Relative weight of the retirement phase ϕ	20^4

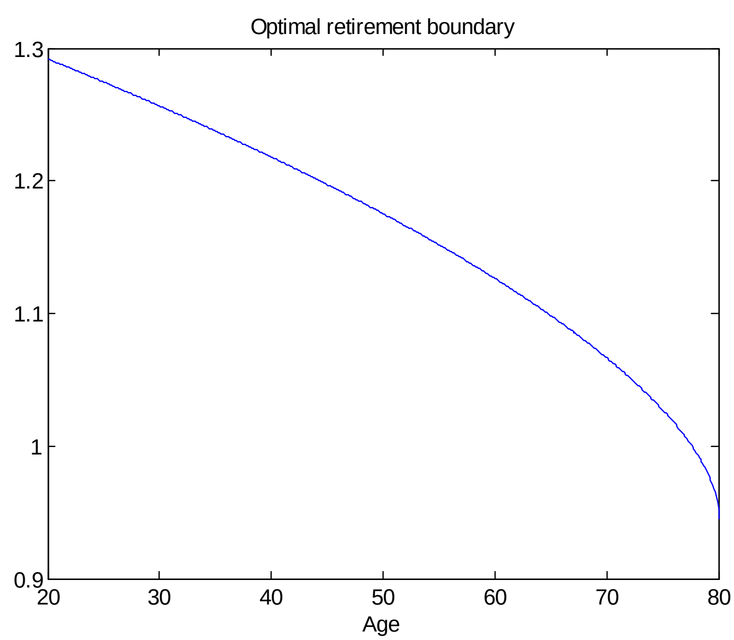

Figure 2·1: This figure shows the optimal retirement boundary for state variable x_t. Initial liquid wealth is 0.

Proposition 8. *The derivative of the boundary* $\frac{\partial B_t}{\partial y}$ *satisfies*

$$
\begin{aligned}
& \frac{R}{1-R} f \frac{\partial B_t}{\partial y} G^B(t,T;\beta,\rho,\eta,\mathcal{A}) \qquad (2.11) \\
& -\frac{R}{1-R} \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta t) B_t^{\gamma} \\
& \times G^B(t,T;\beta,\rho,1,\mathcal{A}) \left(y^{-1} \left(-\frac{1}{R}(1-\gamma) \right) + \gamma \frac{\frac{\partial B_t}{\partial y}}{B_t} \right) \\
& + \frac{R}{1-R} f B_t \frac{\partial}{\partial y} G^B(t,T;\beta,\rho,\eta,\mathcal{A}) + \frac{\partial}{\partial y} G^B(t,T;0,1,0,\mathcal{A}) \\
& -\frac{R}{1-R} \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta t) B_t^{\gamma} \frac{\partial}{\partial y} G^B(t,T;\beta,\rho,1,\mathcal{A}) \\
= \; & 0,
\end{aligned}
$$

with boundary condition

$$
\frac{\partial B_T}{\partial y} = \frac{\phi^{\frac{1}{R}} \left(-\frac{1}{R}(1-\gamma) \right) y^{-\frac{1}{R}(1-\gamma)-1} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta T) B_T^{\gamma}}{f - \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta T) \gamma B_T^{\gamma-1}}, \qquad (2.12)
$$

where

$$
\begin{aligned}
& \frac{\partial}{\partial y} G^B(t,T;\beta,\rho,\eta,\mathcal{A}) \\
= \; & \int_t^T \exp(-A(\beta,\rho,\eta)(v-t)) \, n(-d(B_t,B_v,v;C(\rho,\eta))) \\
& \times \frac{1}{\sigma_x \sqrt{v-t}} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial B_t}{\partial y}}{B_t} \right) dv.
\end{aligned}
$$

Proof. See appendix. ♦

Instead of solving the backward equation (2.11) using similar Newton-Raphson recursive algorithm, a much easier way to compute the derivative of the boundary with respect to the multiplier is to increase the multiplier y by a small amount Δy, for this new multiplier $y + \Delta y$, use the same algorithm that solves equation (2.10) to compute a new boundary $B_t(y + \Delta y)$. The derivative can be approximated by $\frac{\partial B_t}{\partial y} \approx$

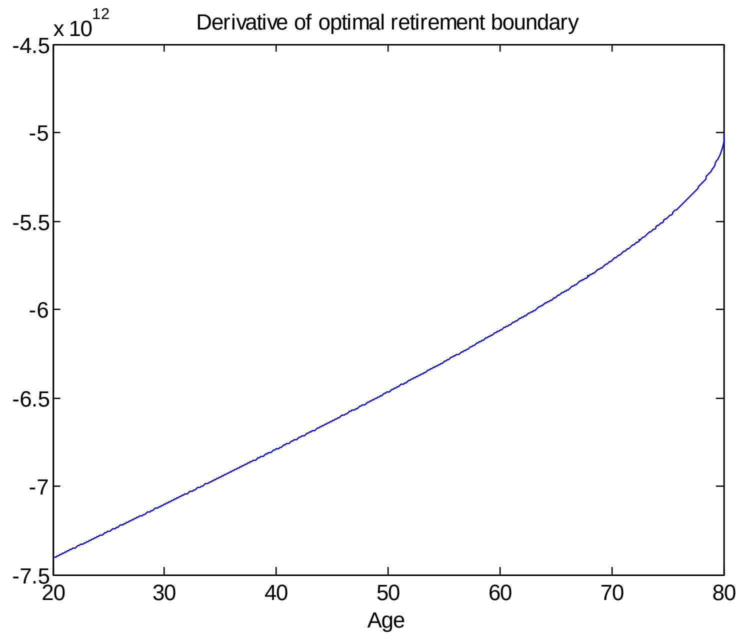

Figure 2·2: This figure shows the derivative of the optimal retirement boundary with respect to the multiplier y.

$(B_t(y + \Delta y) - B_t(y)) / \Delta y$. Figure 2.2 gives the derivative of the optimal retirement boundary with respect to the multiplier y for initial liquid wealth 0. In general, $\frac{\partial B_t}{\partial y}$ may be positive or negative depending on the parameter values, see the terminal condition (2.12). For the parameter values in Table 1.1, $\frac{\partial B_t}{\partial y}$ is negative and the retirement boundary will move higher for a smaller value of the multiplier.

2.5 Optimal liquid wealth, human capital and total wealth

For the optimal liquid wealth X_t, the human capital H_t and the total wealth N_t at time t before retirement, we have

$$\xi_t X_t = E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv \right],$$

$$\xi_t H_t = E_t\left[\int_t^{\tau_t^*} \xi_v w_v dv\right],$$

and

$$\xi_t N_t = E_t\left[\int_t^{\tau_t^*} \xi_v e_v^* dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv\right].$$

The human capital is the present value of the maximal labor income until retirement. The total wealth is the sum of the liquid wealth and the human capital, $N_t = X_t + H_t$. The total wealth finances the total expenditure on consumption and leisure during the accumulation phase and the consumption during the retirement phase. After retirement, human capital is depleted. The total wealth is equal to the liquid wealth, which solely finances the consumption during retirement and satisfies $\xi_t X_t = E_t\left[\int_t^T \xi_v c_v^* dv\right]$. These expressions of the wealth components are not amenable to computational implementation and the next theorem gives the closed form expressions of the liquid wealth, the human capital and the total wealth.

Theorem 9. *We have the following representations of the wealth processes. Before retirement, the optimal liquid wealth X_t satisfies*

$$\begin{aligned}
X_t &= \xi_t^{-1} E_t\left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv\right] \\
&= \xi_t^{-1} E_t\left[\int_t^{\tau_t^*} f a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} w_v^{(1-\eta)\rho} dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} dv\right] \\
&= f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} G(t,T;\beta,\rho,\eta,\mathcal{A}) - w_t G(t,T;0,1,0,\mathcal{A}) \qquad (2.13)\\
&\quad + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G(t,T;\beta,\rho,1,\mathcal{R}) \\
&\quad - \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} \frac{\partial}{\partial y} G(t,T;\beta,\rho,\eta,\mathcal{A}) \\
&\quad - y w_t \frac{\partial}{\partial y} G(t,T;0,1,0,\mathcal{A}) - \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G(t,T;\beta,\rho,1,\mathcal{R}),
\end{aligned}$$

the human capital is

$$\begin{aligned} H_t &= \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v w_v dv \right] \\ &= w_t G(t, T; 0, 1, 0, \mathcal{A}) \\ &\quad + \frac{R^2}{1-R} f a_t^{\frac{1}{R}} y^\rho \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} \frac{\partial}{\partial y} G(t, T; \beta, \rho, \eta, \mathcal{A}) + R y w_t \frac{\partial}{\partial y} G(t, T; 0, 1, 0, \mathcal{A}) \\ &\quad + \frac{R^2}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^\rho \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G(t, T; \beta, \rho, 1, \mathcal{R}), \end{aligned}$$

and the total wealth is,

$$\begin{aligned} N_t &= \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv + \int_{\tau_t^*}^T \xi_v c_v^* dv \right] \\ &= (y\xi_t)^{-1} E_t \left[\int_t^{\tau_t^*} a_v^{\frac{1}{R}} (y\xi_v)^\rho w_v^{(1-\eta)\rho} f dv + \int_{\tau_t^*}^T \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} (y\xi_v)^\rho dv \right] \\ &= f a_t^{\frac{1}{R}} (y\xi_t)^{-\frac{1}{R}} w_t^{(1-\eta)\rho} G(t, T; \beta, \rho, \eta, \mathcal{A}) + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} (y\xi_t)^{-\frac{1}{R}} G(t, T; \beta, \rho, 1, \mathcal{R}) \\ &\quad - R f a_t^{\frac{1}{R}} y^\rho \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} \frac{\partial}{\partial y} G(t, T; \beta, \rho, \eta, \mathcal{A}) - (1-R) y w_t \frac{\partial}{\partial y} G(t, T; 0, 1, 0, \mathcal{A}) \\ &\quad - R \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^\rho \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G(t, T; \beta, \rho, 1, \mathcal{R}). \end{aligned}$$

After retirement, the human capital is exhausted and the liquid wealth is

$$\begin{aligned} X_t &= \xi_t^{-1} E_t \left[\int_t^T \xi_v c_v^* dv \right] = \xi_t^{-1} E_t \left[\int_t^T \xi_v \left(\frac{y\xi_v}{a_v} \right)^{-\frac{1}{R}} \phi^{\frac{1}{R}} dv \right] \\ &= \phi^{\frac{1}{R}} \left(\frac{y\xi_t}{a_t} \right)^{-\frac{1}{R}} G(t, T; \beta, \rho, 1). \end{aligned}$$

Proof. We have

$$\begin{aligned}
& J_t \\
= & \sup_{\tau\in\mathcal{S}} E_t \left[\int_0^\tau \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho w_v^{(1-\eta)\rho} dv + \int_0^\tau y\xi_v w_v dv \right. \\
& \left. + \int_\tau^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho dv \right] \\
= & \int_0^t \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho w_v^{(1-\eta)\rho} dv + \int_0^t y\xi_v w_v dv \\
& + \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho w_v^{(1-\eta)\rho} dv + \int_t^\tau y\xi_v w_v dv \right. \\
& \left. + \int_\tau^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho dv \right].
\end{aligned}$$

Denote

$$L_t \equiv J_t - \left[\int_0^t \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho w_v^{(1-\eta)\rho} dv + \int_0^t y\xi_v w_v dv \right].$$

Thus

$$\begin{aligned}
L_t = & \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho w_v^{(1-\eta)\rho} dv + \int_t^\tau y\xi_v w_v dv \right. \\
& \left. + \int_\tau^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho dv \right] \\
= & E_t \left[\int_t^{\tau_t^*} \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho w_v^{(1-\eta)\rho} dv + \int_t^{\tau_t^*} y\xi_v w_v dv \right. \\
& \left. + \int_{\tau_t^*}^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho dv \right],
\end{aligned}$$

where τ_t^* is the optimal stopping time. Liquid wealth X_t satisfies

$$\xi_t X_t = E_t\left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^T \xi_v c_v^* dv\right].$$

Plug in the values of e_v^* and c_v^*,

$$\xi_t X_t = E_t\left[\int_t^{\tau_t^*} f a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} w_v^{(1-\eta)\rho} dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^T \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} dv\right].$$

Use the same proof in Proposition 2, L_t is a convex function of y and the derivative of L_t with respect to the multiplier y satisfies

$$\begin{aligned} \frac{\partial L_t}{\partial y} &= E_t\left[\int_t^{\tau_t^*} -f a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^{\tau_t^*} \xi_v w_v dv - \int_{\tau_t^*}^T \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} dv\right] \\ &= -\xi_t X_t. \end{aligned}$$

Use Early Exercise Premium representation to write L_t as the following

$$\begin{aligned} L_t &= E_t\left[\int_t^{\tau_t^*} \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^{\tau_t^*} y\xi_v w_v dv \right. \\ &\quad \left. + \int_{\tau_t^*}^{T} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} dv \right] \\ &= E_t\left[\int_t^{T} \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^{T} y\xi_v w_v dv \right] \\ &\quad - E_t\left[\int_t^{T} \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho} 1_{\mathcal{R}(v)} dv + \int_t^{T} y\xi_v w_v 1_{\mathcal{R}(v)} dv \right. \\ &\quad \left. - \int_t^{T} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} 1_{\mathcal{R}(v)} dv \right] \\ &= \frac{R}{1-R} f a_t^{\frac{1}{R}} \left(y\xi_t\right)^{\rho} w_t^{(1-\eta)\rho} G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) + y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\ &\quad + \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} \left(y\xi_t\right)^{\rho} G\left(t,T;\beta,\rho,1,\mathcal{R}\right). \end{aligned} \tag{2.14}$$

Therefore,

$$
\begin{aligned}
& \xi_t X_t \\
= \;& E_t\left[\int_t^{\tau_t^*} f a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} w_v^{(1-\eta)\rho} dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} dv\right] \qquad (2.15) \\
= \;& -\frac{\partial L_t}{\partial y} \\
= \;& f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{\rho} w_t^{(1-\eta)\rho} G\left(t, T; \beta, \rho, \eta, \mathcal{A}\right) - \xi_t w_t G\left(t, T; 0, 1, 0, \mathcal{A}\right) \\
& + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{\rho} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
& - \frac{R}{1-R} f a_t^{\frac{1}{R}} \left(y\xi_t\right)^{\rho} w_t^{(1-\eta)\rho} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, \eta, \mathcal{A}\right) \\
& - y \xi_t w_t \frac{\partial}{\partial y} G\left(t, T; 0, 1, 0, \mathcal{A}\right) \\
& - \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} \left(y\xi_t\right)^{\rho} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right),
\end{aligned}
$$

and wealth process X_t is

$$
\begin{aligned}
& X_t \\
= \;& f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} G\left(t, T; \beta, \rho, \eta, \mathcal{A}\right) - w_t G\left(t, T; 0, 1, 0, \mathcal{A}\right) \\
& + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
& - \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, \eta, \mathcal{A}\right) \\
& - y w_t \frac{\partial}{\partial y} G\left(t, T; 0, 1, 0, \mathcal{A}\right) - \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right).
\end{aligned}
$$

(2.14) $-\frac{R}{1-R}y\times$ (2.15), canceling terms, we get

$$\begin{aligned}
& E_t\left[\int_t^{\tau_t^*} y\xi_v w_v dv + \int_t^{\tau_t^*} \frac{R}{1-R} y\xi_v w_v dv\right] \\
= \; & y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) + \frac{R}{1-R} y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\
& + y\left(\frac{R}{1-R}\right)^2 f a_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho} w_t^{(1-\eta)\rho}\frac{\partial}{\partial y} G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) \\
& + y\frac{R}{1-R} y\xi_t w_t \frac{\partial}{\partial y} G\left(t,T;0,1,0,\mathcal{A}\right) \\
& + y\left(\frac{R}{1-R}\right)^2 \phi^{\frac{1}{R}} a_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}\frac{\partial}{\partial y} G\left(t,T;\beta,\rho,1,\mathcal{R}\right).
\end{aligned}$$

Thus human capital is

$$\begin{aligned}
& \xi_t^{-1} E_t\left[\int_t^{\tau_t^*} \xi_v w_v dv\right] \\
= \; & w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\
& + \frac{R^2}{1-R} f a_t^{\frac{1}{R}} y^{\rho}\xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho}\frac{\partial}{\partial y} G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right) + R y w_t \frac{\partial}{\partial y} G\left(t,T;0,1,0,\mathcal{A}\right) \\
& + \frac{R^2}{1-R}\phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho}\xi_t^{-\frac{1}{R}}\frac{\partial}{\partial y} G\left(t,T;\beta,\rho,1,\mathcal{R}\right).
\end{aligned}$$

(2.14) + $y \times$ (2.15), canceling terms, we get

$$\begin{aligned}
& E_t\left[\int_t^{\tau_t^*}\frac{1}{1-R}y\xi_v e_v^* dv+\int_{\tau_t^*}^{T}\frac{1}{1-R}y\xi_v c_v^* dv\right]\\
= \; & E_t\left[\int_t^{\tau_t^*}\frac{1}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}w_v^{(1-\eta)\rho}f dv+\int_{\tau_t^*}^{T}\frac{1}{1-R}\phi^{\frac{1}{R}}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}dv\right]\\
= \; & \frac{1}{1-R}fa_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}w_t^{(1-\eta)\rho}G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right)\\
& +\frac{1}{1-R}\phi^{\frac{1}{R}}a_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}G\left(t,T;\beta,\rho,1,\mathcal{R}\right)\\
& -y\frac{R}{1-R}fa_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}w_t^{(1-\eta)\rho}\frac{\partial G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right)}{\partial y}-y^2\xi_t w_t\frac{\partial G\left(t,T;0,1,0,\mathcal{A}\right)}{\partial y}\\
& -y\frac{R}{1-R}\phi^{\frac{1}{R}}a_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}\frac{\partial G\left(t,T;\beta,\rho,1,\mathcal{R}\right)}{\partial y}.
\end{aligned}$$

Thus total wealth is

$$\begin{aligned}
& \xi_t^{-1}E_t\left[\int_t^{\tau_t^*}\xi_v e_v^* dv+\int_{\tau_t^*}^{T}\xi_v c_v^* dv\right]\\
= \; & \left(y\xi_t\right)^{-1}E_t\left[\int_t^{\tau_t^*}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}w_v^{(1-\eta)\rho}f dv+\int_{\tau_t^*}^{T}\phi^{\frac{1}{R}}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}dv\right]\\
= \; & fa_t^{\frac{1}{R}}\left(y\xi_t\right)^{-\frac{1}{R}}w_t^{(1-\eta)\rho}G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right)+\phi^{\frac{1}{R}}a_t^{\frac{1}{R}}\left(y\xi_t\right)^{-\frac{1}{R}}G\left(t,T;\beta,\rho,1,\mathcal{R}\right)\\
& -Rfa_t^{\frac{1}{R}}y^{\rho}\xi_t^{-\frac{1}{R}}w_t^{(1-\eta)\rho}\frac{\partial}{\partial y}G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right)-\left(1-R\right)yw_t\frac{\partial}{\partial y}G\left(t,T;0,1,0,\mathcal{A}\right)\\
& -R\phi^{\frac{1}{R}}a_t^{\frac{1}{R}}y^{\rho}\xi_t^{-\frac{1}{R}}\frac{\partial}{\partial y}G\left(t,T;\beta,\rho,1,\mathcal{R}\right). \blacklozenge
\end{aligned}$$

Note that in the representations of the liquid wealth X_t, the human capital H_t and the total wealth N_t, each can be decomposed into two parts. The first part involves the conditional expectations G, and the second part involves the derivative of the conditional expectations with respect to the multiplier $\frac{\partial}{\partial y}G$. Thus the wealth depends on the derivative of the boundary with respect to y. It's not hard to see that the first part that involves the conditional expectations is precisely the representation of the wealth according to early exercise premium representation. For example, total wealth

$$N_t = \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv \right].$$

If we write down the EEP representation for N_t, it is

$$\begin{aligned} & \xi_t^{-1} E_t \left[\int_t^{T} \xi_v e_v^* dv \right] - \xi_t^{-1} E_t \left[\int_t^{T} \xi_v e_v^* 1_{\mathcal{R}(v)} dv - \int_t^{T} \xi_v c_v^* 1_{\mathcal{R}(v)} dv \right] \\ = \; & f a_t^{\frac{1}{R}} \left(y \xi_t \right)^{-\frac{1}{R}} w_t^{(1-\eta)\rho} G\left(t, T; \beta, \rho, \eta, \mathcal{A}\right) + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} \left(y \xi_t \right)^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right), \end{aligned}$$

which is the first part in the representation of the total wealth that involves only the conditional expectations. Therefore, the second part that involves the derivative of the conditional expectations with respect to the multiplier is the discrepancy between the EEP representation of the wealth and their true value. The reason that the EEP representation does not hold for the wealth processes is that the optimal stopping time is not maximizing the wealth process, but maximizing the value function J_t. J_t is a Snell Envelope and the wealth processes are not.

For numerical implementation, we simulate 10 realizations per year from age 20 to age 80, in total 600 data points. Figure 2.3 shows a particular trajectory of the state variable x_t. x_t crosses the optimal retirement boundary (shown previously in Figure 2.1) at about age 72, i.e., the individual under this particular trajectory is optimal to retire at age 72. Figure 2.4 gives the stock market index and wage process under this trajectory. The stock market data is from age 20 to age 80, and the wage process is from age 20 to the retirement age 72. The stock market experiences an increase at about age 40, so does the wage process. The state variable x_t is positively related to the stock market and the wage process if its volatility $\theta/R+((1-\eta)\rho-1)\sigma_w$ is positive. For parameters chosen in Table 2.1, the volatility of x_t is positive. The increase of the state variable x_t at age 40 reduces the distance between x_t and the boundary, but is not sufficient for x_t to touch the boundary. We observe sharp increases in the stock market and the wage process starting about age 60, and the increase of the state variable x_t makes it cross the boundary at about age 72. Figure 2.5 gives the corresponding trajectory of the total wealth, the liquid wealth and the human capital[6]. At age 20, the liquid wealth of the individual is 0, his total wealth is solely in the form of human capital. As stock market and wage increases, the individual's liquid wealth increases and reaches a maximum of about 2 million at about age 40 when the stock market and the wage reaches a local peak (As later shown in the optimal portfolio, the individual under this trajectory is long the market). Human capital is about 3.4 million at age 20 and roughly follows a decreasing path as the remaining work life shortens. Human capital reaches 0 the moment that the individual retires at about age 72. At age 72, the individual has liquid wealth about 1.2 million which is solely used to finance retirement consumption. At time T age 80, all wealth are depleted.

Note that the trajectory of the state variable x_t might cross the boundary multiple times, for example in Figure 2.6. Under this trajectory, the individual is optimal to retire at about age 55. If the individual does not retire at age 55 but continues to work, he has welfare loss at age 55. Of course, at any time t, the optimization problem restarts. At any age between 55 and 59, immediate retirement is optimal. At any age between

[6] For computations of the wealth processes, notice that in the representation of the term $\frac{\partial}{\partial y}G(t,T;\beta,\rho,\eta,\mathcal{A})$, $\sqrt{v-t}$ is in the denominator inside the integral, thus it's divergent when v approaches t. A change of variable is needed for these terms in numerical computation. Let $\sqrt{v-t}=u$, we get $\frac{\partial}{\partial y}G(t,T;\beta,\rho,\eta,\mathcal{A}) = \int_t^T \exp\left(-A(\beta,\rho,\eta)(v-t)\right) n\left(-d\left(x_t,B_v,v;C(\rho,\eta)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv = \int_0^{\sqrt{T-t}} \exp\left(-A(\beta,\rho,\eta)u^2\right) n\left(-d\left(x_t,B_{u^2+t},u^2+t;C(\rho,\eta)\right)\right) \frac{2}{\sigma_x} \left(\frac{\frac{\partial B_{u^2+t}}{\partial y}}{B_{u^2+t}} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) du.$

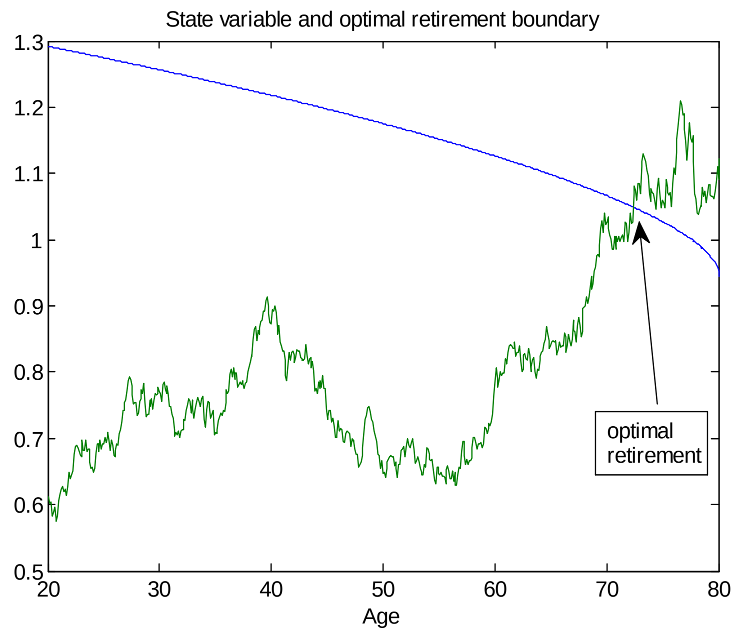

Figure 2·3: This figure shows the state variable crosses the boundary at about age 72.

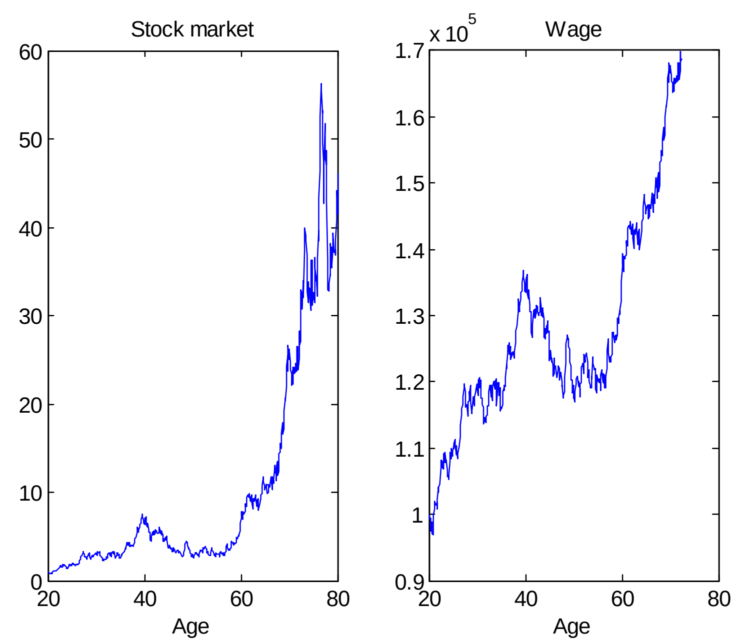

Figure 2·4: This figure shows a trajectory of the stock market from age 20 to 80 and a trajectory of wage from age 20 to retirement.

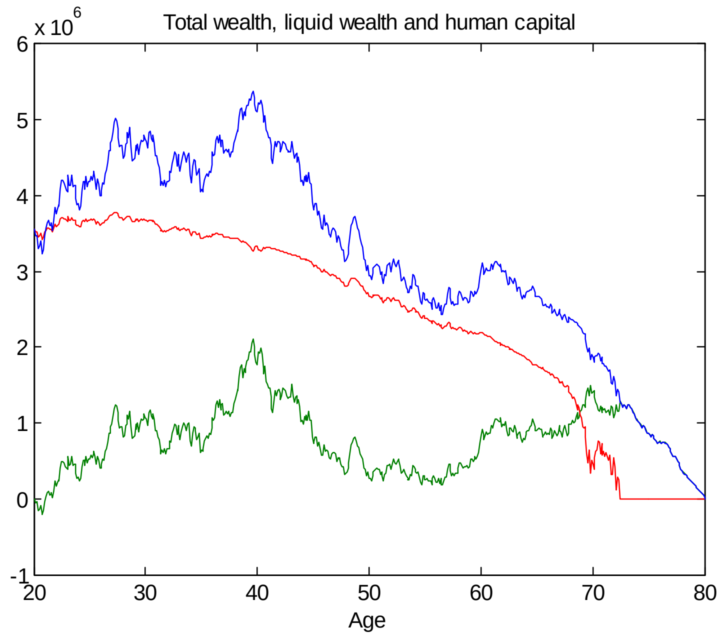

Figure 2·5: This figure shows total wealth (blue path), liquid wealth (green path) and human capital (red path).

59 and 61, the optimal policy is to continue working. After 61, the state variable x_t remains inside the optimal retirement region and it's optimal to retire immediately.

The next theorem gives the closed form expression of the retirement boundary for the liquid wealth. This boundary is more practical and useful because an individual can compare his observed liquid wealth with this boundary and know if it's optimal to retire immediately. Immediate retirement is optimal when his liquid wealth crosses this boundary.

Theorem 10. *For $R > 1$, retirement is optimal when the liquid wealth crosses its boundary, i.e.,*

$$X_t \geq \phi^{\frac{1}{R}} y^{-\frac{1}{R}\left(1+\gamma_\xi/R\right)} a_t^{\frac{1}{R}} G\left(t, T; \beta, \rho, 1\right) w_0^{((1-\eta)\rho-1)\gamma_\xi/R} \exp\left(-\frac{1}{R}\delta_\xi t\right) B_t^{-\gamma_\xi/R}, \quad (2.16)$$

where $\gamma_\xi = \frac{-\theta}{\sigma_x}$, $\delta_\xi = -r - \frac{1}{2}\theta^2 - \gamma_\xi\left(\mu_x - \frac{1}{2}\sigma_x^2\right)$ and B_t is the boundary for the state

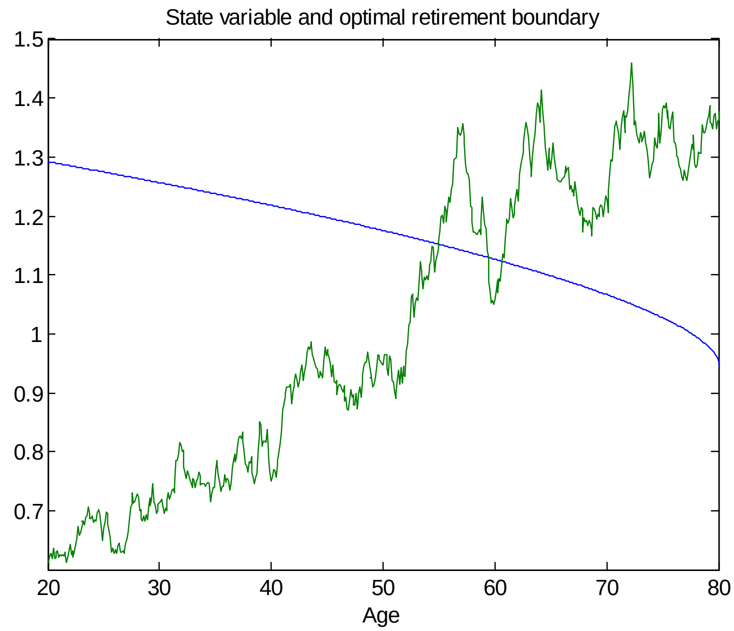

Figure 2·6: This figure shows that the state variable crosses the optimal retirement boundary multiple times.

variable x_t.

Proof. Liquid wealth before retirement is

$$X_t = \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv \right],$$

where τ_t^* is the optimal stopping time. Plug in the values of e_v^* and c_v^*,

$$X_t = \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} w_v^{(1-\eta)\rho} f dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} dv \right].$$

Liquid wealth after retirement is

$$\begin{aligned} X_t &= E_t\left[\int_t^T \xi_{t,v} c_v^* dv\right] = E_t\left[\int_t^T \xi_{t,v}\left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}}\phi^{\frac{1}{R}} dv\right] \\ &= \phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho} dv\right] \\ &= \phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right). \end{aligned}$$

It is shown[7] that retirement is optimal when ξ_t becomes sufficiently low, or w_t becomes sufficiently large, or $x_t = a_t^{\frac{1}{R}}\left(y\xi_t\right)^{-\frac{1}{R}} w_t^{(1-\eta)\rho-1}$ rises above B_t. Consider the form of liquid wealth after retirement $X_t = \phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$, it's also true that retirement is optimal when the value of $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$ is greater than its own boundary. Therefore after we compute the boundary B_t of $x_t = a_t^{\frac{1}{R}}\left(y\xi_t\right)^{-\frac{1}{R}} w_t^{(1-\eta)\rho-1}$, we can transform it to the boundary of $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$. Now we show that the boundary of the term $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$ is exactly the retirement boundary of liquid wealth. Because $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$ is the form of liquid wealth after retirement, when it hits its boundary, i.e. liquid wealth hits the boundary, it's optimal to retire. However, $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$ is not the form of liquid wealth before retirement, liquid wealth before retirement is

$$\xi_t^{-1} E_t\left[\int_t^{\tau_t^*} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} w_v^{(1-\eta)\rho} f dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^T \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} dv\right].$$

Therefore, in order to show that the boundary of $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$ is exactly the retire boundary of liquid wealth, we also need to show that before retirement, liquid wealth is less than the boundary of $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right)$. We know that before

[7] See appendix Proposition A1.

retirement, the value of $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}}G\left(t,T;\beta,\rho,1\right)$ is less than its boundary, thus we just need to show that liquid wealth before retirement is less than or equal to the value of $\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}}G\left(t,T;\beta,\rho,1\right)$. This is to show, at time t before retirement,

$$\begin{aligned} X_{t,b} &\equiv \xi_t^{-1}E_t\left[\int_t^{\tau_t^*} a_v^{\frac{1}{R}}y^{-\frac{1}{R}}\xi_v^{\rho}w_v^{(1-\eta)\rho}f dv - \int_t^{\tau_t^*}\xi_v w_v dv \right. \\ &\quad \left. + \int_{\tau_t^*}^{T}\phi^{\frac{1}{R}}a_v^{\frac{1}{R}}y^{-\frac{1}{R}}\xi_v^{\rho}dv\right] \\ &\leq X_{t,a} \equiv \phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}}G\left(t,T;\beta,\rho,1\right) = \xi_t^{-1}E_t\left[\int_t^{T}\phi^{\frac{1}{R}}a_v^{\frac{1}{R}}y^{-\frac{1}{R}}\xi_v^{\rho}dv\right]. \end{aligned}$$

Optimal value function

$$\begin{aligned} L_t &= E_t\left[\int_t^{\tau_t^*}\frac{R}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}w_v^{(1-\eta)\rho}f dv + \int_t^{\tau_t^*} y\xi_v w_v dv \right. \\ &\quad \left. + \int_{\tau_t^*}^{T}\frac{R}{1-R}\phi^{\frac{1}{R}}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}dv\right]. \end{aligned}$$

Thus we have

$$\frac{R}{1-R}y\xi_t X_{t,b} + \frac{1}{1-R}E_t\left[\int_t^{\tau_t^*} y\xi_v w_v dv\right] = L_t,$$

and

$$L_t \geq E_t\left[\int_t^{T}\frac{R}{1-R}\phi^{\frac{1}{R}}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}dv\right] = \frac{R}{1-R}y\xi_t X_{t,a}.$$

Therefore

$$\frac{R}{1-R}y\xi_t X_{t,b} + \frac{1}{1-R}E_t\left[\int_t^{\tau_t^*} y\xi_v w_v dv\right] \geq \frac{R}{1-R}y\xi_t X_{t,a}.$$

For $R>1$, we get $X_{t,b}\leq X_{t,a}$. To compute the form of the boundary of liquid wealth,

we have

$$X_{t,a} = \phi^{\frac{1}{R}} \left(\frac{y\xi_t}{a_t} \right)^{-\frac{1}{R}} G(t, T; \beta, \rho, 1) = \phi^{\frac{1}{R}} y^{-\frac{1}{R}} a_t^{\frac{1}{R}} G(t, T; \beta, \rho, 1)\, \xi_t^{-\frac{1}{R}}.$$

ξ_t is a transform of x_t,

$$\xi_t = \left(\frac{x_t}{x_0} \right)^{\gamma_\xi} \exp(\delta_\xi t)$$

where $\gamma_\xi = \frac{-\theta}{\sigma_x}$, $\delta_\xi = -r - \frac{1}{2}\theta^2 - \gamma_\xi \left(\mu_x - \frac{1}{2}\sigma_x^2 \right)$, $x_0 = y^{-\frac{1}{R}} w_0^{(1-\eta)\rho - 1}$ and

$$\mu_x = -\frac{\beta}{R} + \frac{1}{R}\left(r + \frac{1}{2}\theta^2 \right) + ((1-\eta)\rho - 1)\left(\mu_w - \frac{1}{2}\sigma_w^2 \right) + \frac{1}{2}\sigma_x^2,$$

$$\sigma_x = \frac{\theta}{R} + ((1-\eta)\rho - 1)\,\sigma_w.$$

Therefore, after we compute the boundary B_t of x_t, the boundary of liquid wealth is

$$\begin{aligned} & \phi^{\frac{1}{R}} y^{-\frac{1}{R}} a_t^{\frac{1}{R}} G(t, T; \beta, \rho, 1) \left(\left(\frac{B_t}{x_0} \right)^{\gamma_\xi} \exp(\delta_\xi t) \right)^{-\frac{1}{R}} \\ = \;& \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1+\gamma_\xi/R)} a_t^{\frac{1}{R}} G(t, T; \beta, \rho, 1)\, w_0^{((1-\eta)\rho-1)\gamma_\xi/R} \exp\left(-\frac{1}{R}\delta_\xi t \right) B_t^{-\gamma_\xi/R}. \; \blacklozenge \end{aligned}$$

Figure 2.7 gives the retirement boundary for the liquid wealth and the trajectory of the liquid wealth (shown previously in Figure 2.5). It shows that the liquid wealth crosses its optimal retirement boundary at about age 72, the same time as the state variable x_t crosses B_t.

2.6 Optimal consumption, optimal labor and optimal expenditure

We derived the optimal consumption, the optimal leisure and the optimal expenditure in proposition 2 and now we know the expressions of the initial budget constraint satisfied by the multiplier, i.e., equation (2.13) at time 0, thus we can find the multiplier y. We have the following solutions for optimal consumption, optimal leisure and optimal

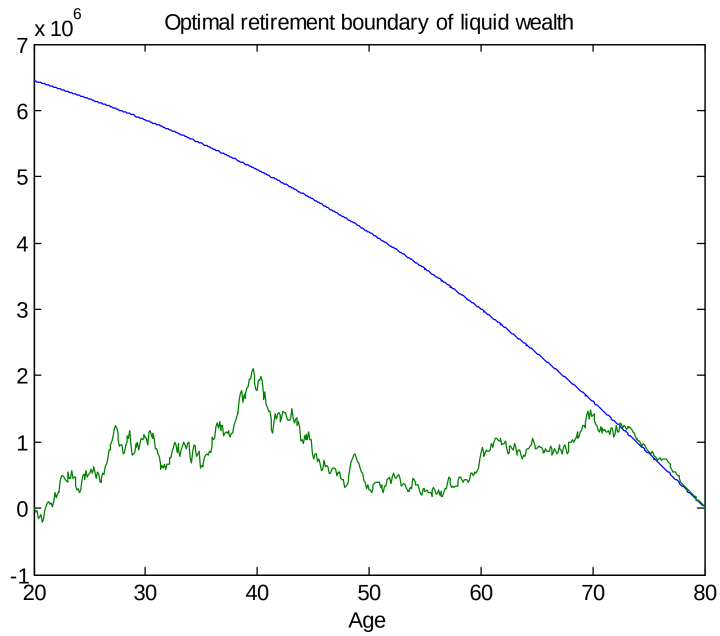

Figure 2·7: This figure shows that the liquid wealth crosses its optimal retirement boundary at about age 72.

expenditure. Optimal labor is the difference between the maximal amount of labor $\overline{h}$ endowed (normalized to be 1) and the optimal leisure l_v^*, i.e., $h_v^* = 1 - l_v^*$. Before retirement, we have

$$c_v^* = \eta \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho} f,$$

$$l_v^* = (1-\eta) \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho} f \frac{1}{w_v},$$

$$e_v^* = \frac{c_v^*}{\eta} = \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} w_v^{(1-\eta)\rho} f,$$

and after retirement we have

$$c_v^* = \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} \phi^{\frac{1}{R}},$$

where $\rho = 1 - \frac{1}{R}$, $f = \frac{1}{\eta}\left(\frac{1-\eta}{\eta}\right)^{-(1-\eta)\rho}$ and the multiplier y satisfies equation (2.13) at time 0.

The optimal policies are derived from the first order conditions to equate marginal benefits and marginal costs. Consumption is increasing with respect to $\xi_v^{-\frac{1}{R}} w_v^{(1-\eta)\rho}$ and consumption volatility $\frac{\theta}{R} + (1-\eta)\rho\sigma_w$ is positive, therefore in a state where the stock market experiences a positive shock, it's optimal to consume more, ceteris paribus. For leisure, its volatility is $\frac{\theta}{R} + ((1-\eta)\rho - 1)\sigma_w$, the effect of stock market shock depends on whether the volatility is positive or negative. When leisure volatility is positive, leisure is positively related to the stock market and when leisure volatility is negative, they are negatively related. After retirement, consumption volatility is $\frac{\theta}{R}$, which is smaller than consumption volatility before retirement and is decreasing with respect to coefficient of relative risk aversion R. When the wage is deterministic, $\sigma_w = 0$, consumption volatility before and after retirement and leisure volatility are all equal to $\frac{\theta}{R}$. When $\sigma_w = 0$, consumption and leisure are both decreasing with respect to state price density ξ_v, thus it pays to consume more and have more leisure when SPD is low, that is, when the stock market experiences positive shocks.

η is the measure of relative weight of consumption and labor. In the total expenditure e_v^*, the percentage that the individual spends on consumption is η, and the percentage that the individual spends on leisure is $1-\eta$ with opportunity cost being the wage w_v. This is due to Cobb-Douglas utility that the individual spends certain percentage on each goods. The larger the value of η, the larger the percentage spent on consumption and the smaller the percentage spent on leisure. When η approaches to 1, leisure will no longer be valued.

ϕ is the coefficient that measures the relative weight of the retirement phase, and it also controls the disruption of the consumption when the individual enters the retirement phase. The disruption is measured by the ratio of the consumption immediately after retirement and the consumption just before retirement,

$$\text{ratio} = \frac{\phi^{\frac{1}{R}}}{\eta w_v^{(1-\eta)\rho} f}.$$

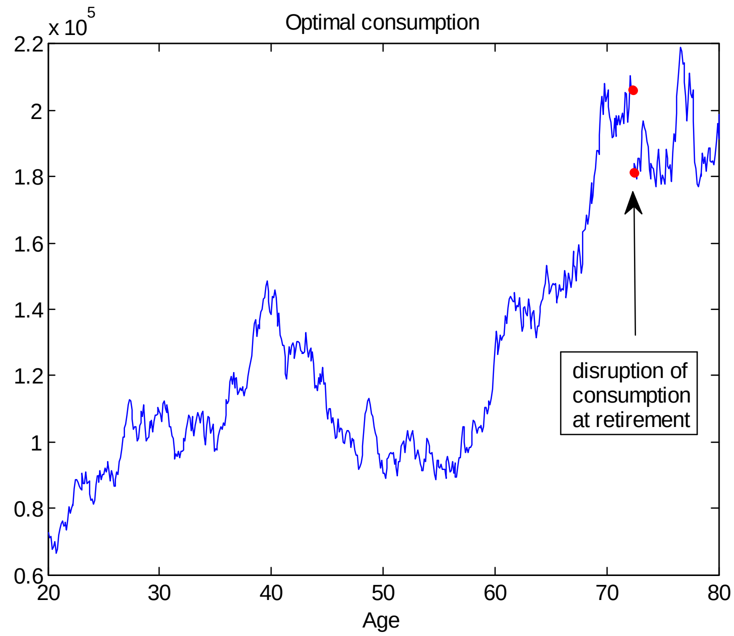

Figure 2·8: This figure shows the optimal consumption before and after retirement.

Figure 2.8 shows the optimal consumption before and after retirement and Figure 2.9 shows the optimal leisure before and after retirement. The optimal consumption and the optimal leisure both have positive volatilities for parameter values in Table 2.1, thus are positively related to the stock market. For this particular trajectory of underlying Brownian motion, the optimal consumption and the optimal leisure both reach a local maximum at around age 40 as the stock market and the wage reach a local peak which boosts the liquid wealth and the total wealth. As time approaches the retirement age 72, the optimal consumption and the optimal leisure are both at high levels, when the liquid wealth reaches a level sufficient high for the individual to retire. Note that at retirement age, the consumption experiences a sudden drop. The disruption is controlled by the ratio $\frac{\phi^{\frac{1}{R}}}{\eta w_v^{(1-\eta)\rho} f}$, which is related to the wage level the moment before retirement. This is due to the structural change in the individual's preferences at the retirement date. Leisure jumps to the level of 1 at retirement and remains at the level as the individual

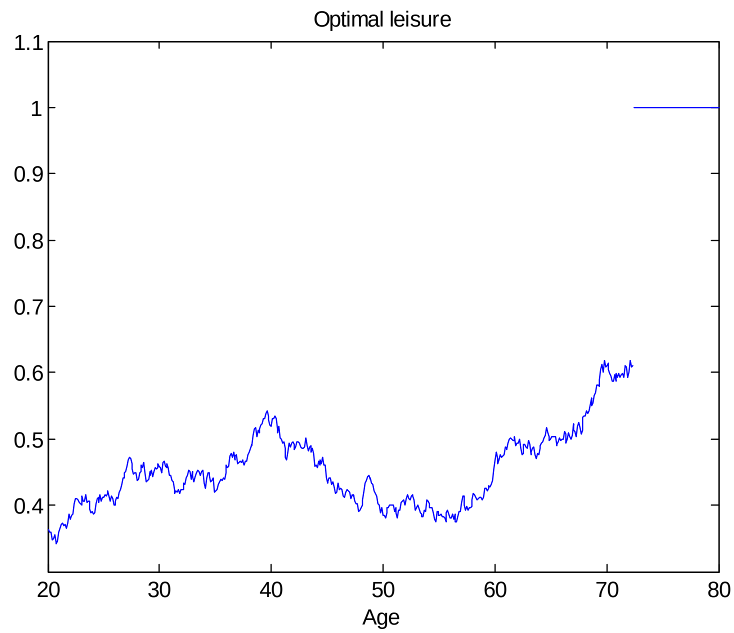

Figure 2·9: This figure shows the optimal leisure before and after retirement.

no longer supplies labor.

Consumption, leisure and total expenditure are increasing with respect to the initial wealth. L_0 is a convex function of the multiplier y. Since $\frac{\partial L_0}{\partial y} = -x$, initial wealth is a decreasing function of y. When initial wealth is higher, the multiplier y becomes smaller, which results in higher consumption, leisure and total expenditure. Table 2.2 shows the impact of the initial liquid wealth on the optimal consumption, optimal leisure and optimal total expenditure. Without loss of generality, the impact on initial optimal consumption, optimal leisure and optimal total expenditure is reported. For example, if initial liquid wealth increases from 0 to $250,000$, initial optimal total expenditure increases from $109,160$ to $116,740$, out of which 66.6% is spent for the optimal consumption and 33.3% is spent for the optimal leisure with opportunity cost being the initial wage $100,000$. When initial liquid wealth increases, at any time before retirement, the optimal consumption, leisure and total expenditures increase. When initial

Table 2.2: The impact of the initial liquid wealth on the optimal consumption, leisure and total expenditure.

Initial liquid wealth x	Consumption c_0	Leisure l_0	Total expenditure e_0
0	$72,774$	0.36387	$109,160$
$50,000$	$73,785$	0.36893	$110,678$
$100,000$	$74,796$	0.37398	$112,194$
$150,000$	$75,807$	0.37903	$113,710$
$200,000$	$76,817$	0.38408	$115,225$
$250,000$	$77,826$	0.38913	$116,740$

liquid wealth increases, at any time after retirement, the optimal consumption, leisure and total expenditures also increase.

Next proposition compares the optimal policies for the individual who has the option to retire with the optimal policies for the individual who has no option to retire (work until time T).

Proposition 11. *If* $x > E\left[\int_0^T \xi_v e_v^* dv - \int_0^T \xi_v w_v dv\right]$*, then for the same initial wealth, the optimal consumption, the optimal leisure and the optimal expenditure before retirement for the individual who has the option to retire are lower than the corresponding optimal policies when the individual has no option to retire (work until time* T*). If* $x \leq E\left[\int_0^T \xi_v e_v^* dv - \int_0^T \xi_v w_v dv\right]$*, then for the same initial wealth, the optimal consumption, the optimal leisure and the optimal expenditure before retirement for the individual who has the option to retire are higher than or equal to the corresponding optimal policies when the individual has no option to retire.*

Proof. L_t is a convex function of the multiplier y. Since $\frac{\partial L_t}{\partial y} = -\xi_t X_t$, X_t is a decreasing function of y. To compare the optimal policies of an individual who has the option to retire before T with the optimal policies of an individual who has no option to retire, since the first order conditions of the optimization problems are the same, we just need to compare the values of the multiplier y for the two problems. If $x > E\left[\int_0^T \xi_v e_v^* dv - \int_0^T \xi_v w_v dv\right]$, then for the same initial wealth, the multiplier for the individual with the option to retire is higher than the multiplier for the individual with no option to retire, therefore the optimal consumption, the optimal labor and the optimal expenditure before retirement for the individual who has the option to retire are lower than the corresponding optimal policies when the individual has no option to retire. And vice versa. ♦

For the parameter values in Table 2.1, we calculate,

$$\begin{aligned} & E\left[\int_0^T \xi_v e_v^* dv - \int_0^T \xi_v w_v dv\right] \\ = \; & f y^{-\frac{1}{R}} w_0^{(1-\eta)\rho} G\left(0, T; \beta, \rho, \eta\right) - w_0 G\left(0, T; 0, 1, 0\right), \end{aligned}$$

which gives $-3,622$. Therefore we have $x = 0 > E\left[\int_0^T \xi_v e_v^* dv - \int_0^T \xi_v w_v dv\right]$. For the same initial wealth $-3,622$, we can calculate the optimal consumption, the optimal leisure and the optimal expenditure for the individual with the retirement option. They are uniformly lower than the corresponding optimal policies for the individual without the retirement option at any time before the first individual retires. Table 2.3 shows the difference in the initial optimal policies between the two individuals. Since the difference between 0 and $-3,622$ is relatively small, the impact of retirement option on the optimal consumption, the optimal leisure and the optimal expenditure is not significant. With retirement option, the individual consumes only 109 less in total expenditure annually.

2.7 Optimal portfolio

The optimal portfolio π_t is derived in the next theorem. Before retirement, the optimal portfolio can be decomposed into two parts π_{1t} and π_{2t}, where π_{1t} is the mean-variance component, and π_{2t} is the hedging demand against fluctuations in wages. When the

Table 2.3: The impact of the retirement option on the optimal consumption, leisure and total expenditure.

	with retirement option	without retirement option
Initial liquid wealth x	$-3,622$	$-3,622$
Consumption c_0	$72,700$	$72,774$
Leisure l_0	0.36350	0.36387
Total expenditure e_0	$109,051$	$109,160$

wage process is deterministic, i.e., $\sigma_w = 0$, the hedging demand π_{2t} vanishes. After retirement, the optimal portfolio only consists of a mean-variance component as the individual no longer earns wages.

Theorem 12. *Before retirement, the optimal portfolio π_t satisfies*

$$\begin{aligned}
& \pi_t \sigma \\
= \; & (\theta/R + (1-\eta)\,\rho\sigma_w)\, f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_1 - \sigma_w w_t f_2 \\
& + (\theta/R) \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} f_3 \\
& - (\theta/R + (1-\eta)\,\rho\sigma_w)\, \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_4 \\
& - \sigma_w y w_t f_5 - (\theta/R)\, \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} f_6 \\
& + (\theta/R + ((1-\eta)\,\rho - 1)\,\sigma_w) \\
& \times \Big[f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_7 - w_t f_8 + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} f_9 \\
& - \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_{10} - y w_t f_{11} - \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} f_{12} \Big] .
\end{aligned}$$

And we have $\pi_t = \pi_{1t} + \pi_{2t}$, *where*

$$
\begin{aligned}
&\pi_{1t} = \frac{1}{R}\sigma^{-1}\theta \\
&\times \Big[f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_1 + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} f_3 \\
&- \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_4 - \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} f_6 \\
&+ f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_7 - w_t f_8 + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} f_9 \\
&- \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_{10} - y w_t f_{11} - \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} f_{12} \Big],
\end{aligned}
$$

$$
\begin{aligned}
&\pi_{2t} = \sigma^{-1}\sigma_w \\
&\times \Big[(1-\eta)\,\rho f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_1 - w_t f_2 \\
&- (1-\eta)\,\rho \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_4 - y w_t f_5 \\
&+ ((1-\eta)\,\rho - 1) \\
&\times (f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_7 - w_t f_8 + \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} f_9 \\
&- \frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} f_{10} - y w_t f_{11} - \frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} f_{12}) \Big].
\end{aligned}
$$

And the explicit expressions of f_1 to f_{12} are,

$$f_1 = \int_t^T \exp\left(-A\left(\beta, \rho, \eta\right)\left(v - t\right)\right) N\left(-d\left(x_t, B_v, v; C\left(\rho, \eta\right)\right)\right) dv,$$

$$f_2 = \int_t^T \exp\left(-A\left(0, 1, 0\right)\left(v - t\right)\right) N\left(-d\left(x_t, B_v, v; C\left(1, 0\right)\right)\right) dv,$$

$$f_3 = \int_t^T \exp\left(-A\left(\beta, \rho, 1\right)\left(v - t\right)\right) N\left(d\left(x_t, B_v, v; C\left(\rho, 1\right)\right)\right) dv,$$

$$f_4 = \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right) n\left(-d\left(x_t, B_v, v; C\left(\rho,\eta\right)\right)\right) \times \frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv,$$

$$f_5 = \int_t^T \exp\left(-A\left(0,1,0\right)\left(v-t\right)\right) n\left(-d\left(x_t, B_v, v; C\left(1,0\right)\right)\right) \times \frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv,$$

$$f_6 = \int_t^T \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right) n\left(d\left(x_t, B_v, v; C\left(\rho,1\right)\right)\right) \times \frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv,$$

$$f_7 = -\int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right) n\left(-d\left(x_t, B_v, v; C\left(\rho,\eta\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$f_8 = -\int_t^T \exp\left(-A\left(0,1,0\right)\left(v-t\right)\right) n\left(-d\left(x_t, B_v, v; C\left(1,0\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$f_9 = \int_t^T \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right) n\left(d\left(x_t, B_v, v; C\left(\rho,1\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$f_{10} = -\int_t^T d\left(x_t, B_v, v; C\left(\rho,\eta\right)\right) \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right) \times n\left(-d\left(x_t, B_v, v; C\left(\rho,\eta\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv,$$

$$f_{11} = -\int_t^T d\left(x_t, B_v, v; C\left(1,0\right)\right) \exp\left(-A\left(0,1,0\right)\left(v-t\right)\right)$$
$$\times n\left(-d\left(x_t, B_v, v; C\left(1,0\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv,$$

$$f_{12} = -\int_t^T d\left(x_t, B_v, v; C\left(\rho,1\right)\right) \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right)$$
$$\times n\left(d\left(x_t, B_v, v; C\left(\rho,1\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv.$$

After retirement, the optimal portfolio is $\pi_t = \frac{1}{R}\sigma^{-1}\theta\phi^{\frac{1}{R}}\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} G\left(t,T;\beta,\rho,1\right).$

Proof. Plugin the expressions of

$$G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right),\ \ G\left(t,T;0,1,0,\mathcal{A}\right),\ \ G\left(t,T;\beta,\rho,1,\mathcal{R}\right),$$

$$\frac{\partial}{\partial y}G\left(t,T;\beta,\rho,\eta,\mathcal{A}\right),\ \ \frac{\partial}{\partial y}G\left(t,T;0,1,0,\mathcal{A}\right),\ \ \frac{\partial}{\partial y}G\left(t,T;\beta,\rho,1,\mathcal{R}\right)$$

in the representation of X_t, we get

$$\begin{aligned}
& X_t \\
= \; & f a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)(v-t)\right) \\
& \times N\left(-d\left(x_t, B_v, v; C\left(\rho,\eta\right)\right)\right) dv \\
& -w_t \int_t^T \exp\left(-A\left(0,1,0\right)(v-t)\right) N\left(-d\left(x_t, B_v, v; C\left(1,0\right)\right)\right) dv \\
& +\phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} \int_t^T \exp\left(-A\left(\beta,\rho,1\right)(v-t)\right) N\left(d\left(x_t, B_v, v; C\left(\rho,1\right)\right)\right) dv \\
& -\frac{R}{1-R} f a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} w_t^{(1-\eta)\rho} \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)(v-t)\right) \\
& \times n\left(-d\left(x_t, B_v, v; C\left(\rho,\eta\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv \\
& -y w_t \int_t^T \exp\left(-A\left(0,1,0\right)(v-t)\right) \\
& \times n\left(-d\left(x_t, B_v, v; C\left(1,0\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv \\
& -\frac{R}{1-R} \phi^{\frac{1}{R}} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \int_t^T \exp\left(-A\left(\beta,\rho,1\right)(v-t)\right) \\
& \times n\left(d\left(x_t, B_v, v; C\left(\rho,1\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} \left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv,
\end{aligned}$$

then apply Clark-Ocone formula[8] on the right hand side of the equation and equate the volatility on both sides. ♦

Figure 2.10 shows the portfolio components, the mean variance component and the hedge against fluctuations in wages[9]. For this particular trajectory, before retirement,

[8] See Detemple et al. (2003) appendix D and Ocone and Karatzas (1991).

[9] For computations of the portfolio components, notice that in the representations of the terms

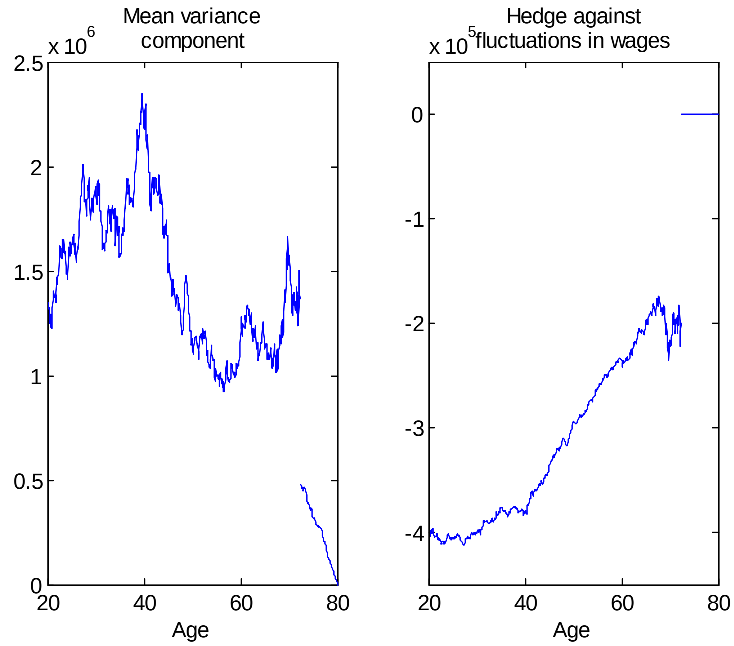

Figure 2·10: This figure shows the portfolio components.

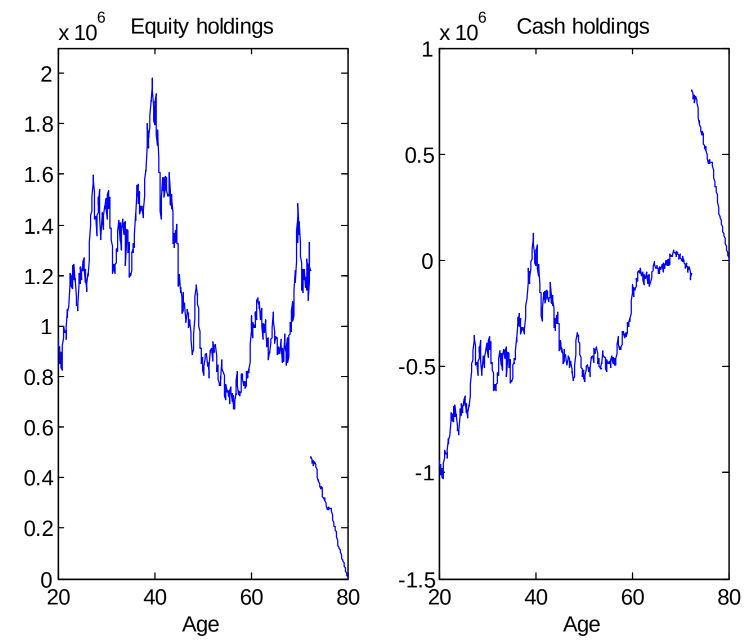

Figure 2·11: This figure shows the optimal portfolio and cash holdings before and after retirement.

the mean variance component dominates the hedge against fluctuations in wages, thus the equity holdings in Figure 2.11 is very similar to the mean variance component. The mean variance component reaches its maximum at about age 40, when the stock market reaches a local peak. We observe that the mean variance component is positive while the wage hedge is negative. After retirement, the mean variance component converges to 0 at age approaches 80. Wage hedge after retirement is 0 as wage is 0. We observe that this individual is long the market all the time and borrowing cash during most of the time before retirement. At the retirement date, both equity holdings and cash holdings experience a discontinuity. The individual sell about 0.8 million stock for cash. At age approaches 80, both equity holdings and cash holdings are converging to 0.

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f_{10}, f_{11}, f_{12}, $v-t$ is in the denominator inside the integral, thus it's divergent when v approaches t. A change of variable is needed for numerically computing these terms. Let $v-t=\exp(u)$, we get
$f_{10} = -\int_{-\infty}^{\log(T-t)} d\left(x_t, B_{\exp(u)+t}, \exp(u)+t; C(\rho,\eta)\right) \exp\left(-A(\beta,\rho,\eta)\exp(u)\right)$
$\times n\left(-d\left(x_t, B_{\exp(u)+t}, \exp(u)+t; C(\rho,\eta)\right)\right) \frac{1}{\sigma_x^2} \left(\frac{\frac{\partial B_{\exp(u)+t}}{\partial y}}{B_{\exp(u)+t}} - \frac{\frac{\partial x_t}{\partial y}}{x_t} \right) du.$

Chapter 3

Optimal consumption, labor, portfolio and retirement for individuals with log utility

3.1 Introduction

For an individual with log utility, the coefficient of relative risk aversion is equal to 1. Choi and Shim (2006) provide an explicit solution for log utility, for the case of infinite horizon where the individual is infinitely lived with disutility from labor before retirement, using dynamic programming approach. We complement the existing literature by providing closed form solutions for the finite horizon using the martingale and convex duality approach.

We find that before retirement, if the initial multiplier of an individual with power utility and the initial multiplier of an individual with log utility are equal, then the optimal consumption and labor of an individual with log utility are the limits of the optimal consumption and labor of an individual with power utility when the coefficient of relative risk aversion converges to 1. Furthermore, if not only the multipliers are equal but also the two individuals have the same optimal retirement boundaries of the state variable, then the optimal liquid wealth, the optimal retirement boundary of the liquid wealth and the optimal portfolio of an individual with log utility are the limits of the optimal liquid wealth, the optimal retirement boundary of the liquid wealth and the optimal portfolio of an individual with power utility when the coefficient of relative risk aversion approaches to 1.

When do the individual with log utility and the individual with power utility have the same optimal retirement boundaries of the state variable? We find that in general, the two individuals have different retirement boundaries of the state variable even if all parameters for the two individuals are the same, the initial multipliers are the same and the coefficient of relative risk aversion converges to 1. However, under the condition that the coefficient ϕ which measures the relative weight of the retirement phase is equal to $1/\eta$ where η is a measure of relative weight of consumption and leisure, the two individuals will have the same optimal retirement boundaries of the state variable if all parameters for the two individuals are the same, the initial multipliers are the same and the coefficient of relative risk aversion converges to 1. And when do they have the same multipliers? Since the multiplier is endogenous and it depends on the initial liquid wealth, varying the level of initial liquid wealth can ensure the two individuals have the same multiplier.

After retirement, as long as the two individuals have the same value of initial multiplier, the optimal consumption, labor, wealth and portfolio of an individual with log utility are the limits of the corresponding optimal policies of an individual with power utility when the coefficient of relative risk aversion converges to 1.

The maximization problem for the individual with log utility is

$$V(x) = \sup_{\tau \in \mathcal{S},\ (c,l,\pi) \in A} E\left[\int_0^\tau a_v u^a(c_v, l_v)\, dv + \phi \int_\tau^T a_v u^r(c_v, 1)\, dv\right]$$

where $u^a(c_v, l_v) = \frac{1}{\eta}(\eta \ln(c_v) + (1-\eta)\ln(l_v))$, $u^r(c_v, 1) = \ln(c_v)$ and $a_v = \exp(-\beta v)$, subject to the static budget constraint

$$E\left[\int_0^\tau \xi_v (c_v - w_v + w_v l_v)\, dv + \int_\tau^T \xi_v c_v dv\right] \leq x.$$

3.2 Pure optimal stopping problem

Before retirement, the utility function is

$$a_v u^a(c_v, l_v) = a_v \frac{1}{\eta}(\eta \ln(c_v) + (1-\eta)\ln(l_v)).$$

First order conditions give

$$a_v u_c^a(c_v, l_v) = a_v c_v^{-1} = y\xi_v, \qquad a_v u_l^a(c_v, l_v) = a_v \frac{1-\eta}{\eta} l_v^{-1} = y\xi_v w_v.$$

Calculations give the following solutions

$$c_v^* = \eta \left(\frac{y\xi_v}{a_v}\right)^{-1} f, \qquad l_v^* = (1-\eta)\left(\frac{y\xi_v}{a_v}\right)^{-1} f \frac{1}{w_v},$$

$$e_v^* \equiv c_v^* + l_v^* w_v = \left(\frac{y\xi_v}{a_v}\right)^{-1} f,$$

where

$$f = 1/\eta \Longleftrightarrow \eta f = 1.$$

After retirement, the utility function is

$$\phi a_v u^r(c_v, 1) = \phi a_v \ln(c_v).$$

First order condition gives

$$\phi a_v u_c^r(c_v, 1) = \phi a_v c_v^{-1} = y\xi_v.$$

Thus

$$c_v^* = \left(\frac{y\xi_v}{a_v}\right)^{-1} \phi.$$

Calculations show

$$\begin{aligned} & \eta \ln\left(c_v^*\right) + (1-\eta)\ln\left(l_v^*\right) \\ = \; & \eta \ln\left(\eta \left(\frac{y\xi_v}{a_v}\right)^{-1} f\right) + (1-\eta)\ln\left((1-\eta)\left(\frac{y\xi_v}{a_v}\right)^{-1} f \frac{1}{w_v}\right) \\ = \; & -\ln\left(\frac{y\xi_v}{a_v}\right) - (1-\eta)\ln\left(w_v\right) + (1-\eta)\ln\left(\frac{1-\eta}{\eta}\right), \end{aligned}$$

$$a_v u^a\left(c_v^*, l_v^*\right) = -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right),$$

$$\begin{aligned} & a_v \widetilde{U}^a(y a_v^{-1}\xi_v, y a_v^{-1}\xi_v w_v) \\ = \; & a_v u^a\left(c_v^*, l_v^*\right) - y\xi_v c_v^* - y\xi_v w_v l_v^* \\ = \; & -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right) - a_v - \frac{1-\eta}{\eta}a_v \\ = \; & -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta}\right). \end{aligned}$$

$a_v \widetilde{U}^a(y a_v^{-1}\xi_v, y a_v^{-1}\xi_v w_v)$ is a convex function of y.

$$\begin{aligned} a_v \widetilde{U}^r(y a_v^{-1}\xi_v) &= \phi a_v u^r\left(c_v^*, 1\right) - y\xi_v c_v^* \\ &= \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln\left(\phi\right)\right) - \phi a_v \\ &= \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln\left(\phi\right) - 1\right). \end{aligned}$$

$a_v\widetilde{U}^r(ya_v^{-1}\xi_v)$ is a convex function of y. Now we have

$$\begin{aligned}
&\widetilde{J}(y;\tau)\\
&= E\left[\int_0^\tau a_v\widetilde{U}^a(ya_v^{-1}\xi_v, ya_v^{-1}\xi_v w_v)dv + y\int_0^\tau \xi_v w_v dv + \int_\tau^T a_v\widetilde{U}^r(ya_v^{-1}\xi_v)dv\right]\\
&= E\left[\int_0^\tau -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln(w_v) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta}\right)dv\right.\\
&\left. \quad +y\int_0^\tau \xi_v w_v dv + \int_\tau^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln(\phi) - 1\right)dv\right].
\end{aligned}$$

$\widetilde{J}(y;\tau)$, as the sum of the three parts above, is also a convex function of y.

$$\begin{aligned}
x_\tau(y) &= E\left[\int_0^\tau (\xi_v c_v^* + \xi_v w_v l_v^*)\,dv - \int_0^\tau \xi_v w_v dv + \int_\tau^T \xi_v c_v^* dv\right]\\
&= E\left[\int_0^\tau \frac{1}{\eta}y^{-1}a_v dv - \int_0^\tau \xi_v w_v dv + \int_\tau^T \phi y^{-1}a_v dv\right].
\end{aligned}$$

$\widetilde{V}(y) = \sup_{\tau\in\mathcal{S}}\widetilde{J}(y;\tau)$ and denote the stopping time that attains the supremum by τ_y^*, i.e., $\widetilde{V}(y) = \widetilde{J}\left(y;\tau_y^*\right)$. The same proof as in Proposition 2 gives $\widetilde{V}'(y) = -x_{\tau_y^*}(y)$ and $V(x) = \inf_{y>0}\left[\widetilde{V}(y) + yx\right]$. This shows that we can first solve the pure optimal stopping problem of $\widetilde{V}(y)$, while treating the multiplier y as a constant.

3.3 Optimal retirement boundary

Recall that

$$\begin{aligned}
D_t \triangleq\ & \int_0^t \left(a_v u^a\left(c_v^*, l_v^*\right) - y\xi_v e_v^*\right)dv + \int_0^t y\xi_v w_v dv\\
&+E_t\left[\int_t^T \left(\phi a_v u^r\left(c_v^*, 1\right) - y\xi_v c_v^*\right)dv\right],
\end{aligned}$$

and its Snell Envelope is

$$\begin{aligned} J_t &\triangleq \sup_{\tau\in\mathcal{S}} E_t\left[D_\tau\right] \\ &= \sup_{\tau\in\mathcal{S}} E_t\left[\int_0^\tau \left(a_v u^a\left(c_v^*, l_v^*\right) - y\xi_v e_v^*\right) dv + \int_0^\tau y\xi_v w_v dv \right. \\ &\quad \left. + \int_\tau^T \left(\phi a_v u^r\left(c_v^*, 1\right) - y\xi_v c_v^*\right) dv\right]. \end{aligned}$$

For log utility, we have

$$\begin{aligned} D_t &= \int_0^t -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta}\right) dv \\ &\quad + y\int_0^t \xi_v w_v dv + E_t\left[\int_t^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln\left(\phi\right) - 1\right) dv\right], \end{aligned}$$

and

$$\begin{aligned} J_t &= \sup_{\tau\in\mathcal{S}} E_t\left[\int_0^\tau -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta}\right) dv \right. \\ &\quad \left. + y\int_0^\tau \xi_v w_v dv + \int_\tau^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln\left(\phi\right) - 1\right) dv\right]. \end{aligned}$$

We derived the following equation satisfied in the immediate retirement region $\mathcal{R}(t)$ in Chapter 2,

$$\begin{aligned} &E_t\left[\int_t^T a_v u^a\left(c_v^*, l_v^*\right) 1_{\mathcal{A}(v)} dv\right] + yE_t\left[\int_t^T \xi_v w_v 1_{\mathcal{A}(v)} dv\right] - yE_t\left[\int_t^T \xi_v e_v^* 1_{\mathcal{A}(v)} dv\right] \\ &-E_t\left[\int_t^T \phi a_v u^r\left(c_v^*, 1\right) 1_{\mathcal{A}(v)} dv\right] + yE_t\left[\int_t^T \xi_v c_v^* 1_{\mathcal{A}(v)} dv\right] = 0. \qquad (3.1) \end{aligned}$$

Next we find the equation satisfied by the state variable $x_t \equiv \left(\frac{y\xi_t}{a_t}\right)^{-1} w_t^{-1}$ in $\mathcal{R}(t)$ for log utility.

Proposition 13. *For log utility, in the immediate retirement region* $R(t)$*, we have*

$$\begin{aligned}
& x_t\left((\phi-1)\ln(w_t)-\left(\frac{1}{\eta}-\phi\right)\ln(x_t)+K\right)P(t,T;\beta,\mathcal{A}) \\
& +x_t\left(\left(\frac{1}{\eta}-\phi\right)F_1(t,T;\beta,\mathcal{A})+\frac{1-\eta}{\eta}F_2(t,T;\beta,\mathcal{A})\right) \\
= \; & G(t,T;0,1,0,\mathcal{A}),
\end{aligned}$$

where

$$K=\left(\phi\ln(\phi)-\left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right)+f-\phi\right),$$

$$x_t \equiv \left(\frac{y\xi_t}{a_t}\right)^{-1} w_t^{-1},$$

$$P(t,T;\beta,\mathcal{A})=\int_t^T a_{t,v}E_t\left[1_{\mathcal{A}(v)}\right]dv,$$

$$F_1(t,T;\beta,\mathcal{A})=E_t\left[\int_t^T a_{t,v}\ln\left(\frac{\xi_{t,v}}{a_{t,v}}\right)1_{\mathcal{A}(v)}dv\right],$$

$$F_2(t,T;\beta,\mathcal{A})=E_t\left[\int_t^T a_{t,v}\ln(w_{t,v})1_{\mathcal{A}(v)}dv\right],$$

and

$$G(t,T;0,1,0,\mathcal{A})=E_t\left[\int_t^T \xi_{t,v}w_{t,v}1_{\mathcal{A}(v)}dv\right].$$

Proof. For log utility, we have

$$a_v u^a(c_v^*,l_v^*)=-a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln(w_v)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right),$$

$$\phi a_v u^r(c_v^*)=\phi a_v\ln(c_v)=\phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln(\phi)\right),$$

thus

$$
\begin{aligned}
& E_t\left[\int_t^T a_v u^a\left(c_v^*, l_v^*\right) dv\right] \\
= & -E_t\left[\int_t^T a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(w_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right) dv\right] \\
= & -a_t E_t\left[\int_t^T a_{t,v}\left(\frac{1}{\eta}\ln\left(\frac{\xi_{t,v}}{a_{t,v}}\right)+\frac{1-\eta}{\eta}\ln\left(w_{t,v}\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right) dv\right] \\
& -a_t\left(\frac{1}{\eta}\ln\left(\frac{y\xi_t}{a_t}\right)+\frac{1-\eta}{\eta}\ln\left(w_t\right)\right) E_t\left[\int_t^T a_{t,v} dv\right] \\
= & -a_t\left(\frac{1}{\eta}\ln\left(\frac{y\xi_t}{a_t}\right)+\frac{1-\eta}{\eta}\ln\left(w_t\right)\right) P\left(t,T;\beta\right) \\
& -a_t\frac{1}{\eta}F_1\left(t,T;\beta\right)-a_t\frac{1-\eta}{\eta}F_2\left(t,T;\beta\right) \\
& +a_t\left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right)P\left(t,T;\beta\right),
\end{aligned}
$$

$$
\begin{aligned}
& \phi E_t\left[\int_t^T a_v u^r\left(c_v^*\right) dv\right] \\
= & -\phi E_t\left[\int_t^T a_v \ln\left(\frac{y\xi_v}{a_v}\right) dv\right]+\phi\ln\left(\phi\right)E_t\left[\int_t^T a_v dv\right] \\
= & -\phi a_t E_t\left[\int_t^T a_{t,v}\left(\ln\left(\frac{\xi_{t,v}}{a_{t,v}}\right)+\ln\left(\frac{y\xi_t}{a_t}\right)\right) dv\right]+\phi a_t\ln\left(\phi\right)E_t\left[\int_t^T a_{t,v} dv\right] \\
= & -\phi a_t\left(\ln\left(\frac{y\xi_t}{a_t}\right)-\ln\left(\phi\right)\right)P\left(t,T;\beta\right)-\phi a_t F_1\left(t,T;\beta\right),
\end{aligned}
$$

where

$$
P\left(t,T;\beta\right)=E_t\left[\int_t^T a_{t,v} dv\right]=\int_t^T e^{-\beta(v-t)} dv=\frac{1-e^{-\beta(T-t)}}{\beta},
$$

$$F_1\left(t,T;\beta\right) = E_t\left[\int_t^T a_{t,v}\ln\left(\frac{\xi_{t,v}}{a_{t,v}}\right)dv\right],$$

and

$$F_2\left(t,T;\beta\right) = E_t\left[\int_t^T a_{t,v}\ln\left(w_{t,v}\right)dv\right].$$

The components of the value function are

$$\begin{aligned} E_t\left[\int_t^T \xi_v e_v^* dv\right] &= E_t\left[\int_t^T \xi_v\left(\frac{y\xi_v}{a_v}\right)^{-1} f dv\right] \\ &= \frac{fa_t}{y}E_t\left[\int_t^T a_{t,v}dv\right] = \frac{fa_t}{y}P\left(t,T;\beta\right), \end{aligned}$$

$$E_t\left[\int_t^T \xi_v w_v dv\right] = \xi_t w_t E_t\left[\int_t^T \xi_{t,v}w_{t,v}dv\right] = \xi_t w_t G\left(t,T;0,1,0\right),$$

$$\begin{aligned} E_t\left[\int_t^T \xi_v c_v^* dv\right] &= E_t\left[\int_t^T \xi_v\left(\frac{y\xi_v}{a_v}\right)^{-1} \phi dv\right] \\ &= \frac{\phi a_t}{y}E_t\left[\int_t^T a_{t,v}dv\right] = \frac{\phi a_t}{y}P\left(t,T;\beta\right). \end{aligned}$$

Therefore we have

$$
\begin{aligned}
& -a_t\left(\frac{1}{\eta}\ln\left(\frac{y\xi_t}{a_t}\right)+\frac{1-\eta}{\eta}\ln\left(w_t\right)\right)P\left(t,T;\beta,\mathcal{A}\right) \\
& -a_t\frac{1}{\eta}F_1\left(t,T;\beta,\mathcal{A}\right)-a_t\frac{1-\eta}{\eta}F_2\left(t,T;\beta,\mathcal{A}\right) \\
& +a_t\left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right)P\left(t,T;\beta,\mathcal{A}\right) \\
& +y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\
& -fa_tP\left(t,T;\beta,\mathcal{A}\right) \\
& +\phi a_t\left(\ln\left(\frac{y\xi_t}{a_t}\right)-\ln\left(\phi\right)\right)P\left(t,T;\beta,\mathcal{A}\right)+\phi a_tF_1\left(t,T;\beta,\mathcal{A}\right) \\
& +\phi a_tP\left(t,T;\beta,\mathcal{A}\right) \\
= \; & 0.
\end{aligned}
$$

Simplify,

$$
\begin{aligned}
& a_t\left(\frac{1-\eta}{\eta}\ln\left(w_t\right)+\left(\frac{1}{\eta}-\phi\right)\ln\left(\frac{y\xi_t}{a_t}\right)\right)P\left(t,T;\beta,\mathcal{A}\right) \\
& +a_t\left(\left(\frac{1}{\eta}-\phi\right)F_1\left(t,T;\beta,\mathcal{A}\right)+\frac{1-\eta}{\eta}F_2\left(t,T;\beta,\mathcal{A}\right)\right) \\
& +a_t\left(\phi\ln\left(\phi\right)-\left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right)+f-\phi\right)P\left(t,T;\beta,\mathcal{A}\right) \\
= \; & y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right).
\end{aligned}
$$

Divided by $y\xi_t w_t$ on both sides, we get

$$
\begin{aligned}
& x_t\left(\frac{1-\eta}{\eta}\ln\left(w_t\right)-\left(\frac{1}{\eta}-\phi\right)\ln\left(x_tw_t\right)\right)P\left(t,T;\beta,\mathcal{A}\right) \\
& +x_t\left(\left(\frac{1}{\eta}-\phi\right)F_1\left(t,T;\beta,\mathcal{A}\right)+\frac{1-\eta}{\eta}F_2\left(t,T;\beta,\mathcal{A}\right)\right) \\
& +x_t\left(\phi\ln\left(\phi\right)-\left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right)+f-\phi\right)P\left(t,T;\beta,\mathcal{A}\right) \\
= \; & G\left(t,T;0,1,0,\mathcal{A}\right),
\end{aligned}
$$

or

$$
\begin{aligned}
& x_t\left((\phi-1)\ln(w_t)-\left(\frac{1}{\eta}-\phi\right)\ln(x_t)+K\right)P(t,T;\beta,\mathcal{A}) \\
& +x_t\left(\left(\frac{1}{\eta}-\phi\right)F_1(t,T;\beta,\mathcal{A})+\frac{1-\eta}{\eta}F_2(t,T;\beta,\mathcal{A})\right) \\
= \; & G(t,T;0,1,0,\mathcal{A}),
\end{aligned}
$$

where

$$
K=\left(\phi\ln(\phi)-\left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right)+f-\phi\right). \blacklozenge
$$

The next lemma gives closed form expressions for conditional expectations of random variables restricted to the event of retirement and their derivatives with respect to the multiplier y. These expressions are used for the representation of the optimal retirement boundary, the optimal wealth and the optimal portfolio. First we specify a standing assumption that the immediate retirement region $\mathcal{R}(t)$ is up connected for the state variable x_t, then we can calculate the conditional expectations involving the event $\{x_t \geq B_t\}$.

Assumption. *The immediate retirement region $R(t)$ is up connected for the state variable x_t.*

Sufficient condition to satisfy this assumption is $\phi \geq \frac{1}{\eta}$. See Proposition A2 and its proof in Appendix.

Lemma 14. *The following formulas apply*

$$
P(t,T;\beta,\mathcal{R})=\int_t^T a_{t,v}E_t\left[1_{\mathcal{R}(v)}\right]dv=\int_t^T a_{t,v}N\left(d\left(x_t,B_v,v;0\right)\right)dv,
$$

$$
P(t,T;\beta,\mathcal{A})=\int_t^T a_{t,v}E_t\left[1_{\mathcal{A}(v)}\right]dv=\int_t^T a_{t,v}N\left(-d\left(x_t,B_v,v;0\right)\right)dv,
$$

$$\begin{aligned} & F_1(t,T;\beta,\mathcal{R}) \\ = & \left(\beta - r - \frac{1}{2}\theta^2\right)\int_t^T a_{t,v}(v-t)\,N(d(x_t,B_v,v;0))\,dv \\ & -\theta\int_t^T a_{t,v}\sqrt{v-t}\,n(-d(x_t,B_v,v;0))\,dv, \end{aligned}$$

$$\begin{aligned} & F_1(t,T;\beta,\mathcal{A}) \\ = & \left(\beta - r - \frac{1}{2}\theta^2\right)\int_t^T a_{t,v}(v-t)\,N(-d(x_t,B_v,v;0))\,dv \\ & +\theta\int_t^T a_{t,v}\sqrt{v-t}\,n(-d(x_t,B_v,v;0))\,dv, \end{aligned}$$

$$\begin{aligned} & F_2(t,T;\beta,\mathcal{R}) \\ = & \left(\mu_w - \frac{1}{2}\sigma_w^2\right)\int_t^T a_{t,v}(v-t)\,N(d(x_t,B_v,v;0))\,dv \\ & +\sigma_w\int_t^T a_{t,v}\sqrt{v-t}\,n(-d(x_t,B_v,v;0))\,dv, \end{aligned}$$

$$\begin{aligned} & F_2(t,T;\beta,\mathcal{A}) \\ = & \left(\mu_w - \frac{1}{2}\sigma_w^2\right)\int_t^T a_{t,v}(v-t)\,N(-d(x_t,B_v,v;0))\,dv \\ & -\sigma_w\int_t^T a_{t,v}\sqrt{v-t}\,n(-d(x_t,B_v,v;0))\,dv, \end{aligned}$$

$$\frac{\partial}{\partial y}P(t,T;\beta,\mathcal{R}) = \int_t^T a_{t,v}n(d(x_t,B_v,v;0))\,\frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right)dv.$$

$$\frac{\partial}{\partial y}P\left(t,T;\beta,\mathcal{A}\right)=\int_{t}^{T}a_{t,v}n\left(-d\left(x_{t},B_{v},v;0\right)\right)\frac{1}{\sigma_{x}\sqrt{v-t}}\left(\frac{\frac{\partial B_{v}}{\partial y}}{B_{v}}-\frac{\frac{\partial x_{t}}{\partial y}}{x_{t}}\right)dv,$$

$$\begin{aligned}
&\frac{\partial}{\partial y}F_{1}\left(t,T;\beta,\mathcal{R}\right)\\
=\;&\left(\beta-r-\frac{1}{2}\theta^{2}\right)\\
&\times\int_{t}^{T}a_{t,v}\sqrt{v-t}n\left(d\left(x_{t},B_{v},v;0\right)\right)\frac{1}{\sigma_{x}}\left(\frac{\frac{\partial x_{t}}{\partial y}}{x_{t}}-\frac{\frac{\partial B_{v}}{\partial y}}{B_{v}}\right)dv\\
&-\theta\int_{t}^{T}a_{t,v}n\left(-d\left(x_{t},B_{v},v;0\right)\right)d\left(x_{t},B_{v},v;0\right)\frac{1}{\sigma_{x}}\left(\frac{\frac{\partial B_{v}}{\partial y}}{B_{v}}-\frac{\frac{\partial x_{t}}{\partial y}}{x_{t}}\right)dv,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial y}F_{1}\left(t,T;\beta,\mathcal{A}\right)\\
=\;&\left(\beta-r-\frac{1}{2}\theta^{2}\right)\\
&\times\int_{t}^{T}a_{t,v}\sqrt{v-t}n\left(-d\left(x_{t},B_{v},v;0\right)\right)\frac{1}{\sigma_{x}}\left(\frac{\frac{\partial B_{v}}{\partial y}}{B_{v}}-\frac{\frac{\partial x_{t}}{\partial y}}{x_{t}}\right)dv\\
&+\theta\int_{t}^{T}a_{t,v}n\left(-d\left(x_{t},B_{v},v;0\right)\right)d\left(x_{t},B_{v},v;0\right)\frac{1}{\sigma_{x}}\left(\frac{\frac{\partial B_{v}}{\partial y}}{B_{v}}-\frac{\frac{\partial x_{t}}{\partial y}}{x_{t}}\right)dv,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial y}F_{2}\left(t,T;\beta,\mathcal{R}\right)\\
=\;&\left(\mu_{w}-\frac{1}{2}\sigma_{w}^{2}\right)\\
&\times\int_{t}^{T}a_{t,v}\sqrt{v-t}n\left(d\left(x_{t},B_{v},v;0\right)\right)\frac{1}{\sigma_{x}}\left(\frac{\frac{\partial x_{t}}{\partial y}}{x_{t}}-\frac{\frac{\partial B_{v}}{\partial y}}{B_{v}}\right)dv\\
&+\sigma_{w}\int_{t}^{T}a_{t,v}n\left(-d\left(x_{t},B_{v},v;0\right)\right)d\left(x_{t},B_{v},v;0\right)\frac{1}{\sigma_{x}}\left(\frac{\frac{\partial B_{v}}{\partial y}}{B_{v}}-\frac{\frac{\partial x_{t}}{\partial y}}{x_{t}}\right)dv.
\end{aligned}$$

$$
\begin{aligned}
& \frac{\partial}{\partial y} F_2(t, T; \beta, \mathcal{A}) \\
= & \left(\mu_w - \frac{1}{2}\sigma_w^2\right) \\
& \times \int_t^T a_{t,v} \sqrt{v-t} n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t} \right) dv \\
& -\sigma_w \int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) d\left(x_t, B_v, v; 0\right) \frac{1}{\sigma_x} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t} \right) dv.
\end{aligned}
$$

Proof. See appendix. ♦

Now we can derive the backward recursive equation satisfied by the optimal retirement boundary of the state variable x_t.

Theorem 15. *The boundary B_t satisfies*

$$
\begin{aligned}
& B_t \left((\phi - 1) \ln \left(w_0 \left(\frac{B_t}{x_0} \right)^\gamma \exp(\delta t) \right) - \left(\frac{1}{\eta} - \phi \right) \ln(B_t) + K \right) \\
& \times P^B(t, T; \beta, \mathcal{A}) + B_t \left(\left(\frac{1}{\eta} - \phi \right) F_1^B(t, T; \beta, \mathcal{A}) + \frac{1-\eta}{\eta} F_2^B(t, T; \beta, \mathcal{A}) \right) \\
= & \; G^B(t, T; 0, 1, 0, \mathcal{A}).
\end{aligned} \tag{3.2}
$$

with limiting condition

$$
B_T \left((\phi - 1) \ln \left(w_0 \left(\frac{B_T}{x_0} \right)^\gamma \exp(\delta T) \right) - \left(\frac{1}{\eta} - \phi \right) \ln(B_T) + K \right) = 1,
$$

where $K = \phi \ln(\phi) - \frac{1-\eta}{\eta} \ln\left(\frac{1-\eta}{\eta}\right) + f - \phi$, $\gamma = \frac{\sigma_w}{\sigma_x}$, $\delta = \mu_w - \frac{1}{2}\sigma_w^2 - \gamma\left(\mu_x - \frac{1}{2}\sigma_x^2\right)$ *and*

$$
P^B(t, T; \beta, \mathcal{A}), \;\; F_1^B(t, T; \beta, \mathcal{A}), \;\; F_2^B(t, T; \beta, \mathcal{A}), \;\; G^B(t, T; 0, 1, 0, \mathcal{A})
$$

are the notations for

$$
P(t, T; \beta, \mathcal{A}), \;\; F_1(t, T; \beta, \mathcal{A}), \;\; F_2(t, T; \beta, \mathcal{A}), \;\; G(t, T; 0, 1, 0, \mathcal{A})
$$

when substituting x_t by B_t, respectively.

Proof. We derived that in the retirement region,

$$\begin{aligned} & x_t\left((\phi-1)\ln(w_t)-\left(\frac{1}{\eta}-\phi\right)\ln(x_t)+K\right)P(t,T;\beta,\mathcal{A}) \\ & +x_t\left(\left(\frac{1}{\eta}-\phi\right)F_1(t,T;\beta,\mathcal{A})+\frac{1-\eta}{\eta}F_2(t,T;\beta,\mathcal{A})\right) \\ = \; & G(t,T;0,1,0,\mathcal{A}). \end{aligned}$$

For $x_t=\left(\frac{y\xi_t}{a_t}\right)^{-1}w_t^{-1}$, we have

$$\mu_x=-\beta+\left(r+\frac{1}{2}\theta^2\right)-\left(\mu_w-\frac{1}{2}\sigma_w^2\right)+\frac{1}{2}\sigma_x^2,$$

$$\sigma_x=\theta-\sigma_w.$$

w_t is a transform of x_t,

$$w_t=w_0\left(\frac{x_t}{x_0}\right)^{\gamma}\exp(\delta t),$$

where $\gamma=\frac{\sigma_w}{\sigma_x}$, $\delta=\mu_w-\frac{1}{2}\sigma_w^2-\gamma\left(\mu_x-\frac{1}{2}\sigma_x^2\right)$. Denote

$$P(t,T;\beta,\mathcal{A}),F_1(t,T;\beta,\mathcal{A}),F_2(t,T;\beta,\mathcal{A}) \text{ and } G(t,T;0,1,0,\mathcal{A})$$

when substituting x_t by B_t as

$$P^B(t,T;\beta,\mathcal{A}),F_1^B(t,T;\beta,\mathcal{A}),F_2^B(t,T;\beta,\mathcal{A}) \text{ and } G^B(t,T;0,1,0,\mathcal{A})$$

respectively. Thus the boundary B_t satisfies

$$\begin{aligned} & B_t\left((\phi-1)\ln\left(w_0\left(\frac{B_t}{x_0}\right)^{\gamma}\exp(\delta t)\right)-\left(\frac{1}{\eta}-\phi\right)\ln(B_t)+K\right)P^B(t,T;\beta,\mathcal{A}) \\ & +B_t\left(\left(\frac{1}{\eta}-\phi\right)F_1^B(t,T;\beta,\mathcal{A})+\frac{1-\eta}{\eta}F_2^B(t,T;\beta,\mathcal{A})\right) \\ = \; & G^B(t,T;0,1,0,\mathcal{A}). \end{aligned}$$

To derive the limiting condition for the boundary, let the integrand which is the instantaneous gain minus the instantaneous loss be equal to 0.

$$-a_v u^a\left(c_v^*, l_v\right) + \phi a_v u^r\left(c_v^*, 1\right) - y\xi_v w_v + y\xi_v e_v^* - y\xi_v c_v^* = 0,$$

$$\begin{aligned} & a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right) \\ & + \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln\left(\phi\right)\right) - y\xi_v w_v + f a_v - \phi a_v = 0, \end{aligned}$$

$$\begin{aligned} & a_v\left(\left(\frac{1}{\eta} - \phi\right)\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right)\right) \\ & + a_v\left(\phi\ln\left(\phi\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right) + f - \phi\right) - y\xi_v w_v = 0, \end{aligned}$$

$$\begin{aligned} & x_v\left(\left(\frac{1}{\eta} - \phi\right)\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right)\right) \\ & + x_v\left(\phi\ln\left(\phi\right) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right) + f - \phi\right) - 1 = 0, \end{aligned}$$

$$\begin{aligned} & x_v\left(-\left(\frac{1}{\eta} - \phi\right)\ln\left(x_v w_v\right) + \frac{1-\eta}{\eta}\ln\left(w_v\right) + K\right) = 1, \\ & x_v\left(\left(\phi - 1\right)\ln\left(w_v\right) - \left(\frac{1}{\eta} - \phi\right)\ln\left(x_v\right) + K\right) = 1. \end{aligned}$$

Thus the limiting condition for the boundary is B_T satisfies

$$B_T\left(\left(\phi - 1\right)\ln\left(w_0\left(\frac{B_T}{x_0}\right)^\gamma \exp\left(\delta T\right)\right) - \left(\frac{1}{\eta} - \phi\right)\ln\left(B_T\right) + K\right) = 1. \blacklozenge$$

What is the relationship between the optimal retirement boundary of an individual with log utility and the optimal retirement boundary of an individual with power utility?

Comment. *In general, the optimal retirement boundary* B_t *of an individual with log utility and the optimal retirement boundary* B_t *of an individual with power utility* *are different even if all parameters for the two individuals are the same, the initial multipliers are the same and the coefficient of relative risk aversion converges to* 1*. However, under the condition* $\phi = 1/\eta$*, the two individuals will have the same optimal retirement boundaries if all parameters for the two individuals are the same, the initial multipliers are the same and the coefficient of relative risk aversion converges to* 1*.*

Proof. We consider equation (3.1) satisfied within the retirement region which is used to derive the optimal retirement boundaries for both individual with power utility and individual with log utility. From the candidate solutions of e_v^* and c_v^*, it's straightforward to see that if the multiplier y is the same, e_v^* and c_v^* for power utility converges to e_v^* and c_v^* for log utility as R approaches to 1. $u^a(c_v^*, l_v^*)$ and $\phi u^r(c_v^*, 1)$ for power utility do not converge to $u^a(c_v^*, l_v^*)$ and $\phi u^r(c_v^*, 1)$ for log utility in general. However, if we subtract 1 from the numerators of both

$$u^a(c_v^*, l_v^*) = \frac{\left((c_v^*)^\eta (l_v^*)^{1-\eta}\right)^{1-R}}{\eta(1-R)} \quad \text{and} \quad \phi u^r(c_v^*, 1) = \phi\frac{(c_v^*)^{1-R}}{1-R},$$

then they will converge to

$$u^a(c_v^*, l_v^*) = \frac{1}{\eta}\left(\eta \ln(c_v^*) + (1-\eta)\ln(l_v^*)\right) \quad \text{and} \quad \phi u^r(c_v^*, 1) = \phi \ln(c_v^*),$$

as R approaches to 1. Therefore, if $\phi = 1/\eta$, then $u^a(c_v^*, l_v^*) - \phi u^r(c_v^*, 1)$ for power utility converges to $u^a(c_v^*, l_v^*) - \phi u^r(c_v^*, 1)$ for log utility as R approaches to 1. Notice also that the state variable we choose for log utility $\left(\frac{y\xi_t}{a_t}\right)^{-1} w_t^{-1}$ is the limit of the state variable for power utility $\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} w_t^{(1-\eta)\rho-1}$ as R approaches to 1. Therefore equation (3.1) for power utility converges to equation (3.1) for log utility as R approaches to 1. With the same state variable, the same backward recursive equation will be derived from equation (3.1) if all parameters for the two individuals are the same, the initial multipliers are the same and the coefficient of relative risk aversion converges to 1. ♦

Equation (3.2) is a backward recursive equation. The same algorithm as in the power utility case can be implemented to solve the equation. First use trapezoidal rule to discretize the integrals and recursively solve the boundary starting from the boundary condition. And at each time step, solve the nonlinear equation using Newton-Raphson iteration. Next proposition gives the equation satisfied by the derivative of the retirement boundary with respect to the multiplier y. $\frac{\partial B_t}{\partial y}$ is needed for the expressions of the optimal wealth and the optimal portfolio.

Proposition 16. *The derivative of the boundary* $\frac{\partial B_t}{\partial y}$ *satisfies*

$$\begin{aligned}
& \frac{\partial B_t}{\partial y}\left((\phi-1)\ln\left(w_0\left(\frac{B_t}{x_0}\right)^{\gamma}\exp(\delta t)\right)-\left(\frac{1}{\eta}-\phi\right)\ln(B_t)+K\right)P^B(t,T;\beta,\mathcal{A}) \\
& +B_t\left((\phi-1)\gamma\frac{1}{B_t}\frac{\partial B_t}{\partial y}+(\phi-1)\gamma\frac{1}{y}-\left(\frac{1}{\eta}-\phi\right)\frac{1}{B_t}\frac{\partial B_t}{\partial y}\right)P^B(t,T;\beta,\mathcal{A}) \\
& +\frac{\partial B_t}{\partial y}\left(\left(\frac{1}{\eta}-\phi\right)F_1^B(t,T;\beta,\mathcal{A})+\frac{1-\eta}{\eta}F_2^B(t,T;\beta,\mathcal{A})\right) \\
& +B_t\left((\phi-1)\ln\left(w_0\left(\frac{B_t}{x_0}\right)^{\gamma}\exp(\delta t)\right)-\left(\frac{1}{\eta}-\phi\right)\ln(B_t)+K\right)\frac{\partial}{\partial y}P^B(t,T;\beta,\mathcal{A}) \\
& +B_t\left(\left(\frac{1}{\eta}-\phi\right)\frac{\partial}{\partial y}F_1^B(t,T;\beta,\mathcal{A})+\frac{1-\eta}{\eta}\frac{\partial}{\partial y}F_2^B(t,T;\beta,\mathcal{A})\right) \\
& -\frac{\partial}{\partial y}G^B(t,T;0,1,0,\mathcal{A}) \\
= \;& 0,
\end{aligned}$$

with boundary condition

$$\frac{\partial B_T}{\partial y}=\frac{-B_T(\phi-1)\gamma\frac{1}{y}}{(\phi-1)\ln\left(y^{\gamma}w_0^{1+\gamma}B_T^{\gamma}\exp(\delta T)\right)-\left(\frac{1}{\eta}-\phi\right)\ln(B_T)+K+(\phi-1)\gamma-\left(\frac{1}{\eta}-\phi\right)}.$$

Proof. See appendix. ♦

Figure 3.1 shows the convergence of the retirement boundary for power utility to the retirement boundary for log utility as R approaches to 1, under the condition $\phi = 1/\eta$, all parameters for the two individuals are the same and the initial multipliers are the same. Parameter values are chosen as the same in Table 2.1, except now we change ϕ

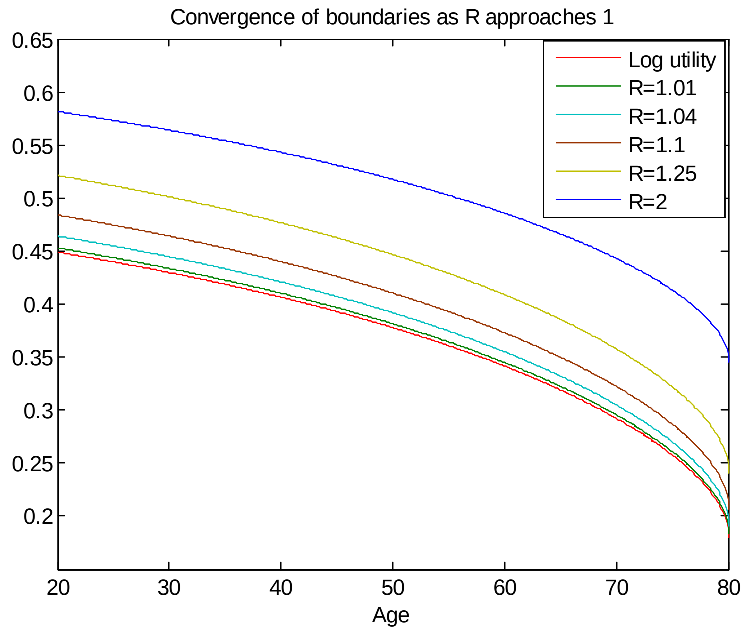

Figure 3·1: This figure shows the convergence of the optimal retirement boundaries of power utility to the optimal retirement boundary of log utility.

to be 3/2, in order for it to be equal to $1/\eta$. The initial multiplier y is chosen to be the same as derived in the numerical studies in Chapter 2, $y = 0.7130688 * 10^{-14}$. The boundary (red) at the bottom is the optimal retirement boundary for an individual with log utility computed from Equation (3.2). All other boundaries are optimal retirement boundaries for an individual with power utility for different coefficients of relative risk aversion R, computed from (2.10). As R approaches to 1, the boundaries of power utility converges to the boundary of log utility. Figure 3.2 shows that the derivatives of the optimal retirement boundaries $\frac{\partial B_t}{\partial y}$ of power utility converge to the derivative of the optimal retirement boundary $\frac{\partial B_t}{\partial y}$ of log utility (red) as R approaches to 1. The derivatives are computed by a more direct approach. Increase the multiplier y by a small amount Δy, for this new multiplier $y + \Delta y$, compute a new boundary $B_t(y + \Delta y)$. The derivative is approximated by $\frac{\partial B_t}{\partial y} \approx (B_t(y + \Delta y) - B_t(y))/\Delta y$.

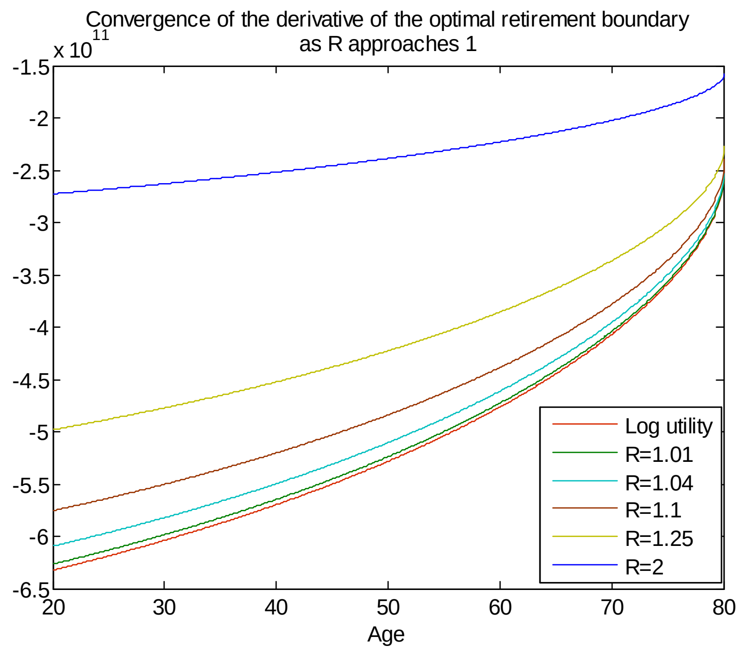

Figure 3·2: This figure shows that the derivatives of the optimal retirement boundaries $\frac{\partial B_t}{\partial y}$ of power utility converge to the derivative of the optimal retirement boundary $\frac{\partial B_t}{\partial y}$ of log utility.

3.4 Optimal liquid wealth

The closed form expression of the optimal liquid wealth is derived in the next theorem.

Theorem 17. *Before retirement, the optimal liquid wealth* X_t *satisfies*

$$\begin{aligned}
& X_t \\
= \; & \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv \right] \\
= \; & \xi_t^{-1} \left[a_t \frac{1}{\eta} \frac{1}{y} P(t, T; \beta, \mathcal{A}) - \xi_t w_t G(t, T; 0, 1, 0, \mathcal{A}) + \phi a_t \frac{1}{y} P(t, T; \beta, \mathcal{R}) \right. \qquad (3.3) \\
& + a_t \left(\frac{1}{\eta} \ln \left(\frac{y \xi_t}{a_t} \right) + \frac{1-\eta}{\eta} \ln(w_t) - \left(\frac{1-\eta}{\eta} \right) \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) \frac{\partial}{\partial y} P(t, T; \beta, \mathcal{A}) \\
& + a_t \frac{1}{\eta} \frac{\partial}{\partial y} F_1(t, T; \beta, \mathcal{A}) + a_t \frac{1-\eta}{\eta} \frac{\partial}{\partial y} F_2(t, T; \beta, \mathcal{A}) - y \xi_t w_t \frac{\partial}{\partial y} G(t, T; 0, 1, 0, \mathcal{A}) \\
& \left. - \phi a_t \left(-\ln \left(\frac{y \xi_t}{a_t} \right) + \ln(\phi) - 1 \right) \frac{\partial}{\partial y} P(t, T; \beta, \mathcal{R}) + \phi a_t \frac{\partial}{\partial y} F_1(t, T; \beta, \mathcal{R}) \right].
\end{aligned}$$

After retirement, the liquid wealth is

$$\begin{aligned}
X_t &= \xi_t^{-1} E_t \left[\int_t^T \xi_v c_v^* dv \right] = \xi_t^{-1} E_t \left[\int_t^T \xi_v \left(\frac{y \xi_v}{a_v} \right)^{-1} \phi dv \right] \\
&= \phi \left(\frac{y \xi_t}{a_t} \right)^{-1} \frac{1 - \exp(-\beta(T-t))}{\beta}.
\end{aligned}$$

Proof. For log utility, we have

$$\begin{aligned}
D_t = & \int_0^t -a_v \left(\frac{1}{\eta} \ln \left(\frac{y \xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln(w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \\
& + y \int_0^t \xi_v w_v dv + E_t \left[\int_t^T \phi a_v \left(-\ln \left(\frac{y \xi_v}{a_v} \right) + \ln(\phi) - 1 \right) dv \right],
\end{aligned}$$

and

$$\begin{aligned} J_t &\equiv \sup_{\tau \in \mathcal{S}} E_t \left[D_\tau \right] \\ &= \sup_{\tau \in \mathcal{S}} E_t \left[\int_0^\tau -a_v \left(\frac{1}{\eta} \ln \left(\frac{y\xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \right. \\ &\quad \left. + y \int_0^\tau \xi_v w_v dv + \int_\tau^T \phi a_v \left(- \ln \left(\frac{y\xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv \right]. \end{aligned}$$

Denote

$$\begin{aligned} L_t &\equiv J_t - E_t \left[\int_0^t -a_v \left(\frac{1}{\eta} \ln \left(\frac{y\xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \right. \\ &\quad \left. + \int_0^t y\xi_v w_v dv \right]. \end{aligned}$$

Thus

$$\begin{aligned} L_t &= E_t \left[\int_t^{\tau_t^*} -a_v \left(\frac{1}{\eta} \ln \left(\frac{y\xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \right. \\ &\quad \left. + \int_t^{\tau_t^*} y\xi_v w_v dv + \int_{\tau_t^*}^T \phi a_v \left(- \ln \left(\frac{y\xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv \right], \end{aligned}$$

where τ_t^* is the optimal stopping time. Liquid wealth X_t satisfies

$$\xi_t X_t = E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^T \xi_v c_v^* dv \right].$$

Plug in the values of e_v^* and c_v^*,

$$\xi_t X_t = E_t \left[\int_t^{\tau_t^*} \frac{f a_v}{y} dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^T \frac{\phi a_v}{y} dv \right].$$

L_t is a convex function of y. Therefore,

$$\frac{\partial L_t}{\partial y} = E_t \left[\int_t^{\tau_t^*} -\frac{f a_v}{y} dv + \int_t^{\tau_t^*} \xi_v w_v dv - \int_{\tau_t^*}^{T} \frac{\phi a_v}{y} dv \right]$$

i.e. $\frac{\partial L_t}{\partial y} = -\xi_t X_t$ is satisfied. Use Early Exercise Premium representation to write L_t as the following

$$
\begin{aligned}
& L_t \\
= \; & E_t\left[\int_t^{\tau_t^*} -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(w_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)dv\right. \\
& \left.+\int_t^{\tau_t^*} y\xi_v w_v dv+\int_{\tau_t^*}^{T}\phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln\left(\phi\right)-1\right)dv\right] \\
= \; & E_t\left[\int_t^{T} -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(w_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)dv\right. \\
& \left.+\int_t^{T} y\xi_v w_v dv\right] \\
& -E_t\left[\int_t^{T} -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(w_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)1_{\mathcal{R}(v)}dv\right. \\
& \left.+\int_t^{T} y\xi_v w_v 1_{\mathcal{R}(v)}dv-\int_t^{T}\phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln\left(\phi\right)-1\right)1_{\mathcal{R}(v)}dv\right] \\
= \; & E_t\left[\int_t^{T} -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(w_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)1_{\mathcal{A}(v)}dv\right. \\
& \left.+\int_t^{T} y\xi_v w_v 1_{\mathcal{A}(v)}dv+\int_t^{T}\phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln\left(\phi\right)-1\right)1_{\mathcal{R}(v)}dv\right] \\
= \; & -a_t\left(\frac{1}{\eta}\ln\left(\frac{y\xi_t}{a_t}\right)+\frac{1-\eta}{\eta}\ln\left(w_t\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)P\left(t,T;\beta,\mathcal{A}\right) \\
& -a_t\frac{1}{\eta}F_1\left(t,T;\beta,\mathcal{A}\right)-a_t\frac{1-\eta}{\eta}F_2\left(t,T;\beta,\mathcal{A}\right)+y\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) \\
& +\phi a_t\left(-\ln\left(\frac{y\xi_t}{a_t}\right)+\ln\left(\phi\right)-1\right)P\left(t,T;\beta,\mathcal{R}\right)-\phi a_t F_1\left(t,T;\beta,\mathcal{R}\right).
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&\xi_t X_t = -\frac{\partial L_t}{\partial y} \\
= \;& a_t \frac{1}{\eta}\frac{1}{y} P(t,T;\beta,\mathcal{A}) - \xi_t w_t G(t,T;0,1,0,\mathcal{A}) + \phi a_t \frac{1}{y} P(t,T;\beta,\mathcal{R}) \\
&+ a_t \left(\frac{1}{\eta} \ln\left(\frac{y\xi_t}{a_t}\right) + \frac{1-\eta}{\eta} \ln(w_t) - \left(\frac{1-\eta}{\eta}\right) \ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta} \right) \frac{\partial}{\partial y} P(t,T;\beta,\mathcal{A}) \\
&+ a_t \frac{1}{\eta}\frac{\partial}{\partial y} F_1(t,T;\beta,\mathcal{A}) + a_t \frac{1-\eta}{\eta}\frac{\partial}{\partial y} F_2(t,T;\beta,\mathcal{A}) - y\xi_t w_t \frac{\partial}{\partial y} G(t,T;0,1,0,\mathcal{A}) \\
&- \phi a_t \left(-\ln\left(\frac{y\xi_t}{a_t}\right) + \ln(\phi) - 1 \right) \frac{\partial}{\partial y} P(t,T;\beta,\mathcal{R}) + \phi a_t \frac{\partial}{\partial y} F_1(t,T;\beta,\mathcal{R}),
\end{aligned}$$

and the optimal liquid wealth X_t is

$$\begin{aligned}
& X_t \\
= \;& \xi_t^{-1} \left[a_t \frac{1}{\eta}\frac{1}{y} P(t,T;\beta,\mathcal{A}) - \xi_t w_t G(t,T;0,1,0,\mathcal{A}) + \phi a_t \frac{1}{y} P(t,T;\beta,\mathcal{R}) \right. \\
&+ a_t \left(\frac{1}{\eta} \ln\left(\frac{y\xi_t}{a_t}\right) + \frac{1-\eta}{\eta} \ln(w_t) - \left(\frac{1-\eta}{\eta}\right) \ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta} \right) \frac{\partial}{\partial y} P(t,T;\beta,\mathcal{A}) \\
&+ a_t \frac{1}{\eta}\frac{\partial}{\partial y} F_1(t,T;\beta,\mathcal{A}) + a_t \frac{1-\eta}{\eta}\frac{\partial}{\partial y} F_2(t,T;\beta,\mathcal{A}) - y\xi_t w_t \frac{\partial}{\partial y} G(t,T;0,1,0,\mathcal{A}) \\
&\left. - \phi a_t \left(-\ln\left(\frac{y\xi_t}{a_t}\right) + \ln(\phi) - 1 \right) \frac{\partial}{\partial y} P(t,T;\beta,\mathcal{R}) + \phi a_t \frac{\partial}{\partial y} F_1(t,T;\beta,\mathcal{R}) \right]. \blacklozenge
\end{aligned}$$

Before retirement, the optimal liquid wealth X_t of an individual with log utility is the limit of the liquid wealth of an individual with power utility when the coefficient of relative risk aversion R approaches 1, if the optimal retirement boundaries B_t are the same, and the corresponding parameters and the initial multipliers for the two individuals are the same. The reason is in the representation of liquid wealth,

$$\xi_t X_t = E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv \right],$$

as R approaches 1, e_v^*, c_v^* and the state variable x_t of power utility all converge to those of log utility. The optimal stopping time τ_t^* are the same if the boundaries are the same. After retirement, convergence results hold for the optimal liquid wealth when R approaches 1, as long as the multipliers for the two individuals are the same.

Theorem 18. *If $\phi \geq \frac{1}{\eta}$ or $a_v u^a(c_v^*, l_v^*) \leq \phi a_v u^r(c_v^*, 1)$, retirement is optimal when the liquid wealth crosses its boundary, i.e.,*

$$X_t \geq y^{-1-\gamma_\xi} w_0^{-\gamma_\xi} \phi a_t \exp(-\delta_\xi t) \frac{1-\exp(-\beta(T-t))}{\beta} B_t^{-\gamma_\xi}. \tag{3.4}$$

where $\gamma_\xi = \frac{-\theta}{\sigma_x}$, $\delta_\xi = -r - \frac{1}{2}\theta^2 - \gamma_\xi\left(\mu_x - \frac{1}{2}\sigma_x^2\right)$ and B_t is the boundary for the state variable x_t.

Proof. The liquid wealth before retirement is

$$X_t = \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v e_v^* dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \xi_v c_v^* dv \right],$$

where τ_t^* is the optimal stopping time. Plug in the values of e_v^* and c_v^*,

$$X_t = \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \frac{f a_v}{y} dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \frac{\phi a_v}{y} dv \right].$$

The liquid wealth after retirement is

$$\begin{aligned} X_t &= E_t \left[\int_t^T \xi_{t,v} c_v^* dv \right] = E_t \left[\int_t^T \xi_{t,v} \phi \left(\frac{y \xi_v}{a_v} \right)^{-1} dv \right] \\ &= \xi_t^{-1} E_t \left[\int_t^T \frac{\phi a_v}{y} dv \right]. \end{aligned}$$

Next we show that the boundary of the term $\xi_t^{-1} E_t \left[\int_t^T \frac{\phi a_v}{y} dv \right]$ is exactly the retirement boundary of the liquid wealth. It suffices to show that the liquid wealth before retirement is less than or equal to the value of $\xi_t^{-1} E_t \left[\int_t^T \frac{\phi a_v}{y} dv \right]$. This is to show, at time t before retirement,

$$
\begin{aligned}
X_{t,b} &\equiv \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \frac{f a_v}{y} dv - \int_t^{\tau_t^*} \xi_v w_v dv + \int_{\tau_t^*}^{T} \frac{\phi a_v}{y} dv \right] \\
&\leq X_{t,a} \equiv \xi_t^{-1} E_t \left[\int_t^{T} \frac{\phi a_v}{y} dv \right],
\end{aligned}
$$

or

$$
E_t \left[\int_t^{\tau_t^*} \frac{f a_v}{y} dv - \int_t^{\tau_t^*} \xi_v w_v dv - \int_t^{\tau_t^*} \frac{\phi a_v}{y} dv \right] \leq 0.
$$

One sufficient condition for the above inequality is $f - \phi \leq 0$, or $\phi \geq \frac{1}{\eta}$. Notice $\phi \geq \frac{1}{\eta}$ is also a sufficient condition for the retirement region to be up-connected. To derive another sufficient condition, consider the optimal value function, which is

$$
\begin{aligned}
L_t &= E_t \left[\int_t^{\tau_t^*} -a_v \left(\frac{1}{\eta} \ln \left(\frac{y \xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \right. \\
&\left. + \int_t^{\tau_t^*} y \xi_v w_v dv + \int_{\tau_t^*}^{T} \phi a_v \left(- \ln \left(\frac{y \xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv \right],
\end{aligned}
$$

thus we have

$$
\begin{aligned}
L_t &= E_t \left[\int_t^{\tau_t^*} -a_v \left(\frac{1}{\eta} \ln \left(\frac{y \xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) \right) dv \right. \\
&\left. + \int_{\tau_t^*}^{T} \phi a_v \left(- \ln \left(\frac{y \xi_v}{a_v} \right) + \ln (\phi) \right) dv \right] - y \xi_t X_{t,b},
\end{aligned}
$$

and

$$\begin{aligned} L_t &\geq E_t\left[\int_t^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln(\phi)-1\right)dv\right] \\ &= E_t\left[\int_t^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln(\phi)\right)dv\right]-y\xi_t X_{t,a}. \end{aligned}$$

Therefore

$$\begin{aligned} & E_t\left[\int_t^{\tau_t^*} -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln(w_v)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right)dv\right. \\ & \left.+\int_{\tau_t^*}^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln(\phi)\right)dv\right]-y\xi_t X_{t,b} \\ \geq\ & E_t\left[\int_t^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln(\phi)\right)dv\right]-y\xi_t X_{t,a}. \end{aligned}$$

Sufficient condition to derive $X_{t,b} \leq X_{t,a}$ is

$$-a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln(w_v)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)\right) \leq \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln(\phi)\right),$$

i.e., $a_v u^a(c_v^*, l_v^*) \leq \phi a_v u^r(c_v^*, 1)$. Next we compute the form of the boundary of the liquid wealth.

$$X_{t,a} = \xi_t^{-1} E_t\left[\int_t^T \frac{\phi a_v}{y} dv\right].$$

ξ_t is a transform of x_t,

$$\xi_t = \left(\frac{x_t}{x_0}\right)^{\gamma_\xi}\exp(\delta_\xi t),$$

where $\gamma_\xi = \frac{-\theta}{\sigma_x}$, $\delta_\xi = -r-\frac{1}{2}\theta^2-\gamma_\xi\left(\mu_x-\frac{1}{2}\sigma_x^2\right)$, $x_0 = y^{-1}w_0^{-1}$ and

$$\mu_x = -\beta+\left(r+\frac{1}{2}\theta^2\right)-\left(\mu_w-\frac{1}{2}\sigma_w^2\right)+\frac{1}{2}\sigma_x^2,$$

$$\sigma_x = \theta - \sigma_w.$$

Therefore the boundary of the liquid wealth is

$$\begin{aligned} & \left(\frac{B_t}{x_0}\right)^{-\gamma_\xi} \exp\left(-\delta_\xi t\right) E_t \left[\int_t^T \frac{\phi a_v}{y} dv\right] \\ = \; & y^{-1-\gamma_\xi} w_0^{-\gamma_\xi} \phi a_t \exp\left(-\delta_\xi t\right) \frac{1 - \exp(-\beta(T-t))}{\beta} B_t^{-\gamma_\xi}. \; \blacklozenge \end{aligned}$$

By inspection of the optimal boundary of liquid wealth for an individual with log utility (3.4) and the optimal boundary of liquid wealth for an individual with power utility (2.16), we find that the former is the limit of the latter when the coefficient of relative risk aversion approaches to 1, if the optimal retirement boundaries B_t are the same, and the corresponding parameters and the initial multipliers of the two individuals are the same.

3.5 Optimal consumption, optimal labor and optimal expenditure

The optimal consumption, the optimal leisure and the optimal expenditure are as follows. Before retirement,

$$c_v^* = \eta \left(\frac{y\xi_v}{a_v}\right)^{-1} f, \qquad l_v^* = (1-\eta)\left(\frac{y\xi_v}{a_v}\right)^{-1} f \frac{1}{w_v},$$

$$e_v^* = c_v^* + l_v^* w_v = \left(\frac{y\xi_v}{a_v}\right)^{-1} f,$$

after retirement,

$$c_v^* = \left(\frac{y\xi_v}{a_v}\right)^{-1} \phi,$$

where $f = 1/\eta$, and the multiplier y satisfies equation (3.3) at time 0. Optimal labor is the difference between the maximal amount of labor $\overline{h}$ endowed (normalized to be 1) and the optimal leisure l_v^*, i.e., $h_v^* = 1 - l_v^*$.

Consumption is inversely related to state price density ξ_v and its volatility is equal to the market price of risk θ. For leisure, it is inversely related to $\xi_v w_v$ and it has volatility $\theta - \sigma_w$. For empirically reasonable estimates of θ and σ_w, $\theta - \sigma_w$ is positive and the consumption and leisure volatility is much higher for the case of log utility than the case of power utility. In a state where the stock market experiences a positive shock, it's optimal to consume more and have more leisure (if assume $\theta - \sigma_w > 0$). After retirement, consumption volatility is θ. When the wage is deterministic, $\sigma_w = 0$, consumption volatility before and after retirement and leisure volatility are all equal to θ, which is a very high volatility.

As in the case of power utility, the optimal consumption, the optimal leisure and the optimal expenditure are increasing with respect to the initial wealth. When initial wealth is higher, the multiplier y becomes smaller, which results in higher consumption, leisure and total expenditure.

η is the measure of relative weight of consumption and labor. In the total expenditure e_v^*, the percentage that the individual spends on consumption is η, and the percentage that the individual spends on leisure is $1 - \eta$ with opportunity cost being the wage w_v. The larger the value of η, the larger the percentage spent on consumption and the smaller the percentage spent on leisure.

ϕ is the coefficient that measures the relative weight of the retirement phase, and it also controls the disruption of the consumption when the individual enters the retirement phase. The disruption is measured by the ratio of the consumption immediately after retirement and the consumption just before retirement ϕ. Note that if ϕ is greater than 1, optimal consumption for an individual with log utility jumps to a higher level at the moment when the individual retires.

The optimal consumption, the optimal leisure and the optimal expenditure of an individual with log utility are the limits of the optimal consumption, the optimal leisure and the optimal expenditure of an individual with power utility when the coefficient of relative risk aversion R converges to 1, provided that the two individuals have the same initial Lagrange multiplier y.

3.6 Optimal portfolio

Theorem 19. *Before retirement, the optimal portfolio π_t satisfies*

$$\begin{aligned}
& \pi_t \sigma \\
= \; & \xi_t^{-1} \Bigg[\theta a_t \frac{1}{\eta} \frac{1}{y} P(t,T;\beta,\mathcal{A}) - \sigma_w \xi_t w_t G(t,T;0,1,0,\mathcal{A}) + \theta \phi a_t \frac{1}{y} P(t,T;\beta,\mathcal{R}) + a_t \\
& \times \left(\frac{1}{\eta} \left(\theta \ln\left(\frac{y\xi_t}{a_t}\right) - \theta \right) + \frac{1-\eta}{\eta} \left(\theta \ln(w_t) + \sigma_w\right) - \theta \left(\frac{1-\eta}{\eta}\right) \ln\left(\frac{1-\eta}{\eta}\right) + \theta \frac{1}{\eta} \right) \\
& \times \frac{\partial}{\partial y} P(t,T;\beta,\mathcal{A}) + \theta a_t \frac{1}{\eta} \frac{\partial}{\partial y} F_1(t,T;\beta,\mathcal{A}) \\
& + \theta a_t \frac{1-\eta}{\eta} \frac{\partial}{\partial y} F_2(t,T;\beta,\mathcal{A}) - \sigma_w y \xi_t w_t \frac{\partial}{\partial y} G(t,T;0,1,0,\mathcal{A}) \\
& - \phi a_t \left(-\theta \ln\left(\frac{y\xi_t}{a_t}\right) + \theta \ln(\phi) \right) \frac{\partial}{\partial y} P(t,T;\beta,\mathcal{R}) + \theta \phi a_t \frac{\partial}{\partial y} F_1(t,T;\beta,\mathcal{R}) \Bigg] \\
& + (\theta - \sigma_w) \xi_t^{-1} \Bigg[a_t \frac{1}{\eta} \frac{1}{y} g_1 - \xi_t w_t g_2 + \phi a_t \frac{1}{y} g_3 \\
& + a_t \left(\frac{1}{\eta} \ln\left(\frac{y\xi_t}{a_t}\right) + \frac{1-\eta}{\eta} \ln(w_t) - \left(\frac{1-\eta}{\eta}\right) \ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta} \right) g_4 \\
& + a_t \frac{1}{\eta} g_5 + a_t \frac{1-\eta}{\eta} g_6 - y \xi_t w_t g_7 - \phi a_t \left(-\ln\left(\frac{y\xi_t}{a_t}\right) + \ln(\phi) - 1 \right) g_8 + \phi a_t g_9 \Bigg].
\end{aligned}$$

We have $\pi_t = \pi_{1t} + \pi_{2t}$*, where*

$$\begin{aligned}
&\pi_{1t} = \sigma^{-1}\theta \\
&\times \xi_t^{-1}\left\{ a_t \frac{1}{\eta}\frac{1}{y} P\left(t,T;\beta,\mathcal{A}\right) + \phi a_t \frac{1}{y} P\left(t,T;\beta,\mathcal{R}\right) \right. \\
&+ a_t \left(\frac{1}{\eta}\left(\ln\left(\frac{y\xi_t}{a_t}\right) - 1 \right) + \frac{1-\eta}{\eta}\ln\left(w_t\right) - \left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta} \right) \\
&\times \frac{\partial}{\partial y} P\left(t,T;\beta,\mathcal{A}\right) + a_t \frac{1}{\eta}\frac{\partial}{\partial y} F_1\left(t,T;\beta,\mathcal{A}\right) + a_t \frac{1-\eta}{\eta}\frac{\partial}{\partial y} F_2\left(t,T;\beta,\mathcal{A}\right) \\
&- \phi a_t \left(-\ln\left(\frac{y\xi_t}{a_t}\right) + \ln\left(\phi\right) \right) \frac{\partial}{\partial y} P\left(t,T;\beta,\mathcal{R}\right) + \phi a_t \frac{\partial}{\partial y} F_1\left(t,T;\beta,\mathcal{R}\right) \\
&+ a_t \frac{1}{\eta}\frac{1}{y} g_1 - \xi_t w_t g_2 + \phi a_t \frac{1}{y} g_3 \\
&+ a_t \left(\frac{1}{\eta}\ln\left(\frac{y\xi_t}{a_t}\right) + \frac{1-\eta}{\eta}\ln\left(w_t\right) - \left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta} \right) g_4 \\
&\left. + a_t \frac{1}{\eta} g_5 + a_t \frac{1-\eta}{\eta} g_6 - y\xi_t w_t g_7 - \phi a_t \left(-\ln\left(\frac{y\xi_t}{a_t}\right) + \ln\left(\phi\right) - 1 \right) g_8 + \phi a_t g_9 \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\pi_{2t} = \sigma^{-1}\sigma_w \\
&\times \xi_t^{-1}\left\{ -\xi_t w_t G\left(t,T;0,1,0,\mathcal{A}\right) + a_t \left(\frac{1-\eta}{\eta}\right) \frac{\partial}{\partial y} P\left(t,T;\beta,\mathcal{A}\right) \right. \\
&- y\xi_t w_t \frac{\partial}{\partial y} G\left(t,T;0,1,0,\mathcal{A}\right) \\
&- \left[a_t \frac{1}{\eta}\frac{1}{y} g_1 - \xi_t w_t g_2 + \phi a_t \frac{1}{y} g_3 \right. \\
&+ a_t \left(\frac{1}{\eta}\ln\left(\frac{y\xi_t}{a_t}\right) + \frac{1-\eta}{\eta}\ln\left(w_t\right) - \left(\frac{1-\eta}{\eta}\right)\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta} \right) g_4 \\
&\left.\left. + a_t \frac{1}{\eta} g_5 + a_t \frac{1-\eta}{\eta} g_6 - y\xi_t w_t g_7 - \phi a_t \left(-\ln\left(\frac{y\xi_t}{a_t}\right) + \ln\left(\phi\right) - 1 \right) g_8 + \phi a_t g_9 \right] \right\}.
\end{aligned}$$

π_{1t} *is the mean-variance hedge, and* π_{2t} *is the hedge against fluctuations in wages. And*

$$g_1 = -\int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$g_2 = -\int_t^T \exp\left(-A\left(0,1,0\right)\left(v-t\right)\right) n\left(-d\left(x_t, B_v, v; C\left(1,0\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$g_3 = \int_t^T a_{t,v} n\left(d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$g_4 = -\int_t^T d\left(x_t, B_v, v; 0\right) a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv,$$

$$\begin{aligned} & g_5 \\ = \; & -\left(\beta - r - \frac{1}{2}\theta^2\right) \int_t^T d\left(x_t, B_v, v; 0\right) a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x^2} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv \\ & +\theta \int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \left(-d^2\left(x_t, B_v, v; 0\right) + 1\right) \frac{1}{\sigma_x^2\sqrt{v-t}} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv, \end{aligned}$$

$$\begin{aligned} & g_6 \\ = \; & -\left(\mu_w - \frac{1}{2}\sigma_w^2\right) \int_t^T d\left(x_t, B_v, v; 0\right) a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x^2} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv \\ & -\sigma_w \int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \left(-d^2\left(x_t, B_v, v; 0\right) + 1\right) \frac{1}{\sigma_x^2\sqrt{v-t}} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv, \end{aligned}$$

$$\begin{aligned} g_7 \;=\; & -\int_t^T d\left(x_t, B_v, v; C\left(1,0\right)\right) \exp\left(-A\left(0,1,0\right)\left(v-t\right)\right) \\ & \times n\left(-d\left(x_t, B_v, v; C\left(1,0\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv, \end{aligned}$$

$$g_8 = -\int_t^T d\left(x_t, B_v, v; 0\right) a_{t,v} n\left(d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv,$$

$$
\begin{aligned}
& g_9 \\
= \; & -\left(\beta - r - \frac{1}{2}\theta^2\right) \int_t^T d\left(x_t, B_v, v; 0\right) a_{t,v} n\left(d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x^2} \left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v} \right) dv \\
& -\theta \int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \left(-d^2\left(x_t, B_v, v; 0\right) + 1\right) \frac{1}{\sigma_x^2 \sqrt{v-t}} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t} \right) dv.
\end{aligned}
$$

After retirement, the optimal portfolio is $\pi_t = \sigma^{-1}\theta\phi\left(\frac{y\xi_t}{a_t}\right)^{-1} \frac{1-\exp(-\beta(T-t))}{\beta}$.

Proof. Apply Clark-Ocone formula to the right hand side in equation (3.3) and equate the volatility on both sides. ♦

Before retirement, the optimal portfolio of an individual with log utility is the limit of the optimal portfolio of an individual with power utility when the coefficient of relative risk aversion R approaches to 1, if the optimal retirement boundaries of the state variables are the same, and the corresponding parameters and the initial multipliers of the two individuals are the same. The reason is that under these conditions, the wealth processes are equal when R approaches 1. After retirement, convergence results hold for the optimal portfolio when R approaches 1, as long as the multipliers for the two individuals are the same.

Part II

Asset liability management with optimal liquidation

Chapter 4

Asset liability management for defined benefit pension plans: optimal liquidation and asset allocation policies

4.1 Introduction

For a defined benefit pension fund, of which the asset-liability management involves the choices of dividend-contribution, risk-reward and continuation-liquidation, we investigate the behavior of the optimal policies.

Trillions of dollars of defined benefit pension asset have been accumulated over the years. However, management of these assets is not the comparative advantage of corporations. The pension fund faces a stream of intermediate liability and a terminal liability, which are benefit payments to the pension plan participants. The value and the risk of this liability side of the balance sheet of a pension fund, are crucial for its optimal asset allocation of pension asset. In other words, the whole balance sheet of the pension fund should be taken into consideration for optimal policies. An asset allocation strategy focusing exclusively on the pension asset while neglecting either the intermediate liability or the terminal liability, is bound to be suboptimal. Moreover, every time the economy experiences a severe recession, both sides of the balance sheet of defined benefit pension fund will normally be hit. The fall of stock prices reduces the value of the assets and the fall of interest rates increases the value of the liability. The resulting shortfall in the pension fund severely disrupts the normal business operation of the fund sponsor. Many firms start to switch from defined benefit pension plan to defined contribution plan so as to transfer this risk which is not related to their core business to the employees. In light of these challenges, we study a model of optimal dividend-contribution, portfolio and liquidation from the viewpoint of a defined benefit pension fund.

Relatively limited attention has been devoted to applying the theory of dynamic optimal asset allocation to study asset-liability management, in spite of the large amount of defined benefit pension asset accumulated over the years currently under management. Boulier et al. (1995), Cairns (2000), Rudolf and Ziemba (2004), Detemple and Rindisbacher (2008), Detemple et al. (2010) and Van Binsbergen and Brandt (2012) are examples of recent contributions in the asset-liability management for defined benefit pension fund literature. A key difference in modeling among these papers is in the objective function of the fund sponsor. Boulier et al. (1995) minimize the disutility which is a power function of contributions from the sponsor to the pension fund. Cairns (2000) extends to general loss function and derives optimal policies for power and exponential loss function. Other papers follow the tradition of Merton (1971). Rudolf and Ziemba (2004) study an intertemporal model where utility is defined on surplus of pension assets over liabilities. In Van Binsbergen and Brandt (2012), the sponsor's utility is a power function of the funding ratio at terminal date, penalized by the stream of contributions. Detemple and Rindisbacher (2008) consider a model where preference of the sponsor is defined over intermediate dividends and terminal excess of pension asset over a fraction of liabilities, and this preference formulation is suitable to model the case when there is a funding shortfall from the full liability at the terminal date. Detemple et al. (2010) develop a model where the fund sponsor's utility is derived from dividends or contributions, depending on if outlay from the fund is higher than or lower than the liability. This preference structure has the advantage that optimal dividends and contributions are jointly determined and it also captures the feature that contributions in excess of a threshold that might hurt the normal business of the sponsor are not tolerated. We extend Detemple et al. (2010) to incorporate endogenous optimal liquidation in the asset-liability management model. The optimal liquidation date, or the optimal date to switch from a defined benefit pension plan to a defined contribution pension plan, is modeled as an optimal stopping time chosen by the fund sponsor. The liquidation date of the pension fund is the terminal date of the investment horizon, and the sponsor cares about the lump sum amount of dividend or contribution at this date.

We derive the recursive integral equation of the optimal liquidation boundary for an endogenous variable, which is related to the terminal liability and the state price density. We provide closed-form solutions of the optimal net cash flow (dividends or contributions), the optimal liquid wealth and the optimal portfolio. As in the individuals' life-cycle problem, early exercise premium representation affords interesting interpretations in terms of net local gains from early liquidation or delayed liquidation. Optimal net cash flows are shown to be positive (dividends) when state price density is sufficiently low and be negative (contributions) when state price density is sufficiently high. We also derive the liquidation boundary for the pension assets and identify in the optimal portfolio the hedges against fluctuations in the intermediate liability and in the terminal liability.

4.2 The model

$(\Omega, \mathcal{F}, P)$ is a complete probability space. W_t, $t \in [0, T]$ is a Brownian motion on the probability space. The flow of information $\mathcal{F}_t$, $t \in [0, T]$ is the filtration generated by W_t.

The market consists of a riskless asset and a risky asset. Riskless asset is a money market account with a constant interest rate $r > 0$. Risky asset has instantaneous return dR_t which satisfies $dR_t = \mu dt + \sigma dW_t$. μ is the expected rate of return, σ is the volatility of the return. μ and σ are both constants and positive. The market price of risk is $\theta = (\mu - r)/\sigma$. The state price density process is $\xi_t = \exp\left(-rt - \frac{1}{2}\theta^2 t - \theta W_t\right)$. This structure implies that the Brownian motion risk is hedgeable, thus the market is complete and there are no arbitrage opportunities.

The sponsor faces a stream of intermediate liability and a terminal liability. Intermediate liability of the pension fund at time t is l_t and l_t satisfies $dl_t = l_t\left(\mu_l dt + \sigma_l dW_t\right)$, where μ_l is the expected growth rate of the intermediate liability and σ_l is the volatility of the growth rate. Terminal liability is L_T and L_t satisfies $dL_t = L_t\left(\mu_L dt + \sigma_L dW_t\right)$, where μ_L is the expected growth rate of the terminal liability and σ_L is the volatility of the growth rate. The sponsor's contribution at time t is p_t and terminal contribution is P_T. Dividend paid out from the fund to the sponsor at time t are d_t and terminal dividend are D_T. Liquid wealth in the fund is X_t. c_t is the amount withdrawn from (when positive) or injected in (when negative) the portfolio. Define intermediate net cash flow $f_t \equiv d_t - p_t$ and terminal net cash flow $F_T \equiv D_T - P_T$. Accounting balance ensures that at any time t, $c_t = d_t + l_t - p_t$. If $c_t \geq l_t$, net cash flow $f_t = d_t = c_t - l_t$ and $p_t = 0$, net cash flow is in the form of dividend; if $c_t < l_t$, net cash flow $f_t = -p_t = c_t - l_t$ and $d_t = 0$, net cash flow is in the form of contribution. At terminal time T, $X_T = D_T + L_T - P_T$. If $X_T \geq L_T$, $F_T = D_T = X_T - L_T$ and $P_T = 0$; if $X_T < L_T$, $F_T = -P_T = X_T - L_T$ and $D_T = 0$. Let π_t be the dollar amount invested in the risky asset. Liquid wealth X_t satisfies

$$\begin{aligned} dX_t &= \left(X_t - \pi_t\right) r dt - c_t dt + \pi_t\left(\mu dt + \sigma dW_t\right) \\ &= \left(rX_t - c_t\right) dt + \pi_t \sigma\left(\theta dt + dW_t\right), \end{aligned}$$

starting from $X_0 = x$.

Preference ordering for the sponsor is represented by Von Neumann-Morgenstein expected utility.

$$\mathcal{U} = E\left[\int_0^{\tau} a_v u\left(f_v + gl_v, l_v, v\right) dv + a_\tau U\left(F_\tau + sL_\tau, L_\tau, \tau\right)\right],$$

where

$$u\left(f_v + gl_v, l_v, v\right) = \frac{\left(f_v + gl_v\right)^{1-R}}{1-R}, \;\; U\left(F_\tau + sL_\tau, L_\tau, \tau\right) = \frac{\left(F_\tau + sL_\tau\right)^{1-R}}{1-R}$$

and $a_v = \exp\left(-\beta v\right)$. R is the coefficient of relative risk aversion. β is the subjective discount rate. τ is the liquidation date chosen by the sponsor. $\tau \in \mathcal{S}$, where $\mathcal{S}$ is the collection of stopping times with values in $[0, T]$. Preference of the sponsor is defined over net cash flow (dividends or contributions) depending on whether outlay from the fund is higher than or lower than the liability. The formulations of the intermediate utility and the terminal utility ensure, as derived in the optimal policy below, that the intermediate contribution is limited to threshold gl_v and the terminal contribution is limited to threshold sL_τ. This allows us to model that large contributions to the fund in excess of the threshold which harm the normal business operations are not allowed. Since the intermediate liability is usually much smaller than the terminal liability, we assume that $g \geq 1$ and $s \leq 1$, i.e., the intermediate contributions to the fund can be a multiple of the intermediate liability, but the terminal contribution is limited to only a fraction of the terminal liability.

4.3 Pure optimal stopping problem

The policy is said to be admissible: $(f, F, \pi) \in A$, if no-bankruptcy condition is satisfied, i.e., liquid wealth is nonnegative. Thus (f, F, π) is admissible if $X_t \geq 0$. Static budget constraint is

$$E\left[\int_0^\tau \xi_v\left(f_v+l_v\right)dv+\xi_\tau\left(F_\tau+L_\tau\right)\right]\leq x. \tag{4.1}$$

The optimization problem can be reduced to maximizing $\mathcal{U}$ subject to the static budget constraint.

Lemma 20. *If (f,F,π) is admissible, then (f,F) satisfies the static budget constraint (4.1). If (f,F) satisfies the static budget constraint (4.1), then $\exists\ \pi$, such that (f,F,π) is admissible.*

Proof. See Appendix. ♦

The maximization problem is now

$$V(x)=\sup_{\tau\in\mathcal{S},\ (f,F,\pi)\in A} E\left[\int_0^\tau a_v u\left(f_v+gl_v,l_v,v\right)dv+a_\tau U\left(F_\tau+sL_\tau,L_\tau,\tau\right)\right],$$

subject to

$$E\left[\int_0^\tau \xi_v\left(f_v+l_v\right)dv+\xi_\tau\left(F_\tau+L_\tau\right)\right]\leq x.$$

Recall the Legendre-Fenchel transform of utility function and inequality (2.4). Choose $x_0=f_v$, $y_0=ya_v^{-1}\xi_v$ and $U(x_0)=u\left(f_v+gl_v,l_v,v\right)$ in (2.4), we have

$$u\left(f_v+gl_v,l_v,v\right)\leq\widetilde{u}(ya_v^{-1}\xi_v)+ya_v^{-1}\xi_v f_v,$$

or

$$a_v u\left(f_v+gl_v,l_v,v\right)\leq a_v\widetilde{u}(ya_v^{-1}\xi_v)+y\xi_v f_v.$$

Choose $x_0=F_\tau$, $y_0=ya_\tau^{-1}\xi_\tau$ and $U(x_0)=U\left(F_\tau+sL_\tau,L_\tau,\tau\right)$ in (2.4), we have

$$U\left(F_\tau+sL_\tau,L_\tau,\tau\right)\leq\widetilde{U}(ya_\tau^{-1}\xi_\tau)+ya_\tau^{-1}\xi_\tau F_\tau,$$

or

$$a_\tau U\left(F_\tau + sL_\tau, L_\tau, \tau\right) \leq a_\tau \widetilde{U}(ya_\tau^{-1}\xi_\tau) + y\xi_\tau F_\tau.$$

Therefore,

$$\begin{aligned}
& E\left[\int_0^\tau a_v u\left(f_v + gl_v, l_v, v\right) dv + a_\tau U\left(F_\tau + sL_\tau, L_\tau, \tau\right)\right] \\
\leq\ & E\left[\int_0^\tau a_v \widetilde{u}(ya_v^{-1}\xi_v) dv + a_\tau \widetilde{U}(ya_\tau^{-1}\xi_\tau)\right] \\
& + yE\left[\int_0^\tau \xi_v f_v dv + \int_\tau^T \xi_\tau F_\tau dv\right] \\
=\ & E\left[\int_0^\tau \left(a_v \widetilde{u}(ya_v^{-1}\xi_v) - y\xi_v l_v\right) dv + a_\tau \widetilde{U}(ya_\tau^{-1}\xi_\tau) - y\xi_\tau L_\tau\right] \\
& + yE\left[\int_0^\tau \left(\xi_v f_v + \xi_v l_v\right) dv + \xi_\tau F_\tau + \xi_\tau L_\tau\right] \\
\leq\ & E\left[\int_0^\tau \left(a_v \widetilde{u}(ya_v^{-1}\xi_v) - y\xi_v l_v\right) dv + a_\tau \widetilde{U}(ya_\tau^{-1}\xi_\tau) - y\xi_\tau L_\tau\right] + yx.
\end{aligned}$$

Define

$$\widetilde{J}\left(y;\tau\right) \triangleq E\left[\int_0^\tau \left(a_v \widetilde{u}(ya_v^{-1}\xi_v) - y\xi_v l_v\right) dv + a_\tau \widetilde{U}(ya_\tau^{-1}\xi_\tau) - y\xi_\tau L_\tau\right].$$

Thus we have

$$E\left[\int_0^\tau a_v u\left(f_v + gl_v, l_v, v\right) dv + a_\tau U\left(F_\tau + sL_\tau, L_\tau, \tau\right)\right] \leq \inf_{y>0}\left[\widetilde{J}\left(y;\tau\right) + yx\right],$$

and

$$V\left(x\right) \leq \sup_{\tau\in\mathcal{S}}\inf_{y>0}\left[\widetilde{J}\left(y;\tau\right) + yx\right].$$

Define

$$\begin{aligned}\widetilde{V}(y) &\triangleq \sup_{\tau\in\mathcal{S}}\widetilde{J}(y;\tau)\\ &= \sup_{\tau\in\mathcal{S}}E\left[\int_0^\tau \left(a_v\widetilde{u}(ya_v^{-1}\xi_v)-y\xi_v l_v\right)dv + a_\tau\widetilde{U}(ya_\tau^{-1}\xi_\tau)-y\xi_\tau L_\tau\right].\end{aligned}$$

Therefore, we have

$$V(x)\le \sup_{\tau\in\mathcal{S}}\inf_{y>0}\left[\widetilde{J}(y;\tau)+yx\right]\le \inf_{y>0}\left[\sup_{\tau\in\mathcal{S}}\widetilde{J}(y;\tau)+yx\right]=\inf_{y>0}\left[\widetilde{V}(y)+yx\right].$$

The next proposition uses the same technique as in Chapter 2 to show that the inequality above is actually an equality. Therefore we can first solve the pure optimal stopping problem of $\widetilde{V}(y)$, while treating y as a constant.

Proposition 21.

$$V(x)=\inf_{y>0}\left[\widetilde{V}(y)+yx\right].$$

Proof. We have

$$u(f_v+gl_v,l_v,v)=\frac{(f_v+gl_v)^{1-R}}{1-R},$$

and

$$U(F_\tau+sL_\tau,L_\tau,\tau)=\frac{(F_\tau+sL_\tau)^{1-R}}{1-R}.$$

First order conditions are

$$a_v u_f(f_v+gl_v,l_v,v)=a_v(f_v+gl_v)^{-R}=y\xi_v,$$

$$a_\tau U_F(F_\tau+sL_\tau,L_\tau,\tau)=a_\tau(F_\tau+sL_\tau)^{-R}=y\xi_\tau.$$

Calculation gives the following solutions

$$f_v^*=\left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}}-gl_v,$$

$$F_\tau^* = \left(\frac{y\xi_\tau}{a_\tau}\right)^{-\frac{1}{R}} - sL_\tau.$$

Now we have

$$\begin{aligned} a_v\widetilde{u}(ya_v^{-1}\xi_v) &= a_v u\left(f_v^* + gl_v, l_v, v\right) - y\xi_v f_v^* \\ &= \frac{1}{1-R}a_v\left(\frac{y\xi_v}{a_v}\right)^{1-\frac{1}{R}} - y\xi_v\left(\left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} - gl_v\right) \\ &= \frac{R}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^\rho + y\xi_v gl_v. \end{aligned}$$

$a_v\widetilde{u}(ya_v^{-1}\xi_v)$ is a convex function of y.

$$\begin{aligned} a_\tau\widetilde{U}(ya_\tau^{-1}\xi_\tau) &= a_\tau U\left(F_\tau^* + sL_\tau, L_\tau, \tau\right) - y\xi_\tau F_\tau^* \\ &= \frac{1}{1-R}a_\tau\left(\frac{y\xi_\tau}{a_\tau}\right)^{1-\frac{1}{R}} - y\xi_\tau\left(\left(\frac{y\xi_\tau}{a_\tau}\right)^{-\frac{1}{R}} - sL_\tau\right) \\ &= \frac{R}{1-R}a_\tau^{\frac{1}{R}}\left(y\xi_\tau\right)^\rho + y\xi_\tau sL_\tau. \end{aligned}$$

$a_\tau\widetilde{U}(ya_\tau^{-1}\xi_\tau)$ is a convex function of y.

$$\begin{aligned} \widetilde{J}\left(y;\tau\right) &= E\left[\int_0^\tau \left(a_v\widetilde{u}(ya_v^{-1}\xi_v) - y\xi_v l_v\right)dv + a_\tau\widetilde{U}(ya_\tau^{-1}\xi_\tau) - y\xi_\tau L_\tau\right] \\ &= E\left[\int_0^\tau \left(\frac{R}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^\rho + y\xi_v gl_v - y\xi_v l_v\right)dv\right. \\ &\quad \left. + \frac{R}{1-R}a_\tau^{\frac{1}{R}}\left(y\xi_\tau\right)^\rho + y\xi_\tau sL_\tau - y\xi_\tau L_\tau\right]. \end{aligned}$$

$\widetilde{J}\left(y;\tau\right)$ is also a convex function of y. For any given $\tau \in \mathcal{S}$ and a given y, define

$$x_\tau\left(y\right) \triangleq E\left[\int_0^\tau \xi_v\left(f_v^* + l_v\right)dv + \xi_\tau\left(F_\tau^* + L_\tau\right)\right].$$

Plug in the values of f_v^* and F_τ^*,

$$\begin{aligned} x_\tau(y) &= E\left[\int_0^\tau \xi_v\left(\left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} - gl_v + l_v\right)dv + \xi_\tau\left(\left(\frac{y\xi_\tau}{a_\tau}\right)^{-\frac{1}{R}} - sL_\tau + L_\tau\right)\right] \\ &= E\left[\int_0^\tau \left(a_v^{\frac{1}{R}} y^{-\frac{1}{R}}\xi_v^\rho - \xi_v gl_v + \xi_v l_v\right)dv + a_\tau^{\frac{1}{R}} y^{-\frac{1}{R}}\xi_\tau^\rho - \xi_\tau sL_\tau + \xi_\tau L_\tau\right]. \end{aligned}$$

Denote the stopping time that attains the supremum in $\widetilde{V}(y) = \sup_{\tau\in\mathcal{S}}\widetilde{J}(y;\tau)$ by τ_y^*, i.e., $\widetilde{V}(y) = \widetilde{J}\left(y;\tau_y^*\right)$. We have $\widetilde{V}'(y) = -x_{\tau_y^*}(y)$ and $V(x) = \inf_{y>0}\left[\widetilde{V}(y) + yx\right]$. ♦

4.4 Optimal liquidation boundary

Now the problem is a pure optimal stopping problem

$$\begin{aligned} &\widetilde{V}(y) \\ =\ &\sup_{\tau\in\mathcal{S}} E\left[\int_0^\tau \left(a_v\widetilde{u}(ya_v^{-1}\xi_v) - y\xi_v l_v\right)dv + a_\tau\widetilde{U}(ya_\tau^{-1}\xi_\tau) - y\xi_\tau L_\tau\right] \\ =\ &\sup_{\tau\in\mathcal{S}} E\left[\int_0^\tau \left(a_v u\left(f_v^* + gl_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right)dv\right. \\ &\left. + a_\tau U\left(F_\tau^* + sL_\tau, L_\tau, \tau\right) - y\xi_\tau F_\tau^* - y\xi_\tau L_\tau\right]. \end{aligned}$$

Let

$$\begin{aligned} &D_t \\ \triangleq\ &\int_0^t \left(a_v u\left(f_v^* + gl_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right)dv \\ &+ a_t U\left(F_t^* + sL_t, L_t, t\right) - y\xi_t F_t^* - y\xi_t L_t, \end{aligned}$$

and the Snell Envelope of D_t is

$$
\begin{aligned}
& J_t \\
\triangleq\ & \sup_{\tau \in \mathcal{S}} E_t \left[D_\tau \right] \\
=\ & \sup_{\tau \in \mathcal{S}} E_t \left[\int_0^\tau \left(a_v u \left(f_v^* + g l_v, l_v, v \right) - y \xi_v f_v^* - y \xi_v l_v \right) dv \right. \\
& \left. + a_\tau U \left(F_\tau^* + s L_\tau, L_\tau, \tau \right) - y \xi_\tau F_\tau^* - y \xi_\tau L_\tau \right].
\end{aligned}
$$

J_t has the following representation.

Proposition 22.

$$J_t = J_t^n + J_t^a,$$

where

$$
\begin{aligned}
& J_t^n \\
=\ & E_t \left[\int_0^T \left(a_v u \left(f_v^* + g l_v, l_v, v \right) - y \xi_v f_v^* - y \xi_v l_v \right) dv \right. \\
& \left. + a_T U \left(F_T^* + s L_T, L_T, T \right) - y \xi_T F_T^* - y \xi_T L_T \right],
\end{aligned}
$$

$$
\begin{aligned}
& J_t^a \\
=\ & -E_t \left[\int_t^T \left(a_v u \left(f_v^* + g l_v, l_v, v \right) - y \xi_v f_v^* - y \xi_v l_v \right) 1_{\mathcal{R}(v)} dv \right] \\
& -E_t \left[\int_t^T 1_{\mathcal{R}(v)} d \left(a_v U \left(F_v^* + s L_v, L_v, v \right) - y \xi_v F_v^* - y \xi_v L_v \right) \right].
\end{aligned}
$$

and $R(v)$ is the immediate liquidation region at time v.

Proof. The same proof as in Proposition 3.

$$\begin{aligned} J_t &= E_t\left[J_T\right] - E_t\left[\int_t^T 1_{\{v=\tau_v^*\}} dD_v\right] \\ &= J_t^n + J_t^a, \end{aligned}$$

where $J_t^n = E_t\left[J_T\right]$ and $J_t^a = -E_t\left[\int_t^T 1_{\{v=\tau_v^*\}} dD_v\right]$. ♦

In the early exercise premium representation of J_t, J_t^n is the value of J_t when the optimal liquidation date is the terminal date T and J_t^a is the early liquidation premium. The net local gain from early liquidation is

$$-a_v u\left(f_v^* + gl_v, l_v, v\right) + y\xi_v f_v^* + y\xi_v l_v - d\left(a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v\right).$$

$y\xi_v f_v^*$, $y\xi_v l_v$, $y\xi_v F_v^*$ and $y\xi_v L_v$ are transformations form monetary terms to utility terms by the marginal cost $y\xi_v$. $-a_v u\left(f_v^* + gl_v, l_v, v\right)$ represents instantaneous utility loss upon early liquidation. $y\xi_v f_v^*$ and $y\xi_v l_v$ represent the utility loss from intermediate net cash flow and intermediate liability avoided by the sponsor from early liquidation. $a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v$ is the net utility gain at the liquidation date. $-d\left(a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v\right)$ represents the possible appreciation in the net utility gain forgone or the possible depreciation in the net utility gain avoided by the sponsor upon early liquidation. The pension fund will never be liquidated prior to T if

$$-a_v u\left(f_v^* + gl_v, l_v, v\right) + y\xi_v f_v^* + y\xi_v l_v - d\left(a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v\right) \leq 0,$$

for all $v \in [0, T]$.

Delayed exercise premium representation gives the following proposition that emphasizes the local gains from delaying liquidation.

Proposition 23.

$$J_t = D_t + J_t^d,$$

where D_t is the immediate liquidation value function

$$\begin{aligned} & D_t \\ = & \int_0^t \left(a_v u\left(f_v^* + gl_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) dv \\ & + a_t U\left(F_t^* + sL_t, L_t, t\right) - y\xi_t F_t^* - y\xi_t L_t, \end{aligned}$$

and the delayed liquidition premium is

$$\begin{aligned} J_t^d &= E_t\left[\int_t^T \left(a_v u\left(f_v^* + gl_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) 1_{\{v<\tau_v^*\}} dv\right] \\ & + E_t\left[\int_t^T d\left(a_t U\left(F_t^* + sL_t, L_t, t\right) - y\xi_t F_t^* - y\xi_t L_t\right) 1_{\{v<\tau_v^*\}} dv\right]. \end{aligned}$$

Proof. As in Proposition 4,

$$\begin{aligned} J_t^d &= E_t\left[D_{\tau_t^*} - D_t\right] \\ &= E_t\left[\int_t^T \left(a_v u\left(f_v^* + gl_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) 1_{\{v<\tau_v^*\}} dv\right] \\ & + E_t\left[\int_t^T d\left(a_t U\left(F_t^* + sL_t, L_t, t\right) - y\xi_t F_t^* - y\xi_t L_t\right) 1_{\{v<\tau_v^*\}} dv\right]. \blacklozenge \end{aligned}$$

The net local gains from delaying retirement is

$$a_v u\left(f_v^* + gl_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v + d\left(a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v\right).$$

$a_v u\left(f_v^* + gl_v, l_v, v\right)$ represents instantaneous utility gain from delaying liquidation. $-y\xi_v f_v^*$ and $-y\xi_v l_v$ represent the utility loss from intermediate net cash flow and intermediate liability incurred by the sponsor from delaying liquidation.

$d\left(a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v\right)$ represents the possible appreciation in the net utility gain collected or the possible depreciation in the net utility gain incurred by the sponsor from delaying liquidation.

Now we specify a standing assumption that the immediate liquidation region $\mathcal{R}(t)$ is up connected for the state variable $x_t = a_t^{\frac{1}{R}}\left(y\xi_t\right)^{-\frac{1}{R}} L_t^{-1}$, this assumption is used to calculate the conditional expectations of random variables restricted to the event of liquidation $\{x_t \geq B_t\}$.

Assumption. *The immediate liquidation region $R(t)$ is up connected for the state variable x_t.*

Sufficient conditions to satisfy this assumption are $\sigma_l \leq \theta/R$, $\sigma_L < \theta/R$ *and* $-r + \mu_L - \sigma_L\theta \leq 0$. See Proposition A3 and its proof in Appendix. In the case that the liquidation region is down connected under the conditions $\sigma_l \leq \theta/R$, $\sigma_L > \theta/R$ and $-r + \mu_L - \sigma_L\theta \geq 0$, only a change of the term $d\left(x_t, B_v, v; C\left(\rho, \eta\right)\right)$ to its opposite $-d\left(x_t, B_v, v; C\left(\rho, \eta\right)\right)$ is needed to derive the formulas under this case. See the derivation of the conditional expectations of random variables restricted to $\{x_t \geq B_t\}$ in proof of Lemma 6 in Appendix.

The next theorem gives the backward recursive equation satisfied by the optimal liquidation boundary.

Theorem 24. *The optimal liquidation boundary B_t satisfies the following backward* *recursive equation*

$$\begin{aligned}
&\frac{R}{1-R}B_t G^B\left(t,T;\beta,\rho,1,\mathcal{A}\right) \\
&+\left(g-1\right)l_0 L_0^{\gamma-1}y^{\frac{\gamma}{R}}\exp\left(\delta t\right)B_t^{\gamma}G^B\left(l;t,T;0,1,0,\mathcal{A}\right) \\
&+\frac{R}{1-R}B_t\exp\left(-A\left(\beta,\rho,1\right)\left(T-t\right)\right)+\left(s-1\right)\exp\left(-A\left(L;0,1,0\right)\left(T-t\right)\right) \\
&-\left(\frac{R}{1-R}B_t+s-1\right) \\
&+A\left(\beta,\rho,1\right)\frac{R}{1-R}B_t G^B\left(t,T;\beta,\rho,1,\mathcal{R}\right) \\
&-\left(\mu_L-r-\sigma_L\theta\right)\left(s-1\right)G^B\left(L;t,T;0,1,0,\mathcal{R}\right) \\
=\ &0.
\end{aligned} \tag{4.2}$$

with limiting condition of the boundary B_T *that satisfies*

$$\begin{aligned}
&\left(g-1\right)z_0\left(\frac{B_T}{x_0}\right)^{\gamma}\exp\left(\delta T\right)+\frac{R}{1-R}B_T\left(1-A\left(\beta,\rho,1\right)\right) \\
&+\left(s-1\right)\left(\mu_L-r-\sigma_L\theta\right) \\
=\ &0,
\end{aligned}$$

where

$$G\left(l;t,T;\beta,\rho,\eta\right)\equiv E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}l_{t,v}^{(1-\eta)\rho}dv\right],$$

$$G\left(L;t,T;\beta,\rho,\eta\right)\equiv E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}L_{t,v}^{(1-\eta)\rho}dv\right],$$

and $G^B(\cdot)$ *is the representation of* $G(\cdot)$ *when substituting* x_t *by* B_t*. The coefficients* γ *and* δ *are*

$$\gamma=\frac{\sigma_z}{\sigma_x},\ \ \delta=\mu_z-\frac{1}{2}\sigma_z^2-\gamma\left(\mu_x-\frac{1}{2}\sigma_x^2\right),$$

where

$$\mu_x=-\frac{\beta}{R}+\frac{1}{R}\left(r+\frac{1}{2}\theta^2\right)-\mu_L+\frac{1}{2}\sigma_L^2+\frac{1}{2}\sigma_x^2,$$

$$\sigma_x = \frac{\theta}{R} - \sigma_L,$$

$$\mu_z = \mu_l - \frac{1}{2}\sigma_l^2 - \mu_L + \frac{1}{2}\sigma_L^2 + \frac{1}{2}\sigma_z^2,$$

$$\sigma_z = \sigma_l - \sigma_L.$$

Proof. When $\tau_t^* = t$, immediate liquidation value function is

$$\begin{aligned} D_t &= \int_0^t \left(a_v u\left(f_v^* + g l_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) dv \\ &\quad + a_t U\left(F_t^* + sL_t, L_t, t\right) - y\xi_t F_t^* - y\xi_t L_t. \end{aligned}$$

In the immediate liquidation region $\mathcal{R}(t)$, $J_t = J_t^n + J_t^a = D_t$. Plug in the representations of J_t^n, J_t^a and D_t, we get

$$\begin{aligned} & E_t \left[\int_0^T \left(a_v u\left(f_v^* + g l_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) dv \right. \\ & \left. + a_T U\left(F_T^* + sL_T, L_T, T\right) - y\xi_T F_T^* - y\xi_T L_T \right] \\ & - E_t \left[\int_t^T \left(a_v u\left(f_v^* + g l_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) 1_{\mathcal{R}(v)} dv \right] \\ & - E_t \left[\int_t^T 1_{\mathcal{R}(v)} d\left(a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v\right) \right] \\ = & \int_0^t \left(a_v u\left(f_v^* + g l_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) dv \\ & + a_t U\left(F_t^* + sL_t, L_t, t\right) - y\xi_t F_t^* - y\xi_t L_t. \end{aligned}$$

Simplifying,

$$\begin{aligned}
& E_t\left[\int_t^T \left(a_v u\left(f_v^* + g l_v, l_v, v\right) - y\xi_v f_v^* - y\xi_v l_v\right) 1_{\mathcal{A}(v)} dv\right] \\
& + E_t\left[a_T U\left(F_T^* + sL_T, L_T, T\right) - y\xi_T F_T^* - y\xi_T L_T\right] \\
& - \left(a_t U\left(F_t^* + sL_t, L_t, t\right) - y\xi_t F_t^* - y\xi_t L_t\right) \\
& - E_t\left[\int_t^T 1_{\mathcal{R}(v)} d\left(a_v U\left(F_v^* + sL_v, L_v, v\right) - y\xi_v F_v^* - y\xi_v L_v\right)\right] \\
= \;& 0.
\end{aligned}$$

Plug in the values of f_v^* and F^*,

$$\begin{aligned}
& E_t\left[\int_t^T \left(\frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho + y\xi_v g l_v - y\xi_v l_v\right) 1_{\mathcal{A}(v)} dv\right] \\
& + E_t\left[\frac{R}{1-R} a_T^{\frac{1}{R}} \left(y\xi_T\right)^\rho + y\xi_T s L_T - y\xi_T L_T\right] \\
& - \left(\frac{R}{1-R} a_t^{\frac{1}{R}} \left(y\xi_t\right)^\rho + y\xi_t s L_t - y\xi_t L_t\right) \\
& - E_t\left[\int_t^T 1_{\mathcal{R}(v)} d\left(\frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho + y\xi_v s L_v - y\xi_v L_v\right)\right] \\
= \;& 0.
\end{aligned}$$

We have

$$\begin{aligned}
E_t\left[\int_t^T \frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v\right)^\rho dv\right] &= \frac{R}{1-R} a_t^{\frac{1}{R}} \left(y\xi_t\right)^\rho E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}} \xi_{t,v}^\rho dv\right] \\
&= \frac{R}{1-R} a_t^{\frac{1}{R}} \left(y\xi_t\right)^\rho G\left(t, T; \beta, \rho, 1\right),
\end{aligned}$$

$$E_t\left[\int_t^T (g-1)\,y\xi_v l_v dv\right] = (g-1)\,y\xi_t l_t E_t\left[\int_t^T \xi_{t,v} l_{t,v} dv\right] = (g-1)\,y\xi_t l_t G\left(l;t,T;0,1,0\right),$$

$$\begin{aligned} & E_t\left[\frac{R}{1-R} a_T^{\frac{1}{R}}\left(y\xi_T\right)^\rho + y\xi_T s L_T - y\xi_T L_T\right] \\ = & \frac{R}{1-R} a_t^{\frac{1}{R}}\left(y\xi_t\right)^\rho E_t\left[a_{t,T}^{\frac{1}{R}} \xi_{t,T}^\rho\right] + (s-1)\,y\xi_t L_t E_t\left[\xi_{t,T} L_{t,T}\right] \\ = & \frac{R}{1-R} a_t^{\frac{1}{R}}\left(y\xi_t\right)^\rho \exp\left(-A\left(\beta,\rho,1\right)\left(T-t\right)\right) \\ & +(s-1)\,y\xi_t L_t \exp\left(-A\left(L;0,1,0\right)\left(T-t\right)\right), \end{aligned}$$

$$\begin{aligned} & E_t\left[\int_t^T d\left(\frac{R}{1-R} a_v^{\frac{1}{R}}\left(y\xi_v\right)^\rho + y\xi_v s L_v - y\xi_v L_v\right)\right] \\ = & E_t\left[\int_t^T \left(\frac{R}{1-R} a_v^{\frac{1}{R}}\left(y\xi_v\right)^\rho\left(-A\left(\beta,\rho,1\right)\right) + (s-1)\,y\xi_v L_v\left(\mu_L - r - \sigma_L\theta\right)\right) dv\right] \\ = & -A\left(\beta,\rho,1\right)\frac{R}{1-R} a_t^{\frac{1}{R}}\left(y\xi_t\right)^\rho G\left(t,T;\beta,\rho,1\right) \\ & +\left(\mu_L - r - \sigma_L\theta\right)(s-1)\,y\xi_t L_t G\left(L;t,T;0,1,0\right). \end{aligned}$$

Therefore, in the immediate retirement region $\mathcal{R}(t)$,

$$\begin{aligned}
& \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{\rho} G(t,T;\beta,\rho,1,\mathcal{A}) + (g-1) y\xi_t l_t G(l;t,T;0,1,0,\mathcal{A}) \\
& + \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{\rho} \exp(-A(\beta,\rho,1)(T-t)) \\
& + (s-1) y\xi_t L_t \exp(-A(L;0,1,0)(T-t)) \\
& - \left(\frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{\rho} + y\xi_t s L_t - y\xi_t L_t \right) \\
& + A(\beta,\rho,1) \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{\rho} G(t,T;\beta,\rho,1,\mathcal{R}) \\
& - (\mu_L - r - \sigma_L \theta)(s-1) y\xi_t L_t G(L;t,T;0,1,0,\mathcal{R}) \\
= \; & 0.
\end{aligned}$$

Divided by $y\xi_t L_t$ on both sides, we get

$$\begin{aligned}
& \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{-\frac{1}{R}} L_t^{-1} G(t,T;\beta,\rho,1,\mathcal{A}) + (g-1) l_t L_t^{-1} G(l;t,T;0,1,0,\mathcal{A}) \\
& + \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{-\frac{1}{R}} L_t^{-1} \exp(-A(\beta,\rho,1)(T-t)) \\
& + (s-1) \exp(-A(L;0,1,0)(T-t)) \\
& - \left(\frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{-\frac{1}{R}} L_t^{-1} + s - 1 \right) \\
& + A(\beta,\rho,1) \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{-\frac{1}{R}} L_t^{-1} G(t,T;\beta,\rho,1,\mathcal{R}) \\
& - (\mu_L - r - \sigma_L \theta)(s-1) G(L;t,T;0,1,0,\mathcal{R}) \\
= \; & 0.
\end{aligned}$$

Let $x_t = a_t^{\frac{1}{R}} (y\xi_t)^{-\frac{1}{R}} L_t^{-1}$, $z_t = l_t L_t^{-1}$. We have

$$x_t = y^{-\frac{1}{R}} L_0^{-1} \exp\left(\left(-\frac{\beta}{R} + \frac{1}{R} \left(r + \frac{1}{2}\theta^2 \right) - \mu_L + \frac{1}{2}\sigma_L^2 \right) t + \left(\frac{\theta}{R} - \sigma_L \right) W_t \right),$$

$$z_t = l_0 L_0^{-1} \exp\left(\left(\mu_l - \frac{1}{2}\sigma_l^2 - \mu_L + \frac{1}{2}\sigma_L^2 \right) t + (\sigma_l - \sigma_L) W_t \right).$$

For x_t and z_t,

$$\mu_x = -\frac{\beta}{R} + \frac{1}{R}\left(r + \frac{1}{2}\theta^2\right) - \mu_L + \frac{1}{2}\sigma_L^2 + \frac{1}{2}\sigma_x^2,$$

$$\sigma_x = \frac{\theta}{R} - \sigma_L,$$

$$\mu_z = \mu_l - \frac{1}{2}\sigma_l^2 - \mu_L + \frac{1}{2}\sigma_L^2 + \frac{1}{2}\sigma_z^2,$$

$$\sigma_z = \sigma_l - \sigma_L.$$

z_t is a transform of x_t,

$$z_t = z_0 \left(\frac{x_t}{x_0}\right)^{\gamma} \exp(\delta t),$$

where $\gamma = \frac{\sigma_z}{\sigma_x}$, $\delta = \mu_z - \frac{1}{2}\sigma_z^2 - \gamma\left(\mu_x - \frac{1}{2}\sigma_x^2\right)$. The liquidation boundary B_t satisfies the following backward recursive equation

$$\begin{aligned}
& \frac{R}{1-R} B_t G^B(t, T; \beta, \rho, 1, \mathcal{A}) \\
& + (g-1) l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}} \exp(\delta t) B_t^{\gamma} G^B(l; t, T; 0, 1, 0, \mathcal{A}) \\
& + \frac{R}{1-R} B_t \exp(-A(\beta, \rho, 1)(T-t)) + (s-1)\exp(-A(L; 0, 1, 0)(T-t)) \\
& - \left(\frac{R}{1-R} B_t + s - 1\right) \\
& + A(\beta, \rho, 1) \frac{R}{1-R} B_t G^B(t, T; \beta, \rho, 1, \mathcal{R}) \\
& - (\mu_L - r - \sigma_L \theta)(s-1) G^B(L; t, T; 0, 1, 0, \mathcal{R}) \\
= \; & 0.
\end{aligned}$$

In the early exercise premium representation, the instantaneous gain minus instantaneous loss is,

$$
\begin{aligned}
& -\left(a_v u\left(f_v^*+g l_v, l_v, v\right)-y \xi_v f_v^*-y \xi_v l_v\right) \\
& -\left(\frac{R}{1-R} a_v^{\frac{1}{R}}\left(y \xi_v\right)^\rho(-A(\beta, \rho, 1))+(s-1) y \xi_v L_v\left(\mu_L-r-\sigma_L \theta\right)\right) \\
= & -\left(\frac{R}{1-R} a_v^{\frac{1}{R}}\left(y \xi_v\right)^\rho+y \xi_v g l_v-y \xi_v l_v\right) \\
& -\left(\frac{R}{1-R} a_v^{\frac{1}{R}}\left(y \xi_v\right)^\rho(-A(\beta, \rho, 1))+(s-1) y \xi_v L_v\left(\mu_L-r-\sigma_L \theta\right)\right) .
\end{aligned}
$$

Let it be equal to 0,

$$
\begin{aligned}
& \frac{R}{1-R} a_v^{\frac{1}{R}}\left(y \xi_v\right)^\rho+y \xi_v g l_v-y \xi_v l_v \\
& +\frac{R}{1-R} a_v^{\frac{1}{R}}\left(y \xi_v\right)^\rho(-A(\beta, \rho, 1))+(s-1) y \xi_v L_v\left(\mu_L-r-\sigma_L \theta\right) \\
= & \ 0 .
\end{aligned}
$$

Divided by $y\xi_v L_v$, we get

$$
(g-1) z_v+\frac{R}{1-R} x_v(1-A(\beta, \rho, 1))+(s-1)\left(\mu_L-r-\sigma_L \theta\right)=0 .
$$

Since

$$
z_v=z_0\left(\frac{x_v}{x_0}\right)^\gamma \exp (\delta v),
$$

we have

$$
\begin{aligned}
& (g-1) z_0\left(\frac{x_v}{x_0}\right)^\gamma \exp (\delta v)+\frac{R}{1-R} x_v(1-A(\beta, \rho, 1)) \\
& +(s-1)\left(\mu_L-r-\sigma_L \theta\right) \\
= & \ 0 .
\end{aligned}
$$

Thus the limiting condition for B_T is

$$\begin{aligned} & (g-1)\, z_0 \left(\frac{B_T}{x_0}\right)^{\gamma} \exp\left(\delta T\right) + \frac{R}{1-R} B_T \left(1 - A\left(\beta, \rho, 1\right)\right) \\ & + (s-1)\left(\mu_L - r - \sigma_L \theta\right) \\ = \; & 0. \blacklozenge \end{aligned}$$

Proposition 25. *The derivative of the boundary* $\frac{\partial B_t}{\partial y}$ *satisfies*

$$\begin{aligned} & \frac{R}{1-R}\frac{\partial B_t}{\partial y} G^B\left(t, T; \beta, \rho, 1, \mathcal{A}\right) \\ & + (g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}} \exp\left(\delta t\right) B_t^{\gamma} G^B\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \left(y^{-1}\frac{\gamma}{R} + \gamma \frac{\frac{\partial B_t}{\partial y}}{B_t}\right) \\ & + \frac{R}{1-R}\frac{\partial B_t}{\partial y} \exp\left(-A\left(\beta, \rho, 1\right)\left(T - t\right)\right) - \frac{R}{1-R}\frac{\partial B_t}{\partial y} \\ & + A\left(\beta, \rho, 1\right) \frac{R}{1-R}\frac{\partial B_t}{\partial y} G^B\left(t, T; \beta, \rho, 1, \mathcal{R}\right) + \frac{R}{1-R} B_t \frac{\partial}{\partial y} G^B\left(t, T; \beta, \rho, 1, \mathcal{A}\right) \\ & + (g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}} \exp\left(\delta t\right) B_t^{\gamma} \frac{\partial}{\partial y} G^B\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \\ & + A\left(\beta, \rho, 1\right) \frac{R}{1-R} B_t \frac{\partial}{\partial y} G^B\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\ & - \left(\mu_L - r - \sigma_L \theta\right)(s-1) \frac{\partial}{\partial y} G^B\left(L; t, T; 0, 1, 0, \mathcal{R}\right) \\ = \; & 0, \end{aligned}$$

with boundary condition

$$\frac{\partial B_T}{\partial y} = \frac{-\frac{\gamma}{R}(g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}-1} B_T^{\gamma} \exp\left(\delta T\right)}{\frac{R}{1-R}\left(1 - A\left(\beta, \rho, 1\right)\right) + \gamma (g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}} B_T^{\gamma-1} \exp\left(\delta T\right)}.$$

Proof. See appendix. ♦

We numerically implement our model. Figure 4.1 gives the optimal liquidation boundary of the state variable x_t and its derivative with respect to the multiplier when the initial liquid wealth is 3 with parameter values in Table 4.1. As shown in Proposition

Table 4.1: Parameter values.

Interest rate r	0.03
Volatility of the market return σ	0.2
Market price of risk θ	0.3
Expected growth rate of intermediate liability μ_l	0.02
Volatility of growth rate of intermediate liability σ_l	0.04
Initial value of intermediate liability l_0	1
Maximal intermediate shortfall tolerated g	2
Expected growth rate of terminal liability μ_L	0.02
Volatility of growth rate of terminal liability σ_L	0.06
Initial value of terminal liability σ_L	10
Maximal terminal shortfall tolerated s	0.5
Coefficient of relative risk aversion R	4
Subjective discount rate β	0

A3, when $\sigma_l \leq \theta/R$, $\sigma_L < \theta/R$ and $-r + \mu_L - \sigma_L\theta \leq 0$, the liquidation region is up-connected. The fund is optimal to liquidate the moment that the state variable x_t hits the boundary. Later we will construct the optimal liquidation boundary for the liquid wealth which is more practical, as the fund sponsor can observe the value of pension asset and decide whether to liquidate by comparing with its boundary. The right hand side in Figure 4.1 gives the derivative of the boundary with respect to the multiplier. As will be seen in the derivations later, this derivative is needed for computing the optimal liquid wealth and the optimal portfolio.

4.5 Optimal pension assets

We are ready to derive the closed form solution of the liquid wealth X_t.

Theorem 26. *We have the following representations of wealth processes. Liquid wealth*

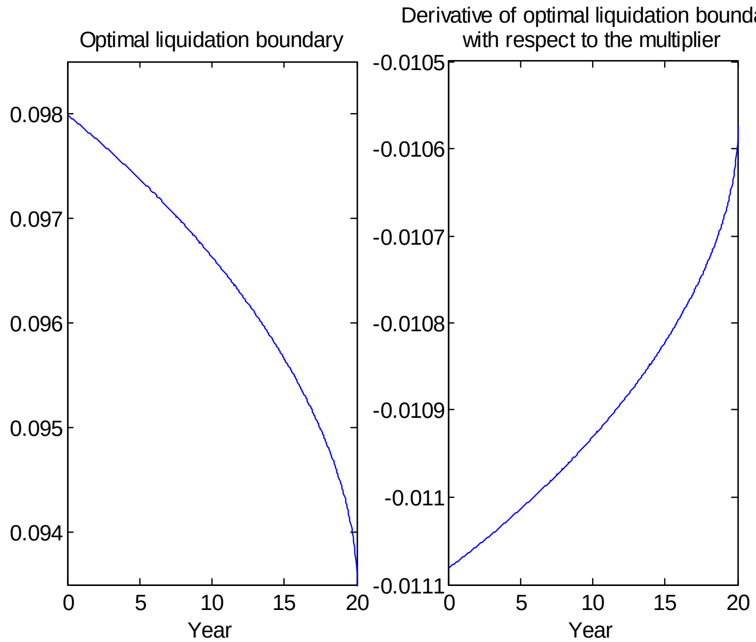

Figure 4·1: This figure shows the optimal liquidation boundary of the state variable x_t and its derivative with respect to the multiplier y. Initial liquid wealth is 3.

X_t satisfies

$$\begin{aligned}
X_t &= \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v \left(f_v^* + l_v \right) dv + \xi_{\tau_t^*} \left(F_{\tau_t^*}^* + L_{\tau_t^*} \right) \right] \\
&= a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{A}\right) - (g-1)\, l_t G\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \qquad (4.3) \\
&\quad + a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} \exp\left(-A\left(\beta, \rho, 1\right)(T-t)\right) \\
&\quad - (s-1)\, L_t \exp\left(-A\left(L; 0, 1, 0\right)(T-t)\right) \\
&\quad + A\left(\beta, \rho, 1\right) a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
&\quad + \left(\mu_L - r - \sigma_L \theta\right)(s-1)\, L_t G\left(L; t, T; 0, 1, 0, \mathcal{R}\right) \\
&\quad - \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{A}\right) - (g-1)\, y l_t \frac{\partial}{\partial y} G\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \\
&\quad - A\left(\beta, \rho, 1\right) \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
&\quad + \left(\mu_L - r - \sigma_L \theta\right)(s-1)\, y L_t \frac{\partial}{\partial y} G\left(L; t, T; 0, 1, 0, \mathcal{R}\right).
\end{aligned}$$

Proof. We have

$$\begin{aligned} & J_t \\ = \ & \sup_{\tau \in \mathcal{S}} E_t \left[\int_0^{\tau} \left(\frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v \right)^{\rho} + y\xi_v g l_v - y\xi_v l_v \right) dv \right. \\ & \left. + \frac{R}{1-R} a_{\tau}^{\frac{1}{R}} \left(y\xi_{\tau} \right)^{\rho} + y\xi_{\tau} s L_{\tau} - y\xi_{\tau} L_{\tau} \right]. \end{aligned}$$

Denote

$$L_t \equiv J_t - \left[\int_0^{t} \left(\frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v \right)^{\rho} + y\xi_v g l_v - y\xi_v l_v \right) dv \right],$$

thus

$$\begin{aligned} & L_t \\ = \ & \sup_{\tau \in \mathcal{S}} E_t \left[\int_t^{\tau} \left(\frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v \right)^{\rho} + y\xi_v g l_v - y\xi_v l_v \right) dv \right. \\ & \left. + \frac{R}{1-R} a_{\tau}^{\frac{1}{R}} \left(y\xi_{\tau} \right)^{\rho} + y\xi_{\tau} s L_{\tau} - y\xi_{\tau} L_{\tau} \right] \\ = \ & E_t \left[\int_t^{\tau_t^*} \left(\frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v \right)^{\rho} + y\xi_v g l_v - y\xi_v l_v \right) dv \right. \\ & \left. + \frac{R}{1-R} a_{\tau_t^*}^{\frac{1}{R}} \left(y\xi_{\tau_t^*} \right)^{\rho} + y\xi_{\tau_t^*} s L_{\tau_t^*} - y\xi_{\tau_t^*} L_{\tau_t^*} \right], \end{aligned}$$

where τ_t^* is the optimal stopping time. Liquid wealth X_t satisfies

$$\xi_t X_t = E_t \left[\int_t^{\tau^*} \xi_v \left(f_v^* + l_v \right) dv + \xi_{\tau_t^*} \left(F_{\tau_t^*}^* + L_{\tau_t^*} \right) \right].$$

Plug in the values of f_v^* and F_τ^*,

$$
\begin{aligned}
&\xi_t X_t \\
= \;& E_t\left[\int_t^{\tau_t^*}\left(a_v^{\frac{1}{R}} y^{-\frac{1}{R}}\xi_v^{\rho} - \xi_v g l_v + \xi_v l_v\right) dv + a_{\tau_t^*}^{\frac{1}{R}} y^{-\frac{1}{R}}\xi_{\tau_t^*}^{\rho} - \xi_{\tau_t^*} s L_{\tau_t^*} + \xi_{\tau_t^*} L_{\tau_t^*}\right].
\end{aligned}
$$

L_t is a convex function of y and the derivative of L_t with respect to the multiplier y satisfies

$$
\begin{aligned}
&\frac{\partial L_t}{\partial y} \\
= \;& E_t\left[\int_t^{\tau_t^*}\left(-a_v^{\frac{1}{R}} y^{-\frac{1}{R}}\xi_v^{\rho} + \xi_v g l_v - \xi_v l_v\right) dv - a_{\tau_t^*}^{\frac{1}{R}} y^{-\frac{1}{R}}\xi_{\tau_t^*}^{\rho} + \xi_{\tau_t^*} s L_{\tau_t^*} - \xi_{\tau_t^*} L_{\tau_t^*}\right] \\
= \;& -\xi_t X_t.
\end{aligned}
$$

Use Early Exercise Premium representation to write L_t as the following

$$
\begin{aligned}
& L_t \\
= \;& E_t\left[\int_t^{\tau_t^*}\left(\frac{R}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}+y\xi_v g l_v-y\xi_v l_v\right)dv\right. \\
& \left.+\frac{R}{1-R}a_{\tau_t^*}^{\frac{1}{R}}\left(y\xi_{\tau_t^*}\right)^{\rho}+y\xi_{\tau_t^*}sL_{\tau_t^*}-y\xi_{\tau_t^*}L_{\tau_t^*}\right] \\
= \;& E_t\left[\int_t^{T}\left(\frac{R}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}+y\xi_v g l_v-y\xi_v l_v\right)dv\right. \\
& \left.+\frac{R}{1-R}a_T^{\frac{1}{R}}\left(y\xi_T\right)^{\rho}+y\xi_T sL_T-y\xi_T L_T\right] \\
& -E_t\left[\int_t^{T}\left(\frac{R}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}+y\xi_v g l_v-y\xi_v l_v\right)1_{\mathcal{R}(v)}dv\right] \\
& -E_t\left[\int_t^{T}1_{\mathcal{R}(v)}d\left(\frac{R}{1-R}a_v^{\frac{1}{R}}\left(y\xi_v\right)^{\rho}+y\xi_v sL_v-y\xi_v L_v\right)\right] \\
= \;& \frac{R}{1-R}a_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}G\left(t,T;\beta,\rho,1,\mathcal{A}\right)+\left(g-1\right)y\xi_t l_t G\left(l;t,T;0,1,0,\mathcal{A}\right) \\
& +\frac{R}{1-R}a_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}\exp\left(-A\left(\beta,\rho,1\right)\left(T-t\right)\right) \\
& +\left(s-1\right)y\xi_t L_t\exp\left(-A\left(L;0,1,0\right)\left(T-t\right)\right) \\
& +A\left(\beta,\rho,1\right)\frac{R}{1-R}a_t^{\frac{1}{R}}\left(y\xi_t\right)^{\rho}G\left(t,T;\beta,\rho,1,\mathcal{R}\right) \\
& -\left(\mu_L-r-\sigma_L\theta\right)\left(s-1\right)y\xi_t L_t G\left(L;t,T;0,1,0,\mathcal{R}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& X_t \\
= \; & \xi_t^{-1}\left(-\frac{\partial L_t}{\partial y}\right) \\
= \; & a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{A}\right) - (g-1)\, l_t G\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \\
& + a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} \exp\left(-A\left(\beta, \rho, 1\right)\left(T-t\right)\right) \\
& - (s-1)\, L_t \exp\left(-A\left(L; 0, 1, 0\right)\left(T-t\right)\right) \\
& + A\left(\beta, \rho, 1\right) a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
& + \left(\mu_L - r - \sigma_L \theta\right)(s-1)\, L_t G\left(L; t, T; 0, 1, 0, \mathcal{R}\right) \\
& - \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{A}\right) - (g-1)\, y l_t \frac{\partial}{\partial y} G\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \\
& - A\left(\beta, \rho, 1\right) \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
& + \left(\mu_L - r - \sigma_L \theta\right)(s-1)\, y L_t \frac{\partial}{\partial y} G\left(L; t, T; 0, 1, 0, \mathcal{R}\right). \; \blacklozenge
\end{aligned}
$$

For numerical implementation of our model, we simulate 20 realizations per year from year 0 to year 20, in total 400 data points. Figure 4.2 shows the simulation of the stock market index, the terminal liability and the intermediate liability for a particular trajectory of underlying Brownian motion with model parameters in Table 4.1. For this particular trajectory, the stock market gradually decreases in the first 4 years, increases sharply between year 4 and 6, then slowly decreases until year 14, finally between year 14 and year 20 experiences a tremendous boom. The terminal liability and the intermediate liability are both assumed to be positively related to the stock market, thus they display similar patterns. Figure 4.3 shows the trajectory of the state variable x_t with its optimal liquidation boundary (shown previously in Figure 4.1). The volatility of the state variable x_t is $\theta/R - \sigma_L$. The state variable is positively related to the stock market as $\theta/R - \sigma_L$ is positive for parameter values in Table 4.1. The evolution

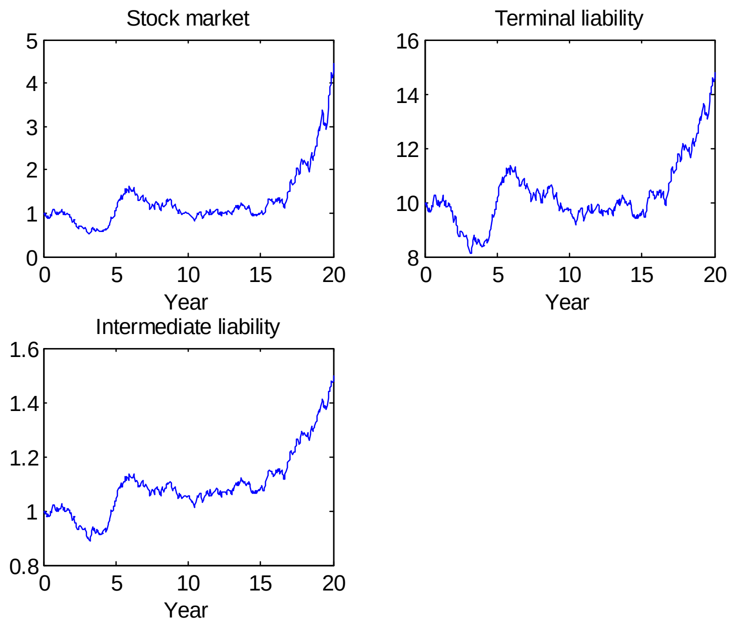

Figure 4·2: This figure shows a trajectory of the stock market index, the terminal liability and the intermediate liability from year 0 to year 20.

of the state variable x_t follows a similar pattern as the stock market index. During the first sharp increase of x_t between year 4 and 6, it moves closer to its optimal liquidation boundary but it's not sufficiently close to hit the boundary. The fund continues to operate and experiences a gradual decline of the stock market until at about year 14, then the state variable x_t increases sharply with the stock market boom and crosses the boundary at about year 19, i.e., the fund under this particular trajectory is optimal to liquidate at year 19.

The next theorem gives the optimal liquidation boundary for the liquid wealth. Immediate liquidation is optimal when the liquid wealth crosses this boundary.

Theorem 27. *If $R > 1$ and $-r + \mu_L - \sigma_L \theta \leq 0$, then liquidation is optimal when the liquid wealth crosses its boundary, i.e.,*

$$X_t \geq y^{\frac{\gamma_L}{R}} L_0^{1+\gamma_L} \exp(\delta_L t) (B_t - s + 1) B_t^{\gamma_L} \tag{4.4}$$

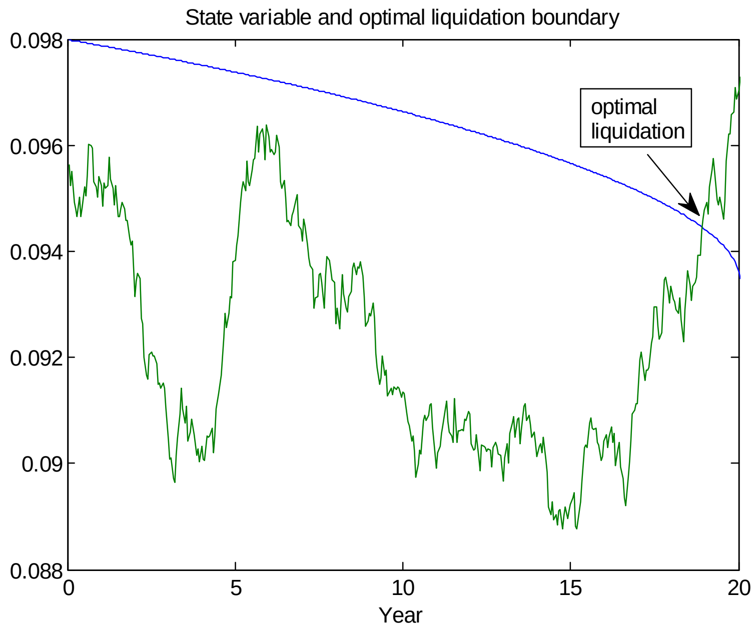

Figure 4·3: This figure shows that the state variable crosses the optimal liquidation boundary at about year 19.

where $\gamma_L = \frac{\sigma_L}{\sigma_x}$, $\delta_L = \mu_L - \frac{1}{2}\sigma_L^2 - \gamma_L\left(\mu_x - \frac{1}{2}\sigma_x^2\right)$ *and* B_t *is the boundary for the state variable* x_t*.*

Proof. The liquid wealth before liquidation is

$$X_t = \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \xi_v \left(f_v^* + l_v\right) dv + \xi_{\tau_t^*}\left(F_{\tau_t^*}^* + L_{\tau_t^*}\right) \right]$$

where τ_t^* is the optimal liquidation time. Plug in the values of f_v^* and F_τ^*,

$$\begin{aligned} & X_t \\ = \; & \xi_t^{-1} E_t \left[\int_t^{\tau_t^*} \left(a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} - \xi_v g l_v + \xi_v l_v \right) dv + a_{\tau_t^*}^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_{\tau_t^*}^{\rho} - \xi_{\tau_t^*} s L_{\tau_t^*} + \xi_{\tau_t^*} L_{\tau_t^*} \right]. \end{aligned}$$

The liquid wealth at liquidation is

$$\begin{aligned} X_t &= F_t^* + L_t \\ &= \left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} - sL_t + L_t \\ &= L_t\left(\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} L_t^{-1} - s + 1\right). \end{aligned}$$

We now show that the boundary of the term $L_t\left(\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} L_t^{-1} - s + 1\right)$ is exactly the liquidation boundary of the liquid wealth. It suffices to show that the liquid wealth before liquidation is less than or equal to the value of $L_t\left(\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} L_t^{-1} - s + 1\right)$. This is to show, at time t before retirement,

$$\begin{aligned} X_{t,b} &\equiv \xi_t^{-1} E_t\left[\int_t^{\tau_t^*} \left(a_v^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_v^{\rho} - \xi_v g l_v + \xi_v l_v\right) dv \right. \\ &\quad \left. + a_{\tau_t^*}^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_{\tau_t^*}^{\rho} - \xi_{\tau_t^*} s L_{\tau_t^*} + \xi_{\tau_t^*} L_{\tau_t^*}\right] \\ &\leq X_{t,a} \equiv L_t\left(\left(\frac{y\xi_t}{a_t}\right)^{-\frac{1}{R}} L_t^{-1} - s + 1\right). \end{aligned}$$

Optimal value function is

$$\begin{aligned} L_t &= E_t\left[\int_t^{\tau_t^*} \left(\frac{R}{1-R} a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} + y\xi_v g l_v - y\xi_v l_v\right) dv \right. \\ &\quad \left. + \frac{R}{1-R} a_{\tau_t^*}^{\frac{1}{R}} \left(y\xi_{\tau_t^*}\right)^{\rho} + y\xi_{\tau_t^*} s L_{\tau_t^*} - y\xi_{\tau_t^*} L_{\tau_t^*}\right]. \end{aligned}$$

Thus we have

$$\frac{R}{1-R} y\xi_t X_{t,b} + \frac{1}{1-R} y E_t\left[\int_t^{\tau_t^*} \left(\xi_v g l_v - \xi_v l_v\right) dv + \xi_{\tau_t^*} s L_{\tau_t^*} - \xi_{\tau_t^*} L_{\tau_t^*}\right] = L_t,$$

and

$$L_t \geq \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^{\rho} + y\xi_t s L_t - y\xi_t L_t = \frac{R}{1-R} y\xi_t X_{t,a} + \frac{1}{1-R} (y\xi_t s L_t - y\xi_t L_t).$$

Therefore

$$\begin{aligned} & \frac{R}{1-R} y\xi_t X_{t,b} \\ & + \frac{1}{1-R} y E_t \left[\int_t^{\tau_t^*} (\xi_v g l_v - \xi_v l_v)\, dv + \xi_{\tau_t^*} s L_{\tau_t^*} - \xi_{\tau_t^*} L_{\tau_t^*} - (\xi_t s L_t - \xi_t L_t) \right] \\ \geq \; & \frac{R}{1-R} y\xi_t X_{t,a}. \end{aligned}$$

Since $g \geq 1$, $\xi_v g l_v - \xi_v l_v \geq 0$. Since $s \leq 1$, if $-r + \mu_L - \sigma_L \theta \leq 0$, we have

$$\begin{aligned} & E_t \left[\xi_{\tau_t^*} s L_{\tau_t^*} - \xi_{\tau_t^*} L_{\tau_t^*} - (\xi_t s L_t - \xi_t L_t) \right] \\ = \; & (s-1) E_t \left[\xi_{\tau_t^*} L_{\tau_t^*} - \xi_t L_t \right] \\ \geq \; & 0. \end{aligned}$$

The reason is that the sufficient condition for $E_t \left[\xi_{\tau_t^*} L_{\tau_t^*} \right] \leq \sup_{\tau \in \mathcal{S}} E_t [\xi_\tau L_\tau] = \xi_t L_t$ is that the integrand in the delayed exercise premium is nonpositive for any time between 0 and T, i.e., $-r + \mu_L - \sigma_L \theta \leq 0$. Now we have

$$E_t \left[\int_t^{\tau_t^*} (\xi_v g l_v - \xi_v l_v)\, dv + \xi_{\tau_t^*} s L_{\tau_t^*} - \xi_{\tau_t^*} L_{\tau_t^*} - (\xi_t s L_t - \xi_t L_t) \right] \geq 0.$$

For $R > 1$, we get $X_{t,b} \leq X_{t,a}$. To compute the form of the boundary of liquid wealth,

$$\begin{aligned} X_{t,a} &= L_t \left(\left(\frac{y\xi_t}{a_t} \right)^{-\frac{1}{R}} L_t^{-1} - s + 1 \right) \\ &= L_t (x_t - s + 1). \end{aligned}$$

L_t is a transform of x_t,

$$L_t = L_0 \left(\frac{x_t}{x_0}\right)^{\gamma_L} \exp\left(\delta_L t\right),$$

where $\gamma_L = \frac{\sigma_L}{\sigma_x}$, $\delta_L = \mu_L - \frac{1}{2}\sigma_L^2 - \gamma_L\left(\mu_x - \frac{1}{2}\sigma_x^2\right)$, $x_0 = y^{-\frac{1}{R}} L_0^{-1}$ and

$$\mu_x = -\frac{\beta}{R} + \frac{1}{R}\left(r + \frac{1}{2}\theta^2\right) - \mu_L + \frac{1}{2}\sigma_L^2 + \frac{1}{2}\sigma_x^2,$$

$$\sigma_x = \frac{\theta}{R} - \sigma_L.$$

Therefore the optimal liquidation boundary of the liquid wealth is

$$y^{\frac{\gamma_L}{R}} L_0^{1+\gamma_L} \exp\left(\delta_L t\right)\left(B_t - s + 1\right) B_t^{\gamma_L}. \ \blacklozenge$$

Figure 4.4 shows the optimal liquidation boundary computed from formula (4.4) and the optimal liquid wealth for the underlying Brownian motion. It shows that the liquid wealth crosses its optimal retirement boundary at about age 19, the same time as the state variable x_t crosses B_t.

4.6 Optimal net cash flow

The optimal intermediate and terminal net cash flow f_v^* and F_τ^* are as follows,

$$f_v^* = \left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}} - gl_v,$$

$$F_\tau^* = \left(\frac{y\xi_\tau}{a_\tau}\right)^{-\frac{1}{R}} - sL_\tau,$$

where the multiplier y satisfies initial budget constraint, i.e., equation (4.3) at time 0.

The optimal net cash flows are derived from the first order conditions to equate marginal benefits and marginal costs. f_v^* and F_τ^* are both decreasing with respect to the state price density ξ. When SPD is sufficiently high, net cash flow is negative and

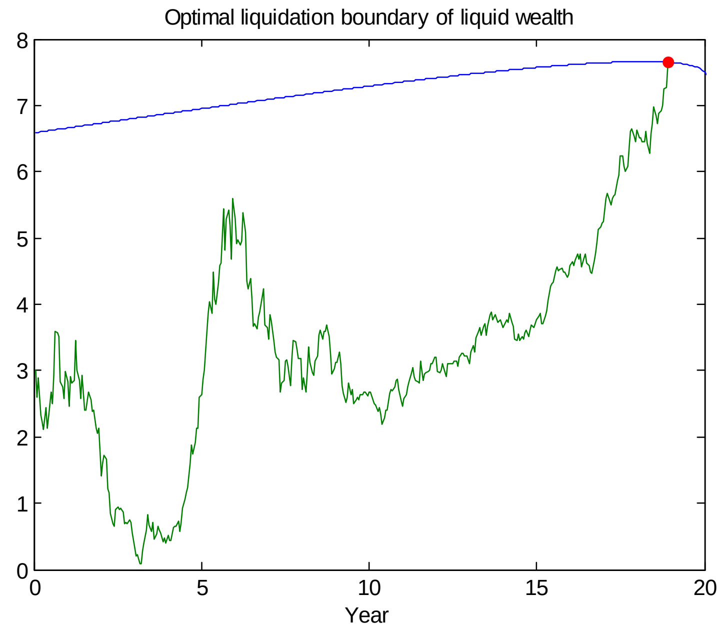

Figure 4·4: This figure shows that the liquid wealth crosses its optimal liquidation boundary at about year 19.

it's optimal to inject funds to the plan. When SPD is sufficiently low, net cash flow is positive and it's optimal to withdraw funds from the plan as dividends. Note that the minimum of intermediate net cash flow f_v^* is $-gl_v$, and the minimum of terminal net cash flow F_τ^* is $-sL_\tau$. Therefore, the intermediate contribution is limited to threshold gl_v and the terminal contribution is limited to threshold sL_τ. This ensures that large contributions to the fund in excess of the threshold which harm the normal business operations are not allowed.

Intermediate net cash flow and terminal net cash flow are both increasing with respect to the initial wealth. Since the initial wealth is a decreasing function of the multiplier y. When initial wealth is higher, the multiplier y becomes smaller, which results in higher intermediate net cash flow and terminal net cash flow.

Figure 4.5 shows the optimal intermediate and terminal net cash flow. The pension fund with initial liquid wealth of 3 is severely underfunded for the parameters of the

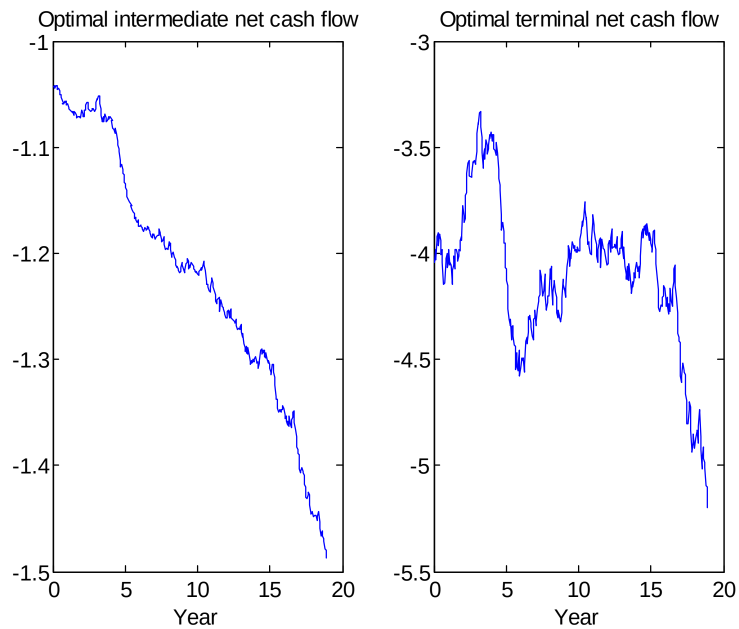

Figure 4·5: This figure shows the optimal intermediate and terminal net cash flow from year 0 to the optimal liquidation date year 19.

intermediate and terminal liability in Table 4.1. As can be seen from Figure 4.5, for this particular trajectory of underlying Brownian motion, both the intermediate net cash flow and the terminal net cash flow are negative, requiring the fund sponsor to inject contribution to the fund. The first component $\left(\frac{y\xi_v}{a_v}\right)^{-\frac{1}{R}}$ in the representation of the optimal intermediate net cash flow is negatively related to the state price density, thus is positively related to the stock market. The second component gl_v is positively related to the intermediate liability, thus is positively related to the stock market. The difference between these two components slightly widens as time increases, which results in the optimal intermediate contribution to gradually increase from 1.05 to 1.48 until liquidation. The magnitude of the second component sL_τ in the optimal terminal net cash flow dominates the magnitude of the first component $\left(\frac{y\xi_\tau}{a_\tau}\right)^{-\frac{1}{R}}$, thus the evolution of F_τ^* follows a similar pattern as $-sL_\tau$. The terminal contribution increases as the stock market increases, and reaches about 5.2 at the optimal liquidation date.

Next proposition compares the optimal net cash flow for the pension fund that has the option to liquidate with the optimal net cash flow for the pension fund that has no option to liquidate (operate until time T).

Proposition 28. *If* $x > E\left[\int_0^T \xi_v \left(f_v^* + l_v\right) dv + \xi_T \left(F_T^* + L_T\right)\right]$, *then for the same initial liquid wealth, the optimal net cash flow for the pension fund that has the option to liquidate is lower than the optimal net cash flow when the pension fund has no option to liquidate. If* $x \leq E\left[\int_0^T \xi_v \left(f_v^* + l_v\right) dv + \xi_T \left(F_T^* + L_T\right)\right]$, *then for the same initial liquid wealth, the optimal net cash flow for the pension fund that has the option to liquidate is higher than or equal to the optimal net cash flow when the pension fund has no option to liquidate.*

Proof. We just need to compare the values of the multiplier y for the two problems. If $x > E\left[\int_0^T \xi_v \left(f_v^* + l_v\right) dv + \xi_T \left(F_T^* + L_T\right)\right]$, then for the same initial liquid wealth, the multiplier for the pension fund with the option to liquidate is higher than the multiplier for the pension fund with no option to liquidate, therefore the optimal net cash flow for the pension fund that has the option to liquidate is lower than the optimal net cash flow when the pension fund has no option to liquidate. And vice versa. ♦

4.7 Optimal portfolio

Optimal portfolio π_t is derived in the next Theorem. π_t can be decomposed to three parts $\pi_t = \pi_{1t} + \pi_{2t} + \pi_{3t}$, where π_{1t} is the mean-variance component, π_{2t} is the hedge against fluctuations in intermediate liability and π_{3t} is the hedge against fluctuations in terminal liability. When the intermediate liability process l_t is deterministic, i.e., $\sigma_l = 0$, the hedging demand π_{2t} vanishes. When the terminal liability process L_t is deterministic, i.e., $\sigma_L = 0$, the hedging demand π_{3t} vanishes.

Theorem 29. *The optimal portfolio π_t satisfies*

$$
\begin{aligned}
& \pi_t \sigma \\
= \; & \frac{\theta}{R} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{A}\right) - \sigma_l \left(g-1\right) l_t G\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \\
& + \frac{\theta}{R} a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} \exp\left(-A\left(\beta, \rho, 1\right)\left(T-t\right)\right) \\
& - \sigma_L \left(s-1\right) L_t \exp\left(-A\left(L; 0, 1, 0\right)\left(T-t\right)\right) \\
& + \frac{\theta}{R} A\left(\beta, \rho, 1\right) a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
& + \sigma_L \left(\mu_L - r - \sigma_L \theta\right)\left(s-1\right) L_t G\left(L; t, T; 0, 1, 0, \mathcal{R}\right) \\
& - \frac{\theta}{R} \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{A}\right) - \sigma_l \left(g-1\right) y l_t \frac{\partial}{\partial y} G\left(l; t, T; 0, 1, 0, \mathcal{A}\right) \\
& - \frac{\theta}{R} A\left(\beta, \rho, 1\right) \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G\left(t, T; \beta, \rho, 1, \mathcal{R}\right) \\
& + \sigma_L \left(\mu_L - r - \sigma_L \theta\right)\left(s-1\right) y L_t \frac{\partial}{\partial y} G\left(L; t, T; 0, 1, 0, \mathcal{R}\right) \\
& + \left(\frac{\theta}{R} - \sigma_L\right) \times \Big[a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} h_1 - \left(g-1\right) l_t h_2 + A\left(\beta, \rho, 1\right) a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} h_3 \\
& + \left(\mu_L - r - \sigma_L \theta\right)\left(s-1\right) L_t h_4 - \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} h_5 - \left(g-1\right) y l_t h_6 \\
& - A\left(\beta, \rho, 1\right) \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} h_7 + \left(\mu_L - r - \sigma_L \theta\right)\left(s-1\right) y L_t h_8 \Big].
\end{aligned}
$$

$\pi_t = \pi_{1t} + \pi_{2t} + \pi_{3t}$, *where*

$$\pi_{1t} = \frac{1}{R}\sigma^{-1}\theta$$
$$\times \Big[a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G(t,T;\beta,\rho,1,\mathcal{A}) + a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} \exp\left(-A(\beta,\rho,1)(T-t)\right)$$
$$+A(\beta,\rho,1)\, a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} G(t,T;\beta,\rho,1,\mathcal{R}) - \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G(t,T;\beta,\rho,1,\mathcal{A})$$
$$-A(\beta,\rho,1)\frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} \frac{\partial}{\partial y} G(t,T;\beta,\rho,1,\mathcal{R})$$
$$+a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} h_1 - (g-1)\, l_t h_2 + A(\beta,\rho,1)\, a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} h_3$$
$$+\left(\mu_L - r - \sigma_L\theta\right)(s-1)\, L_t h_4 - \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} h_5 - (g-1)\, y l_t h_6$$
$$-A(\beta,\rho,1)\frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} h_7 + \left(\mu_L - r - \sigma_L\theta\right)(s-1)\, y L_t h_8\Big],$$

$$\pi_{2t} = \sigma^{-1}\sigma_l \times \left[-(g-1)\, l_t G(l;t,T;0,1,0,\mathcal{A}) - (g-1)\, y l_t \frac{\partial}{\partial y} G(l;t,T;0,1,0,\mathcal{A})\right],$$

$$\pi_{3t} = \sigma^{-1}\sigma_L$$
$$\times \Big\{-(s-1)\, L_t \exp\left(-A(L;0,1,0)(T-t)\right)$$
$$+\left(\mu_L - r - \sigma_L\theta\right)(s-1)\, L_t G(L;t,T;0,1,0,\mathcal{R})$$
$$+\left(\mu_L - r - \sigma_L\theta\right)(s-1)\, y L_t \frac{\partial}{\partial y} G(L;t,T;0,1,0,\mathcal{R})$$
$$-\Big[a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} h_1 - (g-1)\, l_t h_2 + A(\beta,\rho,1)\, a_t^{\frac{1}{R}} y^{-\frac{1}{R}} \xi_t^{-\frac{1}{R}} h_3$$
$$+\left(\mu_L - r - \sigma_L\theta\right)(s-1)\, L_t h_4 - \frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} h_5 - (g-1)\, y l_t h_6$$
$$-A(\beta,\rho,1)\frac{R}{1-R} a_t^{\frac{1}{R}} y^{\rho} \xi_t^{-\frac{1}{R}} h_7 + \left(\mu_L - r - \sigma_L\theta\right)(s-1)\, y L_t h_8\Big]\Big\},$$

and

$$h_1 = -\int_t^T \exp\left(-A(\beta,\rho,1)(v-t)\right) n\left(-d\left(x_t, B_v, v; C(\rho,1)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$h_2 = -\int_t^T \exp\left(-A\left(l;0,1,0\right)\left(v-t\right)\right) n\left(-d\left(l;x_t,B_v,v;C\left(l;1,0\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$h_3 = \int_t^T \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right) n\left(d\left(x_t,B_v,v;C\left(\rho,1\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$h_4 = \int_t^T \exp\left(-A\left(L;0,1,0\right)\left(v-t\right)\right) n\left(d\left(L;x_t,B_v,v;C\left(L;1,0\right)\right)\right) \frac{1}{\sigma_x\sqrt{v-t}} dv,$$

$$\begin{aligned} h_5 &= -\int_t^T d\left(x_t,B_v,v;C\left(\rho,1\right)\right) \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right) \\ &\quad n\left(-d\left(x_t,B_v,v;C\left(\rho,1\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv, \end{aligned}$$

$$\begin{aligned} h_6 &= -\int_t^T d\left(l;x_t,B_v,v;C\left(l;1,0\right)\right) \exp\left(-A\left(l;0,1,0\right)\left(v-t\right)\right) \\ &\quad n\left(-d\left(l;x_t,B_v,v;C\left(l;1,0\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv, \end{aligned}$$

$$\begin{aligned} h_7 &= -\int_t^T d\left(x_t,B_v,v;C\left(\rho,1\right)\right) \exp\left(-A\left(\beta,\rho,1\right)\left(v-t\right)\right) \\ &\quad n\left(d\left(x_t,B_v,v;C\left(\rho,1\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv, \end{aligned}$$

$$\begin{aligned} h_8 &= -\int_t^T d\left(L;x_t,B_v,v;C\left(L;1,0\right)\right) \exp\left(-A\left(L;0,1,0\right)\left(v-t\right)\right) \\ &\quad n\left(d\left(L;x_t,B_v,v;C\left(L;1,0\right)\right)\right) \frac{1}{\sigma_x^2\left(v-t\right)} \left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv. \end{aligned}$$

Proof. Apply Clark-Ocone formula to the representation of X_t in equation (4.3). ♦

Figure 4.6 shows the optimal equity holdings and cash holdings. For this particular trajectory, we observe that the fund is long the market all the time and initially borrows cash to invest in equity. Equity holdings reach the maximum at about year 6 when the stock market reaches a local peak. Then the equity holdings begin to decrease and the cash holdings increase. Figure 4.7 shows the components of the optimal portfolio. The mean variance component dominates the hedges against fluctuations in intermediate liability and in terminal liability, thus the pattern of the mean variance component is very similar to that of the equity holdings in Figure 4.6. The mean variance component reaches its maximum at about year 6, when the stock market reaches a local peak. We observe that the mean variance component is positive while the liability hedges are negative or near 0. The magnitude of the liability hedges follows similar patterns as the magnitude of the liabilities in Figure 4.2 at the beginning when the equity holdings are high. When equity holdings begin to decrease, the magnitude of both liability hedges decreases. At the optimal liquidation date, equity holdings, cash holdings and terminal contribution are used to pay the terminal liability.

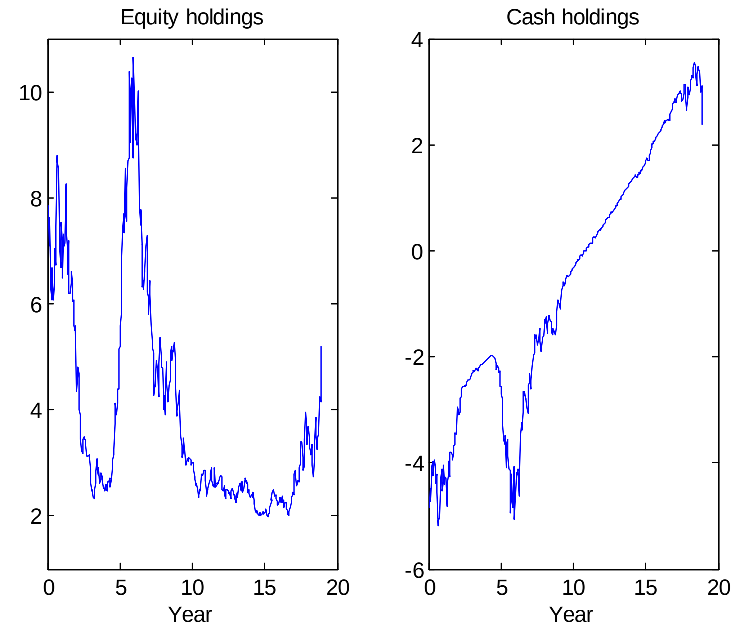

Figure 4·6: This figure shows the amounts of optimal equity holdings and cash holdings before liquidation.

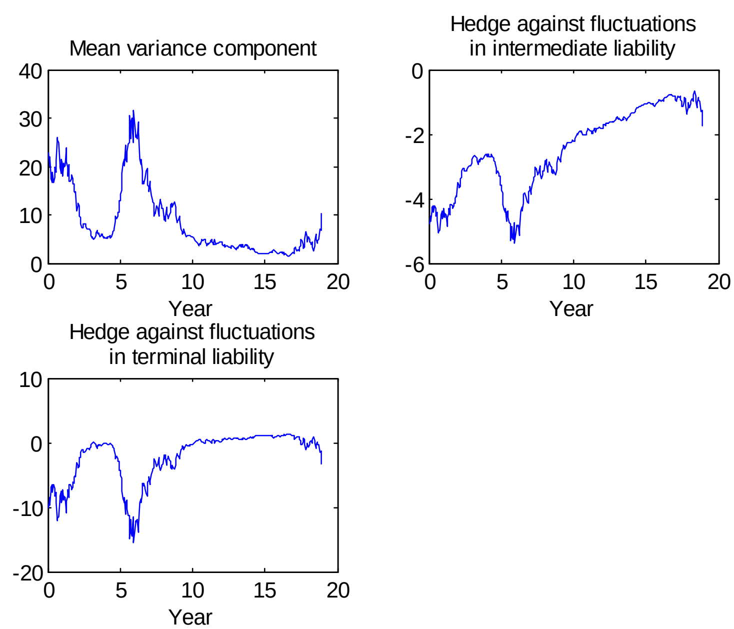

Figure 4·7: This figure shows the portfolio components.

Chapter 5

Conclusion

In this dissertation, we have developed a model of optimal consumption, labor and portfolio choice with endogenous retirement for an individual's life-cycle decisions. Exact solutions of the optimal policies are derived both for an individual with power utility and for an individual with log utility in terms of an optimal retirement boundary. We have also developed a model of optimal dividend-contribution, portfolio and liquidation from the viewpoint of a defined benefit pension fund and the optimal policies depends on an optimal liquidation boundary. In our model the preference is in CRRA class, and the utility is intertemporally additive and independent. An extension of our model is to consider the model with more general preference structures, for example preference that exhibits habit formation. In the life-cycle model, we have assumed that the market is complete and the human capital of an individual can be monetized upfront. It would be interesting to investigate the solutions of the case where the individual faces borrowing constraints thus cannot have negative liquid wealth.

Chapter 6

Appendix

Proof of Lemma 1: Before the retirement, the evolution of the liquid wealth X_v is

$$dX_v = (rX_v - c_v + w_v - w_v l_v)\, dv + \pi_v \sigma\, (\theta dv + dW_v)\,.$$

State price density ξ_v satisfies

$$d\xi_v = -\xi_v\, (rdv + \theta dW_v)\,.$$

Thus,

$$\begin{aligned} d\,(\xi_v X_v) &= \xi_v dX_v + X_v d\xi_v + d\,[\xi, X]_v \\ &= \xi_v\,(rX_v - c_v + w_v - w_v l_v)\, dv + \xi_v \pi_v \sigma\, (\theta dv + dW_v) \\ &\quad -X_v \xi_v\, (rdv + \theta dW_v) - \pi_v \sigma \xi_v \theta dv. \end{aligned}$$

Simplifying,

$$d\,(\xi_v X_v) + \xi_v\,(c_v - w_v + w_v l_v)\, dv = \xi_v\,(\pi_v \sigma - X_v \theta)\, dW_v.$$

Thus we have

$$\xi_t X_t + \int_0^t \xi_v\,(c_v - w_v + w_v l_v)\, dv = x + \int_0^t \xi_v\,(\pi_v \sigma - X_v \theta)\, dW_v. \tag{6.1}$$

If (c, l, π) is admissible, then total wealth is nonnegative. The right hand side in (6.1) is a local martingale and the left hand side is bounded below by a martingale $-E_t\left[\int_0^\tau \xi_v w_v dv\right]$, thus the right hand side is a supermartingale. Use optional stopping theorem, for stopping time $\tau \in \mathcal{S}$,

$$E\left[\xi_\tau X_\tau + \int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv\right] \leq x.$$

After retirement, the evolution of the liquid wealth X_v is

$$\begin{aligned} dX_v &= \left(X_v - \pi_v\right) r dv - c_v dv + \pi_v \left(\mu dv + \sigma dW_v\right) \\ &= \left(r X_v - c_v\right) dv + \pi_v \sigma \left(\theta dv + dW_v\right). \end{aligned}$$

Thus,

$$d\left(\xi_v X_v\right) + \xi_v c_v dv = \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v.$$

Combine with the dynamics of $\xi_t X_v$ before retirement,

$$d\left(\xi_v X_v\right) + \xi_v \left(c_v - w_v + w_v l_v\right) dv = \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v,$$

we get, for $t > \tau$,

$$\begin{aligned} & \xi_t X_t + \int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^t \xi_v c_v dv \\ = \; & x + \int_0^\tau \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v + \int_\tau^t \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v. \end{aligned} \tag{6.2}$$

The right hand side in (6.2) as a function of t is a local martingale and the left hand side is bounded below by a martingale $-E_t\left[\int_0^\tau \xi_v w_v dv\right]$, so the right hand side is a supermartingale. Use optional stopping theorem,

$$\begin{aligned} & E\left[\xi_T X_T + \int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv\right] \\ \leq\ & E\left[\xi_\tau X_\tau + \int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv\right] \\ \leq\ & x. \end{aligned}$$

Thus we have the static constraint

$$E\left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv\right] \leq x.$$

Next we show that if the static budget constraint (2.2) is satisfied, then $\exists\ \pi$, such that (c, l, π) is admissible. Before retirement, use martingale representation theorem,

$$\begin{aligned} & E_t\left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv\right] \\ & -E\left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv\right] \\ =\ & \int_0^t \phi_v dW_v, \text{ for some } \phi_v,\ E\left[\int_0^T \phi_v^2 dv\right] < \infty, \text{ a.s.} \end{aligned}$$

Choose π_v, so that $\xi_v \left(\pi_v \sigma - X_v \theta\right) = \phi_v$. Because

$$d\left(\xi_v X_v\right) + \xi_v \left(c_v - w_v + w_v l_v\right) dv = \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v,$$

we have

$$
\begin{aligned}
& \xi_t X_t + \int_0^t \xi_v \left(c_v - w_v + w_v l_v\right) dv \\
= \;& x + \int_0^t \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v \\
= \;& x + E_t \left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv \right] \\
& - E \left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv \right] \\
\geq \;& E_t \left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv \right].
\end{aligned}
$$

Thus

$$
\xi_t X_t \geq E_t \left[\int_t^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv \right],
$$

or

$$
\xi_t X_t + E_t \left[\int_t^\tau \xi_v w_v \right] \geq E_t \left[\int_t^\tau \xi_v \left(c_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv \right] \geq 0,
$$

i.e., total wealth before retirement is nonnegative. After retirement, use martingale representation theorem,

$$
\begin{aligned}
& E_t \left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv \right] \\
& - E \left[\int_0^\tau \xi_v \left(c_v - w_v + w_v l_v\right) dv + \int_\tau^T \xi_v c_v dv \right] \\
= \;& \int_0^t \phi_v dW_v, \text{ for some } \phi_v,\ E \left[\int_0^T \phi_v^2 dv \right] < \infty, \text{ a.s.}
\end{aligned}
$$

Choose π_v both before retirement and after retirement, so that $\xi_v(\pi_v\sigma - X_v\theta) = \phi_v$. Because before retirement,

$$d(\xi_v X_v) + \xi_v(c_v - w_v + w_v l_v)\,dv = \xi_v(\pi_v\sigma - X_v\theta)\,dW_v,$$

and after retirement

$$d(\xi_v X_v) + \xi_v c_v dv = \xi_v(\pi_v\sigma - X_v\theta)\,dW_v,$$

we have

$$\begin{aligned}
& \xi_t X_t + \int_0^\tau \xi_v(c_v - w_v + w_v l_v)\,dv + \int_\tau^t \xi_v c_v dv \\
= \; & x + \int_0^\tau \xi_v(\pi_v\sigma - X_v\theta)\,dW_v + \int_\tau^t \xi_v(\pi_v\sigma - X_v\theta)\,dW_v \\
= \; & x + E_t\left[\int_0^\tau \xi_v(c_v - w_v + w_v l_v)\,dv + \int_\tau^T \xi_v c_v dv\right] \\
& -E\left[\int_0^\tau \xi_v(c_v - w_v + w_v l_v)\,dv + \int_\tau^T \xi_v c_v dv\right] \\
\geq \; & E_t\left[\int_0^\tau \xi_v(c_v - w_v + w_v l_v)\,dv + \int_\tau^T \xi_v c_v dv\right].
\end{aligned}$$

Thus,

$$\xi_t X_t \geq E_t\left[\int_t^T \xi_v c_v dv\right] \geq 0,$$

i.e. the total wealth after retirement (the liquid wealth) is nonnegative. ♦

In Chapter 1, for $x_t = a_t^{\frac{1}{R}}(y\xi_t)^{-\frac{1}{R}} w_t^{(1-\eta)\rho-1}$, we show that the retirement region is up connected for the state variable x_t.

Proposition A1. *If $R > 1$ and $\sigma_w \leq \theta/R$, then the retirement region is up connected* *for the state variable x_t, i.e., if (x_t, t) is in the retirement region, then $(\lambda x_t, t)$ is also in the retirement region, $\forall \lambda \geq 1$.*

Proof of Proposition A1: At time t, when x_t changes to λx_t, since both ξ_t and w_t depend on the same Brownian motion, we assume ξ_t changes to $p\xi_t$, w_t changes to qw_t. Therefore, $\lambda = p^{-\frac{1}{R}} q^{(1-\eta)\rho-1}$. If (x_t, t) is in the retirement region, then $\tau_t^* = t$,

$$\begin{aligned} J_t &= E_t\left[D_{\tau_t^*}\right] = E_t\left[D_t\right] \\ &= \int_0^t \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + \int_0^t y\xi_v w_v dv \\ &\quad + E_t\left[\int_t^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} dv\right], \end{aligned}$$

and

$$\begin{aligned} & \sup_{\tau \in \mathcal{S}} E_t \left[\int_t^\tau \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^\tau y\xi_v w_v dv \right. \\ & \left. + \int_\tau^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} dv\right] \\ = \; & E_t\left[\int_t^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} dv\right]. \end{aligned}$$

Now for $(\lambda x_t, t)$, we have

$$\begin{aligned} J_t\left(\lambda x_t\right) &= \int_0^t \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + \int_0^t y\xi_v w_v dv \\ &\quad + \sup_{\tau \in \mathcal{S}} E_t \left[\int_t^\tau p^{\rho} q^{(1-\eta)\rho} \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^\tau pqy\xi_v w_v dv \right. \\ &\quad \left. + \int_\tau^T p^{\rho} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v\right)^{\rho} dv\right]. \end{aligned}$$

We want to show that

$$\begin{aligned}
&\sup_{\tau\in\mathcal{S}} E_t\left[\int_t^{\tau} p^{\rho} q^{(1-\eta)\rho} \frac{R}{1-R} f a_v^{\frac{1}{R}} (y\xi_v)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^{\tau} pqy\xi_v w_v dv \right. \\
&\left. + \int_{\tau}^{T} p^{\rho} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} (y\xi_v)^{\rho} dv \right] \\
&\overset{\tau=t}{=} p^{\rho} E_t\left[\int_t^{T} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} (y\xi_v)^{\rho} dv\right],
\end{aligned}$$

i.e., $(\lambda x_t, t)$ is also in the retirement region. We have

$$\begin{aligned}
&\sup_{\tau\in\mathcal{S}} E_t\left[\int_t^{\tau} p^{\rho} q^{(1-\eta)\rho} \frac{R}{1-R} f a_v^{\frac{1}{R}} (y\xi_v)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^{\tau} pqy\xi_v w_v dv \right. \\
&\left. + \int_{\tau}^{T} p^{\rho} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} (y\xi_v)^{\rho} dv \right] \\
= \;& p^{\rho} \sup_{\tau\in\mathcal{S}} E_t\left[\int_t^{\tau} q^{(1-\eta)\rho} \frac{R}{1-R} f a_v^{\frac{1}{R}} (y\xi_v)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^{\tau} p^{1-\rho} qy\xi_v w_v dv \right. \\
&\left. + \int_{\tau}^{T} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} (y\xi_v)^{\rho} dv \right] \\
\leq \;& p^{\rho} \sup_{\tau\in\mathcal{S}} E_t\left[\int_t^{\tau} \frac{R}{1-R} f a_v^{\frac{1}{R}} (y\xi_v)^{\rho} w_v^{(1-\eta)\rho} dv + \int_t^{\tau} y\xi_v w_v dv \right. \\
&\left. + \int_{\tau}^{T} \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} (y\xi_v)^{\rho} dv \right] \\
&+ p^{\rho} \sup_{\tau\in\mathcal{S}} E_t\left[\int_t^{\tau} \left(q^{(1-\eta)\rho} - 1\right) \frac{R}{1-R} f a_v^{\frac{1}{R}} (y\xi_v)^{\rho} w_v^{(1-\eta)\rho} dv \right. \\
&\left. + \int_t^{\tau} \left(p^{1-\rho} q - 1\right) y\xi_v w_v dv \right]
\end{aligned}$$

$$
\begin{aligned}
= \quad & p^{\rho} E_t \left[\int_t^T \frac{R}{1-R} \phi^{\frac{1}{R}} a_v^{\frac{1}{R}} \left(y\xi_v \right)^{\rho} dv \right] \\
& + p^{\rho} \sup_{\tau \in \mathcal{S}} E_t \left[\int_t^{\tau} \left(q^{(1-\eta)\rho} - 1 \right) \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v \right)^{\rho} w_v^{(1-\eta)\rho} dv \right. \\
& \left. + \int_t^{\tau} \left(p^{1-\rho} q - 1 \right) y\xi_v w_v dv \right].
\end{aligned}
$$

Sufficient condition for immediate retirement is

$$
\left(q^{(1-\eta)\rho} - 1 \right) \frac{R}{1-R} f a_v^{\frac{1}{R}} \left(y\xi_v \right)^{\rho} w_v^{(1-\eta)\rho} \leq 0,
$$

and

$$
\left(p^{1-\rho} q - 1 \right) y\xi_v w_v \leq 0
$$

First we consider the case that $R < 1$, then we need $q^{(1-\eta)\rho} - 1 \leq 0$ and $p^{1-\rho}q - 1 \leq 0$. Therefore $q \leq 1$, and $p^{1-\rho}q \leq 1$. We have $\lambda = p^{-\frac{1}{R}} q^{(1-\eta)\rho - 1} = q^{(1-\eta)\rho} p^{-\frac{1}{R}} q^{-1} = q^{(1-\eta)\rho} \left(p^{1-\rho} q \right)^{-1}$, thus for the case that $R < 1$, whether λ is higher than or lower than 1 is ambiguous. Now we consider the empirically relevant case that $R > 1$, then we need $q^{(1-\eta)\rho} - 1 \geq 0$ and $p^{1-\rho}q - 1 \leq 0$. Therefore $q \geq 1$, and $p^{1-\rho}q \leq 1$. We have $\lambda = q^{(1-\eta)\rho} \left(p^{1-\rho} q \right)^{-1} \geq 1$, i.e., the retirement region is up connected. The assumption that $p^{1-\rho}q \leq 1$ is equivalent to $\sigma_w \leq \theta / R$. $q \geq 1$ indicates that the wage process is positively related to the state variable x_t. ♦

Proof of Lemma 6: Calculate

$$
\begin{aligned}
& a_{t,v}^{\frac{1}{R}} \xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho} \\
= \; & \exp\left(-\frac{1}{R}\beta\left(v-t\right)\right)\exp\left(-\rho\left(r+\frac{1}{2}\theta^{2}\right)\left(v-t\right)-\rho\theta\left(W_{v}-W_{t}\right)\right) \\
& \times\exp\left(\left(1-\eta\right)\rho\left(\mu_{w}-\frac{1}{2}\sigma_{w}^{2}\right)\left(v-t\right)+\left(1-\eta\right)\rho\sigma_{w}\left(W_{v}-W_{t}\right)\right) \\
= \; & \exp\left(-\left(\frac{\beta}{R}+\rho\left(r+\frac{1}{2}\theta^{2}\right)-\left(1-\eta\right)\rho\left(\mu_{w}-\frac{1}{2}\sigma_{w}^{2}\right)\right)\left(v-t\right)\right) \\
& \times\exp\left(-\rho\left(\theta-\left(1-\eta\right)\sigma_{w}\right)\left(W_{v}-W_{t}\right)\right),
\end{aligned}
$$

then we have

$$
\begin{aligned}
& E_{t}\left[a_{t,v}^{\frac{1}{R}} \xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho}\right] \\
= \; & E_{t}\left[\exp\left(-\left(\frac{\beta}{R}+\rho\left(r+\frac{1}{2}\theta^{2}\right)-\left(1-\eta\right)\rho\left(\mu_{w}-\frac{1}{2}\sigma_{w}^{2}\right)\right)\left(v-t\right)\right)\right. \\
& \left.\times\exp\left(-\rho\left(\theta-\left(1-\eta\right)\sigma_{w}\right)\left(W_{v}-W_{t}\right)\right)\right] \\
= \; & \int_{-\infty}^{\infty}\left[\exp\left(-\left(\frac{\beta}{R}+\rho\left(r+\frac{1}{2}\theta^{2}\right)-\left(1-\eta\right)\rho\left(\mu_{w}-\frac{1}{2}\sigma_{w}^{2}\right)\right)\left(v-t\right)\right)\right. \\
& \left.\times\exp\left(-\rho\left(\theta-\left(1-\eta\right)\sigma_{w}\right)\sqrt{v-t}z\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^{2}\right)\right]dz \\
= \; & \int_{-\infty}^{\infty}\left[\exp\left(-\left(\frac{\beta}{R}+\rho\left(r+\frac{1}{2}\theta^{2}\right)-\left(1-\eta\right)\rho\left(\mu_{w}-\frac{1}{2}\sigma_{w}^{2}\right)\right)\left(v-t\right)\right)\right. \\
& \times\exp\left(\frac{1}{2}\rho^{2}\left(\theta-\left(1-\eta\right)\sigma_{w}\right)^{2}\left(v-t\right)\right) \\
& \left.\times\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(z+\rho\left(\theta-\left(1-\eta\right)\sigma_{w}\right)\sqrt{v-t}\right)^{2}\right)\right]dz \\
= \; & \int_{-\infty}^{\infty}\exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(z+C\left(\rho,\eta\right)\sqrt{v-t}\right)^{2}\right)dz \\
= \; & \exp\left(-A\left(\beta,\rho,\eta\right)\left(v-t\right)\right),
\end{aligned}
$$

and

$$\begin{aligned} G(t,T;\beta,\rho,\eta) &= E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}dv\right] \\ &= \int_t^T \exp\left(-A(\beta,\rho,\eta)(v-t)\right)dv, \end{aligned}$$

where

$$A(\beta,\rho,\eta)=\frac{\beta}{R}+\rho\left(r+\frac{1}{2}\theta^2\right)-(1-\eta)\rho\left(\mu_w-\frac{1}{2}\sigma_w^2\right)-\frac{1}{2}C(\rho,\eta)^2,$$

and

$$C(\rho,\eta)=\rho\left(\theta-(1-\eta)\sigma_w\right).$$

Also,

$$\begin{aligned} & E_t\left[a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}1_{\mathcal{R}(v)}\right] \\ = \; & E_t\left[a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}\right]E_t\left[\frac{a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}}{E_t\left[a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}\right]}1_{\mathcal{R}(v)}\right] \\ = \; & \exp\left(-A(\beta,\rho,\eta)(v-t)\right)E_t^{C(\rho,\eta)}\left[1_{\mathcal{R}(v)}\right], \end{aligned}$$

where $E_t^{C(\rho,\eta)}[\cdot]$ is the expectation under the measure

$$\frac{dQ_0^{C(\rho,\eta)}}{dP}=\exp\left(-\frac{1}{2}C(\rho,\eta)^2v-C(\rho,\eta)W_v\right).$$

Under this measure, $W_v^{C(\rho,\eta)} = W_v + C(\rho,\eta)\, v$ is a Brownian Motion.

$$\begin{aligned} x_v &\geq B_v \Leftrightarrow x_t \exp\left(\left(\mu_x - \frac{1}{2}\sigma_x^2\right)(v-t) + \sigma_x (W_v - W_t)\right) \geq B_v \\ &\Leftrightarrow \frac{1}{\sqrt{v-t}}\left[(W_v - W_t) + C(\rho,\eta)(v-t)\right] \\ &\geq \frac{1}{\sigma_x\sqrt{v-t}}\left[\log\left(\frac{B_v}{x_t}\right) - \left(\mu_x - \frac{1}{2}\sigma_x^2 - \sigma_x C(\rho,\eta)\right)(v-t)\right] \\ &\Leftrightarrow \frac{W_v^{C(\rho,\eta)} - W_t^{C(\rho,\eta)}}{\sqrt{v-t}} = \widehat{W} \geq -d\left(x_t, B_v, v; C(\rho,\eta)\right), \end{aligned}$$

where $\widehat{W}$ is standard normal distribution under the new measure and

$$d\left(x_t, B_v, v; C(\rho,\eta)\right) = \frac{1}{\sigma_x\sqrt{v-t}}\left[\log\left(\frac{x_t}{B_v}\right) + \left(\mu_x - \frac{1}{2}\sigma_x^2 - \sigma_x C(\rho,\eta)\right)(v-t)\right].$$

Thus we have

$$E_t^{C(\rho,\eta)}\left[1_{\mathcal{R}(v)}\right] = 1 - N\left(-d\left(x_t, B_v, v; C(\rho,\eta)\right)\right) = N\left(d\left(x_t, B_v, v; C(\rho,\eta)\right)\right),$$

$$\begin{aligned} & E_t\left[a_{t,v}^{\frac{1}{R}} \xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho} 1_{\mathcal{R}(v)}\right] \\ = \; & \exp\left(-A(\beta,\rho,\eta)(v-t)\right) N\left(d\left(x_t, B_v, v; C(\rho,\eta)\right)\right), \end{aligned}$$

$$\begin{aligned} G(t,T;\beta,\rho,\eta,\mathcal{R}) &= E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}} \xi_{t,v}^{\rho} w_{t,v}^{(1-\eta)\rho} 1_{\mathcal{R}(v)} dv\right] \\ &= \int_t^T \exp\left(-A(\beta,\rho,\eta)(v-t)\right) N\left(d\left(x_t, B_v, v; C(\rho,\eta)\right)\right) dv, \end{aligned}$$

$$G(t,T;\beta,\rho,\eta,\mathcal{A}) = E_t\left[\int_t^T a_{t,v}^{\frac{1}{R}}\xi_{t,v}^{\rho}w_{t,v}^{(1-\eta)\rho}1_{\mathcal{A}(v)}dv\right]$$
$$= \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)(v-t)\right)N\left(-d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)dv,$$

$$\frac{\partial}{\partial y}G(t,T;\beta,\rho,\eta,\mathcal{R})$$
$$= \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)(v-t)\right)\frac{\partial N\left(d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)}{\partial\left(d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)}$$
$$\times\frac{\partial\left(d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)}{\partial y}dv$$
$$= \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)(v-t)\right)n\left(d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)$$
$$\times\frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial x_t}{\partial y}}{x_t}-\frac{\frac{\partial B_v}{\partial y}}{B_v}\right)dv,$$

and

$$\frac{\partial}{\partial y}G(t,T;\beta,\rho,\eta,\mathcal{A})$$
$$= \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)(v-t)\right)\frac{\partial N\left(-d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)}{\partial\left(-d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)}$$
$$\times\frac{\partial\left(-d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)}{\partial y}dv$$
$$= \int_t^T \exp\left(-A\left(\beta,\rho,\eta\right)(v-t)\right)n\left(-d\left(x_t,B_v,v;C\left(\rho,\eta\right)\right)\right)$$
$$\times\frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v}-\frac{\frac{\partial x_t}{\partial y}}{x_t}\right)dv. \blacklozenge$$

Proof of Proposition 8: Recall that

$$G(t,T;\beta,\rho,\eta,\mathcal{A})=\int_t^T \exp\left(-A(\beta,\rho,\eta)(v-t)\right)N\left(-d\left(x_t,B_v,v;C(\rho,\eta)\right)\right)dv.$$

Denote

$$G^B(t,T;\beta,\rho,\eta,\mathcal{A})=\int_t^T \exp\left(-A(\beta,\rho,\eta)(v-t)\right)N\left(-d\left(B_t,B_v,v;C(\rho,\eta)\right)\right)dv,$$

i.e., substituting x_t using B_t in $G(t,T;\beta,\rho,\eta,\mathcal{A})$. The boundary B_t satisfies

$$\begin{aligned}
&\frac{R}{1-R}fB_tG^B(t,T;\beta,\rho,\eta,\mathcal{A})+G^B(t,T;0,1,0,\mathcal{A})\\
&-\frac{R}{1-R}\phi^{\frac{1}{R}}y^{-\frac{1}{R}(1-\gamma)}w_0^{-1-\gamma((1-\eta)\rho-1)}\exp(\delta t)B_t^{\gamma}G^B(t,T;\beta,\rho,1,\mathcal{A})\\
=\;&0.
\end{aligned}$$

Take derivative with respect to y, we get

$$\begin{aligned}
&\frac{R}{1-R}f\frac{\partial B_t}{\partial y}G^B(t,T;\beta,\rho,\eta,\mathcal{A})\\
&-\frac{R}{1-R}\phi^{\frac{1}{R}}y^{-\frac{1}{R}(1-\gamma)}w_0^{-1-\gamma((1-\eta)\rho-1)}\exp(\delta t)B_t^{\gamma}\\
&\times G^B(t,T;\beta,\rho,1,\mathcal{A})\left(y^{-1}\left(-\frac{1}{R}(1-\gamma)\right)+\gamma\frac{\frac{\partial B_t}{\partial y}}{B_t}\right)\\
&+\frac{R}{1-R}fB_t\frac{\partial}{\partial y}G^B(t,T;\beta,\rho,\eta,\mathcal{A})+\frac{\partial}{\partial y}G^B(t,T;0,1,0,\mathcal{A})\\
&-\frac{R}{1-R}\phi^{\frac{1}{R}}y^{-\frac{1}{R}(1-\gamma)}w_0^{-1-\gamma((1-\eta)\rho-1)}\exp(\delta t)B_t^{\gamma}\frac{\partial}{\partial y}G^B(t,T;\beta,\rho,1,\mathcal{A})\\
=\;&0.
\end{aligned}\tag{6.3}$$

To derive the boundary condition of $\frac{\partial B_T}{\partial y}$, use the limiting condition for boundary B_T,

$$fB_T-\phi^{\frac{1}{R}}z_0\left(\frac{B_T}{x_0}\right)^{\gamma}\exp(\delta T)=1-\frac{1}{R}$$

or

$$fB_T - \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta T) B_T^{\gamma} = 1 - \frac{1}{R}.$$

Take derivative with respect to y, we get

$$\begin{aligned} & f\frac{\partial B_T}{\partial y} - \phi^{\frac{1}{R}}\left(-\frac{1}{R}(1-\gamma)\right) y^{-\frac{1}{R}(1-\gamma)-1} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta T) B_T^{\gamma} \\ & -\phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta T)\gamma B_T^{\gamma-1}\frac{\partial B_T}{\partial y} \\ = & \ 0. \end{aligned}$$

$\frac{\partial B_T}{\partial y}$ can be solved as a function of B_T,

$$\frac{\partial B_T}{\partial y} = \frac{\phi^{\frac{1}{R}}\left(-\frac{1}{R}(1-\gamma)\right) y^{-\frac{1}{R}(1-\gamma)-1} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta T) B_T^{\gamma}}{f - \phi^{\frac{1}{R}} y^{-\frac{1}{R}(1-\gamma)} w_0^{-1-\gamma((1-\eta)\rho-1)} \exp(\delta T)\gamma B_T^{\gamma-1}}. \ \blacklozenge$$

In Chapter 2, let $x_t = \left(\frac{y\xi_t}{a_t}\right)^{-1} w_t^{-1}$. The next proposition shows that the retirement region for state variable x_t is up connected.

Proposition A2. *For $\phi \geq \frac{1}{\eta}$, if (x_t, t) is in the retirement region, then $(\lambda x_t, t)$ is also in the retirement region, $\forall \lambda \geq 1$.*

Proof of Proposition A2: At time t, when x_t changes to λx_t, since both ξ_t and w_t depend on the same Brownian motion, we assume ξ_t changes to $p\xi_t$, w_t changes to qw_t. Therefore, $\lambda = p^{-1}q^{-1}$. If (x_t, t) is in the retirement region, then $\tau_t^* = t$,

$$\begin{aligned} J_t &= E_t\left[D_{\tau_t^*}\right] = E_t\left[D_t\right] \\ &= \int_0^t -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta}\ln(w_v) - \frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta}\right)dv \\ &\quad + y\int_0^t \xi_v w_v dv + E_t\left[\int_t^T \phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln(\phi) - 1\right)dv\right], \end{aligned}$$

and

$$\sup_{\tau\in\mathcal{S}} E_t\left[\int_t^{\tau} -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(w_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)dv\right.$$
$$\left.+y\int_t^{\tau}\xi_v w_v dv+\int_{\tau}^{T}\phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln\left(\phi\right)-1\right)dv\right]$$
$$= E_t\left[\int_t^{T}\phi a_v\left(-\ln\left(\frac{y\xi_v}{a_v}\right)+\ln\left(\phi\right)-1\right)dv\right].$$

Now for $(\lambda x_t, t)$, we have

$$J_t\left(\lambda x_t\right)$$
$$= \int_0^{t} -a_v\left(\frac{1}{\eta}\ln\left(\frac{y\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(w_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)dv$$
$$+y\int_0^{t}\xi_v w_v dv$$
$$+\sup_{\tau\in\mathcal{S}} E_t\left[\int_t^{\tau} -a_v\left(\frac{1}{\eta}\ln\left(\frac{yp\xi_v}{a_v}\right)+\frac{1-\eta}{\eta}\ln\left(qw_v\right)-\frac{1-\eta}{\eta}\ln\left(\frac{1-\eta}{\eta}\right)+\frac{1}{\eta}\right)dv\right.$$
$$\left.+y\int_t^{\tau}p\xi_v q w_v dv+\int_{\tau}^{T}\phi a_v\left(-\ln\left(\frac{yp\xi_v}{a_v}\right)+\ln\left(\phi\right)-1\right)dv\right].$$

We want to show that

$$
\begin{aligned}
& \sup_{\tau \in \mathcal{S}} E_t \Bigg[\int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln \left(\frac{y p \xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (q w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \\
& + y \int_t^{\tau} p \xi_v q w_v dv + \int_{\tau}^{T} \phi a_v \left(- \ln \left(\frac{y p \xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv \Bigg] \\
& \overset{\tau = t}{=} E_t \left[\int_t^{T} \phi a_v \left(- \ln \left(\frac{y p \xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv \right] \\
= \; & E_t \left[\int_t^{T} \phi a_v \left(- \ln \left(\frac{y \xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv + \int_t^{T} \phi a_v \left(- \ln (p) \right) \right],
\end{aligned}
$$

i.e., $(\lambda x_t, t)$ is also in the retirement region.

$$
\begin{aligned}
& \sup_{\tau \in \mathcal{S}} E_t \Bigg[\int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln \left(\frac{y p \xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (q w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \\
& + y \int_t^{\tau} p \xi_v q w_v dv + \int_{\tau}^{T} \phi a_v \left(- \ln \left(\frac{y p \xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv \Bigg] \\
= \; & \sup_{\tau \in \mathcal{S}} E_t \Bigg[\int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln \left(\frac{y \xi_v}{a_v} \right) + \frac{1-\eta}{\eta} \ln (w_v) - \frac{1-\eta}{\eta} \ln \left(\frac{1-\eta}{\eta} \right) + \frac{1}{\eta} \right) dv \\
& + y \int_t^{\tau} \xi_v w_v dv + \int_{\tau}^{T} \phi a_v \left(- \ln \left(\frac{y \xi_v}{a_v} \right) + \ln (\phi) - 1 \right) dv \\
& + \int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln (p) + \frac{1-\eta}{\eta} \ln (q) \right) dv + (pq - 1) \, y \int_t^{\tau} \xi_v w_v dv \\
& + \int_{\tau}^{T} \phi a_v \left(- \ln (p) \right) dv \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln\left(\frac{y\xi_v}{a_v}\right) + \frac{1-\eta}{\eta} \ln(w_v) - \frac{1-\eta}{\eta} \ln\left(\frac{1-\eta}{\eta}\right) + \frac{1}{\eta} \right) dv \right. \\
& \left. + y \int_t^{\tau} \xi_v w_v dv + \int_{\tau}^{T} \phi a_v \left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln(\phi) - 1 \right) dv \right] \\
& + \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln(p) + \frac{1-\eta}{\eta} \ln(q) \right) dv + (pq-1)\, y \int_t^{\tau} \xi_v w_v dv \right. \\
& \left. + \int_{\tau}^{T} \phi a_v (-\ln(p))\, dv \right] \\
=\ & E_t \left[\int_t^{T} \phi a_v \left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln(\phi) - 1 \right) dv \right] \\
& + \sup_{\tau\in\mathcal{S}} \left[E_t \int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln(p) + \frac{1-\eta}{\eta} \ln(q) \right) dv + (pq-1)\, y \int_t^{\tau} \xi_v w_v dv \right. \\
& \left. + \int_{\tau}^{T} \phi a_v (-\ln(p))\, dv \right] \\
=\ & E_t \left[\int_t^{T} \phi a_v \left(-\ln\left(\frac{y\xi_v}{a_v}\right) + \ln(\phi) - 1 \right) dv + \int_t^{T} \phi a_v (-\ln(p)) \right] \\
& + \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln(p) + \frac{1-\eta}{\eta} \ln(q) \right) dv + (pq-1)\, y \int_t^{\tau} \xi_v w_v dv \right. \\
& \left. - \int_t^{\tau} \phi a_v (-\ln(p))\, dv \right].
\end{aligned}
$$

Now we analyze the term

$$
\begin{aligned}
& \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^{\tau} -a_v \left(\frac{1}{\eta} \ln(p) + \frac{1-\eta}{\eta} \ln(q) \right) dv + (pq-1)\, y \int_t^{\tau} \xi_v w_v dv \right. \\
& \left. - \int_t^{\tau} \phi a_v (-\ln(p))\, dv \right].
\end{aligned}
$$

We consider the sufficient conditions for immediate retirement. The term above is the sum of three parts, the second part is stochastic, while the first and the third are both deterministic. We consider them separately. Because for the term $(pq-1)\,y\int_t^\tau \xi_v w_v dv$, $y\xi_v w_v$ is positive, in order to immediately retire, we need $pq \leq 1$, thus we have $\lambda = p^{-1}q^{-1} \geq 1$. Because ξ_t changes to $p\xi_t$, w_t changes to qw_t, we have $p \leq 1$ and $q \geq 1$ if we assume θ is greater than σ_w, which is empirically reasonable. If $p \leq 1$ and $q \geq 1$, we have $\phi a_v\left(-\ln(p)\right) \geq 0$ and $-a_v\left(\frac{1}{\eta}\ln(p) + \frac{1-\eta}{\eta}\ln(q)\right) = -\frac{a_v}{\eta}\left(\ln(pq) - \eta\ln(q)\right) \geq 0$. Both the terms in the first and third integral are deterministic and therefore we can compare them. Immediately retire if $-a_v\left(\frac{1}{\eta}\ln(p) + \frac{1-\eta}{\eta}\ln(q)\right) - \phi a_v\left(-\ln(p)\right) \leq 0$ for all v, i.e. $(\eta\phi - 1)\ln(p) \leq (1-\eta)\ln(q)$. Sufficient condition to satisfy $(\eta\phi-1)\ln(p) \leq (1-\eta)\ln(q)$ is $\eta\phi \geq 1$, i.e. $\phi \geq \frac{1}{\eta}$. ♦

Proof of Lemma 14: Use the definitions

$$C(\rho,\eta) = \rho\left(\theta - (1-\eta)\sigma_w\right),$$

$$A(\beta,\rho,\eta) = \frac{\beta}{R} + \rho\left(r + \frac{1}{2}\theta^2\right) - (1-\eta)\rho\left(\mu_w - \frac{1}{2}\sigma_w^2\right) - \frac{1}{2}C(\rho,\eta)^2,$$

we have

$$C(1,0) = \theta - \sigma_w,$$

$$A(0,1,0) = r + \frac{1}{2}\theta^2 - \left(\mu_w - \frac{1}{2}\sigma_w^2\right) - \frac{1}{2}\left(\theta - \sigma_w\right)^2.$$

$$\begin{aligned}
x_v &\geq B_v \Leftrightarrow x_t \exp\left(\left(\mu_x - \frac{1}{2}\sigma_x^2\right)(v-t) + \sigma_x\left(W_v - W_t\right)\right) \geq B_v \\
&\Leftrightarrow \frac{1}{\sqrt{v-t}}\left[\left(W_v - W_t\right)\right] \geq \frac{1}{\sigma_x\sqrt{v-t}}\left[\log\left(\frac{B_v}{x_t}\right) - \left(\mu_x - \frac{1}{2}\sigma_x^2\right)(v-t)\right] \\
&\Leftrightarrow W \geq -d\left(x_t, B_v, v; 0\right),
\end{aligned}$$

where W is standard normal distribution and

$$d\left(x_t, B_v, v; C\left(\rho,\eta\right)\right) = \frac{1}{\sigma_x\sqrt{v-t}}\left[\log\left(\frac{x_t}{B_v}\right) + \left(\mu_x - \frac{1}{2}\sigma_x^2 - \sigma_x C\left(\rho,\eta\right)\right)(v-t)\right].$$

It follows that

$$P\left(t,T;\beta,\mathcal{R}\right) = \int_t^T a_{t,v} E_t\left[1_{\mathcal{R}(v)}\right] dv = \int_t^T a_{t,v} N\left(d\left(x_t, B_v, v; 0\right)\right) dv,$$

$$P\left(t,T;\beta,\mathcal{A}\right) = \int_t^T a_{t,v} E_t\left[1_{\mathcal{A}(v)}\right] dv = \int_t^T a_{t,v} N\left(-d\left(x_t, B_v, v; 0\right)\right) dv,$$

$$\frac{\partial}{\partial y} P\left(t,T;\beta,\mathcal{R}\right) = \int_t^T a_{t,v} n\left(d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv.$$

$$\frac{\partial}{\partial y} P\left(t,T;\beta,\mathcal{A}\right) = \int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x\sqrt{v-t}}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv,$$

We have,

$$\begin{aligned} E\left[W 1_{\mathcal{R}(v)}\right] &= \int_{-d(x_t,B_v,v;0)}^{\infty} z n(z)\, dz \\ &= -\int_{-d(x_t,B_v,v;0)}^{\infty} n'(z)\, dz = n\left(-d\left(x_t, B_v, v; 0\right)\right), \end{aligned}$$

$$\begin{aligned} E\left[W 1_{\mathcal{A}(v)}\right] &= \int_{-\infty}^{-d(x_t,B_v,v;0)} z n(z)\, dz \\ &= -\int_{-\infty}^{-d(x_t,B_v,v;0)} n'(z)\, dz = -n\left(-d\left(x_t, B_v, v; 0\right)\right), \end{aligned}$$

thus

$$
\begin{aligned}
& F_1\left(t, T; \beta, \mathcal{R}\right) \\
= \;& E_t\left[\int\limits_t^T a_{t,v} \ln\left(\frac{\xi_{t,v}}{a_{t,v}}\right) 1_{\mathcal{R}(v)} dv\right] \\
= \;& E_t\left[\int\limits_t^T a_{t,v}\left(\left(\beta - r - \frac{1}{2}\theta^2\right)(v-t) - \theta W\sqrt{v-t}\right) 1_{\mathcal{R}(v)} dv\right] \\
= \;& \left(\beta - r - \frac{1}{2}\theta^2\right)\int\limits_t^T a_{t,v}\left(v-t\right) E\left[1_{\mathcal{R}(v)}\right] dv - \theta \int\limits_t^T a_{t,v}\sqrt{v-t} E\left[W 1_{\mathcal{R}(v)}\right] dv \\
= \;& \left(\beta - r - \frac{1}{2}\theta^2\right)\int\limits_t^T a_{t,v}\left(v-t\right) N\left(d\left(x_t, B_v, v; 0\right)\right) dv \\
& -\theta \int\limits_t^T a_{t,v}\sqrt{v-t} n\left(-d\left(x_t, B_v, v; 0\right)\right) dv,
\end{aligned}
$$

$$
\begin{aligned}
& F_1\left(t, T; \beta, \mathcal{A}\right) \\
= \;& E_t\left[\int\limits_t^T a_{t,v} \ln\left(\frac{\xi_{t,v}}{a_{t,v}}\right) 1_{\mathcal{A}(v)} dv\right] \\
= \;& E_t\left[\int\limits_t^T a_{t,v}\left(\left(\beta - r - \frac{1}{2}\theta^2\right)(v-t) - \theta W\sqrt{v-t}\right) 1_{\mathcal{A}(v)} dv\right] \\
= \;& \left(\beta - r - \frac{1}{2}\theta^2\right)\int\limits_t^T a_{t,v}\left(v-t\right) E\left[1_{\mathcal{A}(v)}\right] dv - \theta \int\limits_t^T a_{t,v}\sqrt{v-t} E\left[W 1_{\mathcal{A}(v)}\right] dv \\
= \;& \left(\beta - r - \frac{1}{2}\theta^2\right)\int\limits_t^T a_{t,v}\left(v-t\right) N\left(-d\left(x_t, B_v, v; 0\right)\right) dv \\
& +\theta \int\limits_t^T a_{t,v}\sqrt{v-t} n\left(-d\left(x_t, B_v, v; 0\right)\right) dv,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial y} F_1\left(t, T; \beta, \mathcal{R}\right) \\
= \; & \left(\beta - r - \frac{1}{2}\theta^2\right) \\
&\times \int_t^T a_{t,v} \sqrt{v-t}\, n\left(d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x} \left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v} \right) dv \\
&-\theta \int_t^T a_{t,v}\, n\left(-d\left(x_t, B_v, v; 0\right)\right) d\left(x_t, B_v, v; 0\right) \frac{1}{\sigma_x} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t} \right) dv,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial y} F_1\left(t, T; \beta, \mathcal{A}\right) \\
= \; & \left(\beta - r - \frac{1}{2}\theta^2\right) \\
&\times \int_t^T a_{t,v} \sqrt{v-t}\, n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t} \right) dv \\
&+\theta \int_t^T a_{t,v}\, n\left(-d\left(x_t, B_v, v; 0\right)\right) d\left(x_t, B_v, v; 0\right) \frac{1}{\sigma_x} \left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t} \right) dv,
\end{aligned}
$$

$$
\begin{aligned}
& F_2\left(t, T; \beta, \mathcal{R}\right) \\
= \;& E_t\left[\int_t^T a_{t,v} \ln\left(w_{t,v}\right) 1_{\mathcal{R}(v)} dv\right] \\
= \;& E_t\left[\int_t^T a_{t,v}\left(\left(\mu_w - \frac{1}{2}\sigma_w^2\right)(v-t) + \sigma_w W\sqrt{v-t}\right) 1_{\mathcal{R}(v)} dv\right] \\
= \;& \left(\mu_w - \frac{1}{2}\sigma_w^2\right)\int_t^T a_{t,v}\left(v-t\right) E\left[1_{\mathcal{R}(v)}\right] dv + \sigma_w \int_t^T a_{t,v}\sqrt{v-t} E\left[W 1_{\mathcal{R}(v)}\right] dv \\
= \;& \left(\mu_w - \frac{1}{2}\sigma_w^2\right)\int_t^T a_{t,v}\left(v-t\right) N\left(d\left(x_t, B_v, v; 0\right)\right) dv \\
& + \sigma_w \int_t^T a_{t,v}\sqrt{v-t} n\left(-d\left(x_t, B_v, v; 0\right)\right) dv,
\end{aligned}
$$

$$
\begin{aligned}
& F_2\left(t, T; \beta, \mathcal{A}\right) \\
= \;& E_t\left[\int_t^T a_{t,v} \ln\left(w_{t,v}\right) 1_{\mathcal{A}(v)} dv\right] \\
= \;& E_t\left[\int_t^T a_{t,v}\left(\left(\mu_w - \frac{1}{2}\sigma_w^2\right)(v-t) + \sigma_w W\sqrt{v-t}\right) 1_{\mathcal{A}(v)} dv\right] \\
= \;& \left(\mu_w - \frac{1}{2}\sigma_w^2\right)\int_t^T a_{t,v}\left(v-t\right) E\left[1_{\mathcal{A}(v)}\right] dv + \sigma_w \int_t^T a_{t,v}\sqrt{v-t} E\left[W 1_{\mathcal{A}(v)}\right] dv \\
= \;& \left(\mu_w - \frac{1}{2}\sigma_w^2\right)\int_t^T a_{t,v}\left(v-t\right) N\left(-d\left(x_t, B_v, v; 0\right)\right) dv \\
& - \sigma_w \int_t^T a_{t,v}\sqrt{v-t} n\left(-d\left(x_t, B_v, v; 0\right)\right) dv,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial y} F_2\left(t, T ; \beta, \mathcal{R}\right) \\
= & \left(\mu_w - \frac{1}{2}\sigma_w^2\right) \\
& \times \int_t^T a_{t,v}\sqrt{v-t}\, n\left(d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x}\left(\frac{\frac{\partial x_t}{\partial y}}{x_t} - \frac{\frac{\partial B_v}{\partial y}}{B_v}\right) dv \\
& + \sigma_w \int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) d\left(x_t, B_v, v; 0\right) \frac{1}{\sigma_x}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv.
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial y} F_2\left(t, T ; \beta, \mathcal{A}\right) \\
= & \left(\mu_w - \frac{1}{2}\sigma_w^2\right) \\
& \times \int_t^T a_{t,v}\sqrt{v-t}\, n\left(-d\left(x_t, B_v, v; 0\right)\right) \frac{1}{\sigma_x}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv \\
& - \sigma_w \int_t^T a_{t,v} n\left(-d\left(x_t, B_v, v; 0\right)\right) d\left(x_t, B_v, v; 0\right) \frac{1}{\sigma_x}\left(\frac{\frac{\partial B_v}{\partial y}}{B_v} - \frac{\frac{\partial x_t}{\partial y}}{x_t}\right) dv. \blacklozenge
\end{aligned}
$$

Proof of Proposition 16: To derive the equation satisfied by $\frac{\partial B_t}{\partial y}$, take derivative with respect to y in equation (3.2). To derive the boundary condition for $\frac{\partial B_T}{\partial y}$, use the limiting condition

$$
B_T\left(\left(\phi - 1\right)\ln\left(w_0\left(\frac{B_T}{x_0}\right)^{\gamma}\exp\left(\delta T\right)\right) - \left(\frac{1}{\eta} - \phi\right)\ln\left(B_T\right) + K\right) = 1,
$$

or

$$
B_T\left(\left(\phi - 1\right)\ln\left(y^{\gamma} w_0^{1+\gamma} B_T^{\gamma}\exp\left(\delta T\right)\right) - \left(\frac{1}{\eta} - \phi\right)\ln\left(B_T\right) + K\right) = 1,
$$

Take derivative with respect to y,

$$\begin{aligned} & \frac{\partial B_T}{\partial y}\left((\phi-1)\ln\left(y^{\gamma}w_0^{1+\gamma}B_T^{\gamma}\exp(\delta T)\right)-\left(\frac{1}{\eta}-\phi\right)\ln(B_T)+K\right) \\ & +B_T\left((\phi-1)\gamma\frac{1}{B_T}\frac{\partial B_T}{\partial y}+(\phi-1)\gamma\frac{1}{y}-\left(\frac{1}{\eta}-\phi\right)\frac{1}{B_T}\frac{\partial B_T}{\partial y}\right) \\ = & \ 0. \end{aligned}$$

$\frac{\partial B_T}{\partial y}$ can be solved as a function of B_T,

$$\begin{aligned} & \frac{\partial B_T}{\partial y} \\ = & \ \frac{-B_T(\phi-1)\gamma\frac{1}{y}}{(\phi-1)\ln\left(y^{\gamma}w_0^{1+\gamma}B_T^{\gamma}\exp(\delta T)\right)-\left(\frac{1}{\eta}-\phi\right)\ln(B_T)+K+(\phi-1)\gamma-\left(\frac{1}{\eta}-\phi\right)}. \ \blacklozenge \end{aligned}$$

Proof of Lemma 20: The evolution of liquid wealth X_v is

$$dX_v=(rX_v-c_v)\,dv+\pi_v\sigma(\theta dv+dW_v),$$

state price density ξ_v satisfies

$$d\xi_v=-\xi_v(rdv+\theta dW_v).$$

Thus,

$$\begin{aligned} d(\xi_vX_v) &= \xi_v dX_v+X_v d\xi_v+d[\xi,X]_v \\ &= \xi_v(rX_v-c_v)\,dv+\xi_v\pi_v\sigma(\theta dv+dW_v) \\ &\quad -X_v\xi_v(rdv+\theta dW_v)-\pi_v\sigma\xi_v\theta dv. \end{aligned}$$

Simplifying,

$$d(\xi_vX_v)+\xi_vc_vdv=\xi_v(\pi_v\sigma-X_v\theta)\,dW_v.$$

Thus we have

$$\xi_t X_t + \int_0^t \xi_v c_v dv = x + \int_0^t \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v.$$

If (f, F, π) is admissible, then liquid wealth is nonnegative. The right hand side is a local martingale and the left hand side is bounded below, thus the right hand side is a supermartingale. Use optional stopping theorem, for stopping time $\tau \in \mathcal{S}$,

$$E\left[\xi_\tau X_\tau + \int_0^\tau \xi_v c_v dv\right] \leq x,$$

or

$$E\left[\int_0^\tau \xi_v \left(f_v + l_v\right) dv + \xi_\tau \left(F_\tau + L_\tau\right)\right] \leq x.$$

Next we show that if the static budget constraint (4.1) is satisfied, then $\exists\ \pi$, such that (f, F, π) is admissible. Use martingale representation theorem,

$$E_t\left[\int_0^\tau \xi_v c_v dv + \xi_\tau X_\tau\right] - E\left[\int_0^\tau \xi_v c_v dv + \xi_\tau X_\tau\right]$$
$$= \int_0^t \phi_v dW_v, \text{ for some } \phi_v,\ E\left[\int_0^T \phi_v^2 dv\right] < \infty, \text{ a.s.}$$

Choose π_v, so that $\xi_v \left(\pi_v \sigma - X_v \theta\right) = \phi_v$. Because

$$d\left(\xi_v X_v\right) + \xi_v c_v dv = \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v,$$

we have

$$
\begin{aligned}
&\xi_t X_t + \int_0^t \xi_v c_v dv \\
= \;& x + \int_0^t \xi_v \left(\pi_v \sigma - X_v \theta\right) dW_v \\
= \;& x + E_t \left[\int_0^\tau \xi_v c_v dv + \xi_\tau X_\tau \right] - E \left[\int_0^\tau \xi_v c_v dv + \xi_\tau X_\tau \right] \\
\geq \;& E_t \left[\int_0^\tau \xi_v c_v dv + \xi_\tau X_\tau \right].
\end{aligned}
$$

Thus

$$
\xi_t X_t \geq E_t \left[\int_t^\tau \xi_v c_v dv + \xi_\tau X_\tau \right],
$$

and

$$
X_t \geq 0,
$$

i.e., liquid wealth is nonnegative. ♦

In Chapte 3, define the state variable $x_t = a_t^{\frac{1}{R}} \left(y\xi_t\right)^{-\frac{1}{R}} L_t^{-1}$. The next proposition gives the sufficient conditions for the liquidation region to be up connected or down connected for x_t.

Proposition A3. *If $\sigma_l \leq \theta/R$, $\sigma_L < \theta/R$ and $-r + \mu_L - \sigma_L \theta \leq 0$, then the liquidation region is up connected. If $\sigma_l \leq \theta/R$, $\sigma_L > \theta/R$ and $-r + \mu_L - \sigma_L \theta \geq 0$, then the liquidation region is down connected.*

Proof of Proposition A3: At time t, when x_t changes to λx_t, we assume ξ_t changes to $o\xi_t$, L_t changes to pL_t and l_t changes to ql_t. Therefore, $\lambda = o^{-\frac{1}{R}} p^{-1}$. If (x_t, t) is in the liquidation region, then $\tau_t^* = t$,

$$\begin{aligned} J_t &= E_t\left[D_{\tau_t^*}\right] = E_t\left[D_t\right] \\ &= \int_0^t \left(\frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + y\xi_v g l_v - y\xi_v l_v\right) dv \\ &\quad + \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^\rho + y\xi_t s L_t - y\xi_t L_t, \end{aligned}$$

and

$$\begin{aligned} &\sup_{\tau\in\mathcal{S}} E_t\left[\int_t^\tau \left(\frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + y\xi_v g l_v - y\xi_v l_v\right) dv \right. \\ &\left. + \frac{R}{1-R} a_\tau^{\frac{1}{R}} (y\xi_\tau)^\rho + y\xi_\tau s L_\tau - y\xi_\tau L_\tau\right] \\ = \; & \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^\rho + y\xi_t s L_t - y\xi_t L_t. \end{aligned}$$

Now for $(\lambda x_t, t)$, we have

$$\begin{aligned} & J_t(\lambda x_t) \\ = \; & \int_0^t \left(\frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + y\xi_v g l_v - y\xi_v l_v\right) dv \\ & + \sup_{\tau\in\mathcal{S}} E_t\left[\int_t^\tau \left(o^\rho \frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + oqy\xi_v g l_v - oqy\xi_v l_v\right) dv \right. \\ & \left. + o^\rho \frac{R}{1-R} a_\tau^{\frac{1}{R}} (y\xi_\tau)^\rho + opy\xi_\tau s L_\tau - opy\xi_\tau L_\tau\right]. \end{aligned}$$

We want to show that

$$\begin{aligned}
&\sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \left(o^\rho \frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + oqy\xi_v g l_v - oqy\xi_v l_v \right) dv \right. \\
&\left. + o^\rho \frac{R}{1-R} a_\tau^{\frac{1}{R}} (y\xi_\tau)^\rho + opy\xi_\tau s L_\tau - opy\xi_\tau L_\tau \right] \\
&\overset{\tau=t}{=} o^\rho \frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^\rho + opy\xi_t s L_t - opy\xi_t L_t,
\end{aligned}$$

i.e., $(\lambda x_t, t)$ is also in the liquidation region. We have

$$\begin{aligned}
&\sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \left(o^\rho \frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + oqy\xi_v g l_v - oqy\xi_v l_v \right) dv \right. \\
&\left. + o^\rho \frac{R}{1-R} a_\tau^{\frac{1}{R}} (y\xi_\tau)^\rho + opy\xi_\tau s L_\tau - opy\xi_\tau L_\tau \right] \\
= \; & o^\rho \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \left(\frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + o^{1-\rho} qy\xi_v g l_v - o^{1-\rho} qy\xi_v l_v \right) dv \right. \\
&\left. + \frac{R}{1-R} a_\tau^{\frac{1}{R}} (y\xi_\tau)^\rho + o^{1-\rho} py\xi_\tau s L_\tau - o^{1-\rho} py\xi_\tau L_\tau \right] \\
\leq \; & o^\rho \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \left(\frac{R}{1-R} a_v^{\frac{1}{R}} (y\xi_v)^\rho + y\xi_v g l_v - y\xi_v l_v \right) dv \right. \\
&\left. + \frac{R}{1-R} a_\tau^{\frac{1}{R}} (y\xi_\tau)^\rho + y\xi_\tau s L_\tau - y\xi_\tau L_\tau \right] \\
&+ o^\rho \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \left(o^{1-\rho} q - 1 \right) (y\xi_v g l_v - y\xi_v l_v) \, dv \right. \\
&\left. + \left(o^{1-\rho} p - 1 \right) (y\xi_\tau s L_\tau - y\xi_\tau L_\tau) \right] \\
= \; & o^\rho \left(\frac{R}{1-R} a_t^{\frac{1}{R}} (y\xi_t)^\rho + y\xi_t s L_t - y\xi_t L_t \right) \\
&+ o^\rho \sup_{\tau\in\mathcal{S}} E_t \left[\int_t^\tau \left(o^{1-\rho} q - 1 \right) (y\xi_v g l_v - y\xi_v l_v) \, dv \right. \\
&\left. + \left(o^{1-\rho} p - 1 \right) (y\xi_\tau s L_\tau - y\xi_\tau L_\tau) \right].
\end{aligned}$$

Sufficient conditions for immediate liquidation are $\left(o^{1-\rho}q-1\right)\left(y\xi_v g l_v - y\xi_v l_v\right)\leq 0$ and that $\sup\limits_{\tau\in\mathcal{S}} E_t\left[\left(o^{1-\rho}p-1\right)\left(y\xi_\tau s L_\tau - y\xi_\tau L_\tau\right)\right]$ obtains the supremum when $\tau=t$. We have $g\geq 1$ and $s\leq 1$, because we assume intermediate contributions to the fund can be a multiple of the intermediate liability, but terminal contribution is limited to a fraction of the terminal liability. Since $g\geq 1$, for $\left(o^{1-\rho}q-1\right)\left(y\xi_v g l_v - y\xi_v l_v\right)$ to be nonpositive, it's sufficient to require $o^{1-\rho}q\leq 1$, which is equivalent to $\sigma_l\leq\theta/R$ if $p\geq 1$, $q\geq 1$ and $o\leq 1$. For $\sup\limits_{\tau\in\mathcal{S}} E_t\left[\left(o^{1-\rho}p-1\right)\left(y\xi_\tau s L_\tau - y\xi_\tau L_\tau\right)\right]$ to immediately exercise, sufficient condition is that the integrand in the delayed exercise premium is nonpositive for any time between 0 and T, i.e.,

$$\left(o^{1-\rho}p-1\right)(s-1)\left(-r+\mu_L-\sigma_L\theta\right)\leq 0.$$

Since $s\leq 1$, We need $\left(o^{1-\rho}p-1\right)\leq 0$ and $-r+\mu_L-\sigma_L\theta\leq 0$, or $\left(o^{1-\rho}p-1\right)\geq 0$ and $-r+\mu_L-\sigma_L\theta\geq 0$. In order for $\lambda=o^{-\frac{1}{R}}p^{-1}>1$, we need $\left(o^{1-\rho}p-1\right)<0$, which is equivalent to $\sigma_L<\theta/R$. In order for $\lambda=o^{-\frac{1}{R}}p^{-1}<1$, we need $\left(o^{1-\rho}p-1\right)>0$, which is equivalent to $\sigma_L>\theta/R$. ♦

Proof of Proposition 25: To derive the derivative of the boundary $\frac{\partial B_t}{\partial y}$, take derivative of equation (4.2) with respect to the multiplier y. To derive the boundary condition of $\frac{\partial B_T}{\partial y}$, use the limiting condition for boundary B_T,

$$\begin{aligned}&(g-1)\,z_0\left(\frac{B_T}{x_0}\right)^{\gamma}\exp(\delta T)+\frac{R}{1-R}B_T\left(1-A(\beta,\rho,1)\right)\\&+(s-1)\left(\mu_L-r-\sigma_L\theta\right)\\=\;&0,\end{aligned}$$

or

$$\begin{aligned}
& (g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}} B_T^{\gamma} \exp(\delta T) + \frac{R}{1-R} B_T \left(1 - A(\beta, \rho, 1)\right) \\
& + (s-1)(\mu_L - r - \sigma_L \theta) \\
= \; & 0.
\end{aligned}$$

Take derivative with respect to y,

$$\begin{aligned}
& \frac{\gamma}{R}(g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}-1} B_T^{\gamma} \exp(\delta T) + \gamma (g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}} B_T^{\gamma-1} \exp(\delta T) \frac{\partial B_T}{\partial y} \\
& + \frac{R}{1-R} \frac{\partial B_T}{\partial y} \left(1 - A(\beta, \rho, 1)\right) \\
= \; & 0.
\end{aligned}$$

$\frac{\partial B_T}{\partial y}$ can be solved as a function of B_T,

$$\frac{\partial B_T}{\partial y} = \frac{-\frac{\gamma}{R}(g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}-1} B_T^{\gamma} \exp(\delta T)}{\frac{R}{1-R}\left(1 - A(\beta, \rho, 1)\right) + \gamma (g-1)\, l_0 L_0^{\gamma-1} y^{\frac{\gamma}{R}} B_T^{\gamma-1} \exp(\delta T)}. \quad \blacklozenge$$

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Curriculum Vitae

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